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N-vector spin models on the simple-cubic and the body-centered-cubic lattices: A study of the critical behavior of the susceptibility and of the correlation length by high-temperature series extended to order β^{21}

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High-temperature expansions for the free energy, the susceptibility, and the second correlation moment of the classical N-vector model [also known as the $O(N)$ symmetric classical spin-Heisenberg model or as the lattice $O(N)$ nonlinear σ model] on the simple-cubic and the body-centered-cubic lattices are extended to order β^{21} for arbitrary N. The series for the second field derivative of the susceptibility is extended to order β^{17}. We report here on the analysis of the computed series for the susceptibility and the (second moment) correlation length which yields updated estimates of the critical parameters for various values of the spin dimensionality N, including $N=0$ (the self-avoiding walk model), $N=1$ (the Ising spin-1/2 model), $N=2$ (the XY model), and $N=3$ (the classical Heisenberg model). For all values of N we confirm a good agreement with the present renormalization-group estimates. A study of the series for the other observables will appear in a forthcoming paper. [S0163-1829(97)01838-9]

I. INTRODUCTION

The continuing interest in the high-temperature (HT) expansions for the statistical mechanics of lattice spin models (or equivalently in the strong-coupling expansions for Euclidean lattice field theories) has in the last few years been a strong incentive for a substantial extension of series in a variety of models. This valuable computational advance has been made possible not only by the large improvements of the computers performance and the rapid growth of their memory capacity in the last decade, but mainly by a more careful reconsideration of well-known expansion techniques, which have yet to be fully exploited, and by a greater effort in devising and implementing faster algorithms.[1,2]

We have devoted this paper to the widely studied N-vector model[3] on three-dimensional lattices, which is also known as the Heisenberg classical $O(N)$ spin model or, in field-theoretic language, as the lattice $O(N)$ nonlinear σ model. We recall that a HT expansion of the susceptibility χ to order β^{23} has been recently obtained[4] on the simple cubic (sc) lattice for $N=0$ [the self-avoiding walk (SAW) model[5]] by a direct walk counting technique which cannot be extended to different values of N. In the $N=1$ case [the Ising spin-1/2 model] the series $O(\beta^{21})$ on the body-centered-cubic (bcc) lattice for both χ and the second moment of the correlation function μ_2, obtained[6,7] in the pioneering work by Nickel, remain still unsurpassed, but for $N>1$ the series now available are significantly shorter. The general situation before our work of which a brief, partial, and preliminary account has already appeared in Ref. 8, is summarized in Table I listing the longest series published until now[4,6,7,9–19] for specific or generic values of N, on the sc and on the bcc lattices.

This work is part of a sequence devoted to the extension and the analysis of HT series to order β^{21} for the N-vector model on d-dimensional bipartite lattices. The case of the square lattice has already been discussed in Ref. 2. We have chosen to use the (vertex renormalized) linked cluster expansion (LCE) technique[13,20–22] because we have developed algorithms which make it equally efficient in a wide range of space dimensionalities independently of the nature of the site variables, whereas most other methods give the best performances only in very specific situations, such as two-dimensional or low coordination number lattices and for sufficiently simple interactions. Having thus avoided the limitations of previous work, we have been able to produce extensive tables of series-expansion coefficients given as explicit functions of the spin dimensionality N, which summarize in a convenient format a large body of information for an infinite set of universality classes. These tables, which we consider as the main result of our work, are reported in the Appendixes.

We have built upon Ref. 13 where some algorithms for

TABLE I. Longest published HT expansions for the N-vector model on the simple cubic and the bcc lattice before this work.

	Quantities expanded	Parameters	Maximal order	Reference
sc lattice				
	χ	$N=0$	23	4
	χ,μ_2	$N=0$	21	9
	χ	$N=1$	19	10
	μ_2	$N=1$	15	11
	χ,μ_2	$N=2$	17	12
	χ,μ_2	any N	14	13,14
bcc lattice				
	χ,μ_2	$N=0$	16	9
	χ,μ_2	$N=1$	21	6,7
	χ,μ_2	$N=2$	12	9
	χ	$N=3$	11	16
	χ,μ_2	any N	9	19

the automatic LCE calculations have been introduced, and HT expansions of χ, of μ_2, and of the second field derivative of the susceptibility χ_4 for the N-vector model on the sc lattices have been tabulated[13,14] up to β^{14} for dimensions $d=2,3,4$. This calculation has been generalized to the class of d-dimensional bipartite lattices, in particular to the (hyper)sc and (hyper)bcc lattices, for which we have striven to design faster and more efficient algorithms and have introduced some innovations dramatically simplifying established computational schemes. In particular, by taking full advantage of the structural properties of these lattices, we have significantly reduced the fast growth of the combinatorial complexity with the order of expansion which had until now been the main obstacle to the extension of the series. In fact, it is mainly our effort on the algorithmic side that has made progress possible, even in comparison with more recent work[23] using the same hardware resources. Moreover a considerable extension of our calculations is still feasible (and is presently ongoing[24]) since we are far from our computational limits. Our calculation used an ordinary IBM RISC 6000/580H power station with 128 Mb memory capacity and 4 Gb of disk storage. Typical running times in the three-dimensional case were a few hours. In order to give a rough idea of the size of the computation it is sufficient to mention that over 2×10^7 graphs have to be generated and evaluated to complete the expansion of χ and μ_2 through β^{21}. This should be compared with the corresponding figure: 1.1×10^4, of the $O(\beta^{14})$ computation in Ref. 13 or with the figure 5×10^4 occurring in the recent analogous computation of Ref. 23 for the face-centered-cubic lattice to order β^{13}. Approximately 3×10^6 graphs contribute to the computation of χ_4 at order 17.

We are confident that our results are correct, not only because our codes have passed numerous direct and indirect internal tests, but also because N and d enter in the whole computational procedure as parameters, so that a good general verification is achieved if our expansion coefficients, when specialized to $N=$ 0,1,2,3 and ∞, agree with the (more or less) long series already available in various dimensions. Further details on the comparison with the available series, in particular in the limits $N\to0$ and $N\to\infty$, can be found in our paper devoted to the two-dimensional N-vector model.[2]

Note that, strictly speaking, the N-vector model is defined only for positive integer spin dimensionality N. There are, however, infinitely many "analytic interpolations" in the variable N of the HT coefficients and, as a consequence, of all physical quantities. We have performed the "natural" analytic interpolation of the HT coefficients as rational functions of N, which coincides with that used in the $1/N$ expansion as well as in the usual renormalization-group (RG) treatments and is unique in the sense of the Carlson theorem.[25]

It is also worth emphasizing that the LCE technique can be readily adapted to produce HT expansions for the very general class of models [which include the $O(N)$ symmetric $P(\vec{\varphi}^2)$ lattice boson field theories], described by the partition function

$$Z=\int \Pi d\mu(\vec{\varphi}_i^2)\exp\left[\beta\sum_{\langle r,s\rangle}\vec{\varphi}_r\cdot\vec{\varphi}_s\right], \tag{1}$$

where $\vec{\varphi}_i$ is a N-component vector and $d\mu(\vec{\varphi}_i^2)$ is the appropriate single spin measure. If we choose the form $d\mu(\vec{\varphi}_i^2)=\delta(\vec{\varphi}_i^2-1)d\vec{\varphi}_i$ for the single spin measure, Eq. (1) reduces to the partition function of the N-vector model. Also a broad class of other models of interest in statistical mechanics including the general spin-S Ising model, the Blume-Capel model, the double Gaussian model, etc., can be represented in this form. The HT series for some of these models have been extended and will be discussed elsewhere.[24] A wider discussion of Eq. (1) as a full theoretical laboratory for the study of scalar isovector lattice field theories as well as a detailed account of the graph theoretical and the algorithmic part of our work will also be presented elsewhere.[24]

The general interest in a *direct* determination of the critical properties of the classical lattice spin models with increasing reliability is clear. Other good general motivations for such a laborious calculation as a long HT series expansion include more accurate tests of the validity both of the assumption of universality, on which the renormalization-group (RG) approach to critical phenomena is based, and of the various approximation procedures required to estimate universal critical parameters by field theoretic methods. In fact, waiting for rigorous arguments to come, the only crucial tests[26] of the validity of the Borel resummed $\epsilon=4-d$ expansions[26–28] or of the perturbative expansions at fixed dimension (FD),[26,28–31] for which the $O(N)$ model served as a paradigm, are presently limited to a careful comparison with experimental data or with other numerical data. Actually, this comparison mainly concerns numerical data of different origins, both because experiments are difficult and the experimental results, when available, are much less precise (with the very remarkable exception of the exponent ν in the $N=2$ case[32]) and because experimental representatives are known only for a few universality classes.[33] Actually for $N\gtrsim3$ the physical, but not the numerical, interest of the model is somewhat lowered by the observation that the $O(N)$ symmetric fixed point appears to be unstable within the ϵ expansion.[34]

It should also be observed that, in spite of steady progress,[35] the stochastic algorithms do not yet seem ready to completely supersede HT series in the study of models where the site variables have many components and/or the space dimensionality is large, or in the computation of multispin correlation functions. More generally HT series remain valuable subjects of independent study and sources of auxiliary information for other kinds of numerical calculations. Therefore we have also reported in the tables our estimates of nonuniversal critical parameters like the inverse critical temperatures. The computation of the critical amplitudes and of their universal combinations[36] will be discussed elsewhere.[24]

The paper is organized as follows: In Sec. II we present our notation and define the quantities that we shall study. The analysis of the series is presented in Sec. III along with a comparison to some previous analyses, to some results obtained by stochastic methods and to the RG results, both by the ϵ expansion and the FD perturbative techniques. In the Appendixes we have reported the closed-form expressions for the HT series coefficients of χ and μ_2 as functions of the spin dimensionality N and their evaluation for $N=0$ (the SAW model), $N=1$ (the Ising spin-1/2 model), $N=2$ (the

XY model), and $N=3$ (the classical Heisenberg model). The present tabulation extends significantly and supersedes the one to order β^{14} in Ref. 14 which, unfortunately, contains a few misprints. In a forthcoming paper[24] we will present an analysis of the series for the free energy and for its fourth field derivative χ_4.

II. DEFINITIONS AND NOTATIONS

For convenience of the reader we list here our definitions and notations. As the Hamiltonian H of the N-vector model we shall take

$$H\{v\}=-\frac{1}{2}\sum_{\langle\vec{x},\vec{x}'\rangle} v(\vec{x})\cdot v(\vec{x}'). \qquad (2)$$

where $v(\vec{x})$ is a N-component classical spin of unit length at the lattice site $\vec{x}$, and the sum extends to all nearest-neighbor pairs of sites.

The susceptibility is defined as

$$\chi(N,\beta)=\sum_{\vec{x}}\langle v(0)\cdot v(\vec{x})\rangle_c=1+\sum_{r=1}^{\infty} a_r(N)\beta^r, \qquad (3)$$

where $\langle v(0)\cdot v(\vec{x})\rangle_c$ is the connected correlation function between the spin at the origin and the spin at the site $\vec{x}$.

The second moment of the correlation function is defined as

$$\mu_2(N,\beta)=\sum_{\vec{x}} \vec{x}^2\langle v(0)\cdot v(\vec{x})\rangle_c=\sum_{r=1}^{\infty} s_r(N)\beta^r. \qquad (4)$$

In terms of χ and μ_2 we define the second moment correlation length ξ by

$$\xi^2(N,\beta)=\frac{\mu_2(N,\beta)}{6\chi(N,\beta)}. \qquad (5)$$

III. ANALYSIS OF THE SERIES

Let us now turn to a discussion of our updated estimates for the critical temperatures and the critical exponents γ and ν in the $N=2,3,4$ cases where our series are significantly longer (up to 10 more terms) than those previously available. We shall also give some comments on the cases $N=0$ and 1 in which our extension is more modest and for $N>4$ where only a few numerical results are available.

It has become clear from a long experience, mainly gained from the analysis of the Ising model HT expansions,[6,37–41] that in order to achieve a substantial improvement in the precision of the estimates of the critical parameters from the analysis of extended series one should properly allow for the expected nonanalytic corrections[42] (usually also called confluent corrections) to the leading power-law behavior of thermodynamic quantities near a critical point. For instance, we recall that, if we set $\tau=1-\beta/\beta_c$, the susceptibility is expected to behave, in the vicinity of the critical point β_c, as

$$\chi(N,\beta)\simeq C_\chi(N)\tau^{-\gamma(N)}[1+a_\chi(N)\tau^{\theta(N)}+a'_\chi(N)\tau^{2\theta(N)}+\cdots+e_\chi(N)\tau+e'_\chi(N)\tau^2+\cdots], \qquad (6)$$

when $\tau\downarrow 0$. Not only the critical exponent $\gamma(N)$, but also the leading confluent correction exponent $\theta(N)$ is universal (for each N). On the other hand, the critical amplitudes $C_\chi(N), a_\chi(N), a'_\chi(N), e_\chi(N)$, etc., are expected to depend smoothly on the parameters of the Hamiltonian, i.e. they are nonuniversal. Similar considerations apply to ξ (and to the other singular quantities) which, however, contains a different critical exponent and different critical amplitudes $C_\xi(N), a_\xi(N)$, etc., but the same leading confluent exponent $\theta(N)$. It is also known that $\theta(N)\simeq 0.5$ for small values of N (Ref. 26) and $\theta(N)=1+O(1/N)$ for large N.[43]

As experience has indicated, the established ratio extrapolation and Padé approximant (PA) methods are generally inadequate to the difficult numerical problem of determining simultaneously β_c, the critical exponent and the leading confluent exponent in Eq. (6), a task which essentially amounts to an intrinsically unstable double exponential fit. It is considered appropriate, then, to resort to the inhomogeneous differential approximants (DA) method,[44] a generalization of the PA method, which, in principle, can be better suited to represent functions behaving like $\phi_1(x)(x-x_0)^{-\gamma}+\phi_2(x)$ near a singular point x_0, where $\phi_1(x)$ is a regular function of x and $\phi_2(x)$ may contain a (confluent) singularity of strength smaller than γ.

A. Unbiased analysis

To begin with, we have performed a series analysis by DA's, essentially following the protocol suggested in Ref. 45 which is *unbiased* for confluent singularities. For each N, we have computed $\beta_c(N)$ and $\gamma(N)$ by first and second order DA's built in terms of the susceptibility series and have then used this estimate of $\beta_c(N)$ to bias the determination of $\nu(N)$ from the series for the square of the (second moment) correlation length ξ^2. Completely consistent results are also obtained, in general, by the method of critical point renormalization.[44] Also the specific-heat exponent $\alpha(N)$ can be estimated by examining the behavior of χ at $\beta=-\beta_c(N)$, where a weak antiferromagnetic singularity is expected for bipartite lattices. As shown in Ref. 46, having set $\tilde{\tau}=1+\beta/\beta_c$, one has

$$\chi(N,\beta)\simeq\tilde{c}(N)+\tilde{a}(N)\tilde{\tau}^{1-\alpha(N)}+\tilde{b}(N)\tilde{\tau}+\cdots \qquad (7)$$

as $\tilde{\tau}\downarrow 0$. We shall however present this study in a forthcoming paper[24] in order to jointly discuss also the results of the analysis of the free energy.

The results of the present unbiased analysis do not significantly modify those obtained in the similar preliminary study[8] with series $O(\beta^{19})$. They are reported in Table II for $N\leqslant 3$ and in Table III for $N\geqslant 4$ and compared with some of the most accurate recently published estimates [4,28–32,47–58] by various other methods, in particular by RG perturbative methods. Only for the physically most interesting cases, namely for $N=0,1,2,3$, elaborate Borel resummed estimates are available for both the fifth-order ϵ expansion[26–28] and either the six-loop[29] or the seven-loop FD expansion.[30]

The scope of the seven-loop FD computation of Ref. 30 is however slightly limited by the present uncertainty in the

TABLE II. A summary of the estimates of the critical parameters for $0 \leqslant N \leqslant 3$.

N	Method	Reference	β_c	γ	ν
0	HTE sc unbiased	4	0.2134987(10)	1.16193(10)	
	HTE sc unbiased		0.213497(6)	1.161(2)	0.592(2)
	HTE sc θ-biased		0.213493(3)	1.1594(8)	0.5878(6)
	MonteCarlo sc	47	0.2134969(10)		0.5877(6)
	MonteCarlo sc	48	0.213492(1)	1.1575(6)	
	HTE bcc unbiased		0.153131(2)	1.1612(8)	0.591(2)
	HTE bcc θ-biased		0.153128(3)	1.1582(8)	0.5879(6)
	RG FD perturb.	30		1.1569(8)	0.5872(8)
	RG ϵ expansion	26		1.157(3)	0.5880(15)
1	HTE sc unbiased		0.221663(9)	1.244(3)	0.634(2)
	HTE sc θ-biased		[0.2216544(3)]	1.2388(10)	0.6315(8)
	MonteCarlo sc	55	0.2216595(26)		0.6289(8)
	MonteCarlo sc	49	0.2216544(3)	1.237(2)	0.6301(8)
	HTE bcc	39		1.2395(4)	0.632(1)
	HTE bcc	7		1.237(2)	0.6300(15)
	HTE bcc unbiased		0.157379(2)	1.243(2)	0.634(2)
	HTE bcc θ-biased		[0.157373(2)]	1.2384(6)	0.6308(5)
	RG FD perturb.	30		1.2378(12)	0.6301(10)
	RG ϵ expansion	26		1.2390(25)	0.6310(15)
2	Experiment	32			0.6705(6)
	HTE sc unbiased		0.45419(3)	1.327(4)	0.677(3)
	HTE sc θ-biased		0.45419(3)	1.325(3)	0.675(2)
	MonteCarlo sc	51	0.45420(2)	1.308(16)	0.662(7)
	MonteCarlo sc	52	0.4542(1)	1.316(5)	0.670(7)
	MonteCarlo sc	54	0.454165(4)	1.319(2)	0.672(1)
	HTE bcc unbiased		0.320428(3)	1.322(3)	0.674(2)
	HTE bcc θ-biased		0.320427(3)	1.322(3)	0.674(2)
	HTE fcc	15	0.2075(1)	1.323(15)	0.670(7)
	RG FD perturb.	30		1.318(2)	0.6715(15)
	RG ϵ expansion	26		1.315(7)	0.671(5)
3	HTE sc unbiased		0.69303(3)	1.404(4)	0.715(3)
	HTE sc θ-biased		0.69305(4)	1.406(3)	0.716(2)
	MonteCarlo sc	53	0.693035(37)	1.3896(70)	0.7036(23)
	MonteCarlo sc	54	0.693002(12)	1.399(2)	0.7128(14)
	HTE bcc unbiased		0.486805(4)	1.396(3)	0.711(2)
	HTE bcc θ-biased		0.486820(4)	1.402(3)	0.714(2)
	MonteCarlo bcc	53	0.486798(12)	1.385(10)	0.7059(37)
	HTE fcc	16	0.3149(6)	1.40(3)	0.72(1)
	RG FD perturb.	30		1.3926(26)	0.7096(16)
	RG ϵ expansion	26		1.39(1)	0.710(7)

value of the renormalized coupling constant $\overline{g}(N)$ used in the calculation. Therefore, in Ref. 30, the exponent values have been conveniently expressed as the sum of the central estimate corresponding to this approximate value and of a small deviation proportional to the difference between $\overline{g}(N)$ and the true renormalized coupling $g^*(N)$. For simplicity, we have reported in Table II the central estimate and have allowed for the contribution of the possible deviation only by doubling the expected error of the summation procedure: this roughly amounts to assume optimistically an uncertainty in the value of the renormalized coupling of a few parts per thousand. For $N > 3$, no estimates of the exponents by the ϵ-expansion method have been published, while only very recently an extensive computation by the six-loop FD expansion method has appeared.[31] Unfortunately, no estimates of error for the exponents are given in Ref. 31, but, in analogy with the small-N case, we would reasonably expect relative errors of the order of 1%.

From our unbiased analysis of the sc lattice series we obtain exponent estimates essentially consistent, within their

TABLE III. A summary of the estimates of the critical parameters for $4 \leqslant N \leqslant 12$.

N	Method	Reference	β_c	γ	ν
4	HTE sc unbiased		0.93589(6)	1.474(4)	0.750(3)
	HTE sc θ-biased		0.93600(4)	1.491(4)	0.759(3)
	MonteCarlo sc	56	0.9360(1)	1.477(18)	0.7479(90)
	MonteCarlo sc	54	0.935861(8)	1.478(2)	0.7525(10)
	HTE bcc unbiased		0.65531(6)	1.461(4)	0.744(3)
	HTE bcc θ-biased		0.65542(3)	1.484(4)	0.756(3)
	RG FD perturb.	57		1.45(3)	0.74(1)
	RG FD perturb.	31		1.449	0.738
6	HTE sc unbiased		1.42859(6)	1.582(5)	0.804(3)
	HTE sc θ-biased		1.42895(6)	1.614(5)	0.821(3)
	HTE bcc unbiased		0.99613(6)	1.566(4)	0.796(3)
	HTE bcc θ-biased		0.99644(4)	1.608(4)	0.819(3)
	RG FD perturb.	31		1.556	0.790
8	HTE sc unbiased		1.9263(2)	1.656(5)	0.840(3)
	HTE sc θ-biased		1.92705(7)	1.701(4)	0.864(3)
	HTE bcc unbiased		1.33984(7)	1.644(5)	0.833(3)
	HTE bcc θ-biased		1.34040(6)	1.696(4)	0.862(3)
	RG FD perturb.	31		1.637	0.830
	1/N expansion	58		1.6449	0.8355
10	HTE sc unbiased		2.4267(2)	1.712(6)	0.867(4)
	HTE sc θ-biased		2.42792(8)	1.763(4)	0.894(4)
	HTE bcc unbiased		1.68509(8)	1.699(5)	0.860(4)
	HTE bcc θ-biased		1.68586(7)	1.761(4)	0.893(3)
	RG FD perturb.	31		1.697	0.859
	1/N expansion	58		1.7241	0.8731
12	HTE sc unbiased		2.9291(3)	1.759(6)	0.889(4)
	HTE sc θ-biased		2.9304(1)	1.812(5)	0.916(4)
	HTE bcc unbiased		2.03130(8)	1.741(6)	0.881(4)
	HTE bcc θ-biased		2.03230(8)	1.808(5)	0.914(3)
	RG FD perturb.	31		1.743	0.881
	1/N expansion	58		1.7746	0.8969

errors, with the available ϵ-expansion results, but, in general, slightly larger (up to $\simeq 1\%$, or even more for intermediate values of N) than the FD expansion results.

In the case of the bcc lattice, however, our unbiased estimates are also completely compatible with the sixth-order[26,29] FD perturbative results or with the most recent seventh-order[30] results. A similar situation has already been encountered in a previous very accurate unbiased analysis of the $N=1$ case.[45] We see this simply as a confirmation that the series for lattices with larger coordination number have a faster convergence rate (or, in other words, a greater ''effective length''[45]) and also as an indication that the simplest unbiased DA's might be only partially able to describe the confluent singularities. Therefore the larger discrepancies of the sc lattice results should not be interpreted as indicative of universality violations, but rather as a warning that the systematic errors of our analysis due to the finite length of our series and to the confluent singularities are quite likely to be underestimated if we evaluate the uncertainties solely from the scatter of the approximant values obtained using a sufficiently large number of series coefficients. We should also stress that, for $N \geqslant 4$, the sequence of DA estimates for the critical temperature or the exponents which use an increasing number of coefficients, show evident residual trends which indicate the presence of important confluent corrections to scaling, so that some ''reasonable'' extrapolation of the results becomes necessary. Since this inevitably involves some assumption on the confluent exponent and therefore introduces some biasing, it will be more appropriately dealt with in the next paragraph. In order to distinguish clearly the effects of the various assumptions, we have chosen not to perform any further extrapolation in our ''unbiased estimates'' reported in Tables II and III, although it is clear that neglecting residual trends is a source of sizable systematic error.

Even within these limitations, we have improved the precision of the values of the critical parameters as obtained from HT series by unbiased methods, and, so far, have not inferred from this analysis any indication of a serious inconsistency with the estimates from RG.

In conclusion, within the *unbiased* approach to series analysis, the influence of the confluent singularities can be assessed more accurately and the results of the analysis can be better reconciled with the estimates from the RG methods probably only by computing still longer series, as it has been already recognized for the Ising model.[6,38,39] On the other hand, if we are ready to assume universality from the beginning, then, as suggested by work on the bcc lattice in the $N=1$ case[7,39] whose results are reported in Table II, a more accurate determination of the critical exponents could be obtained, even without further extending the series, by a study of appropriately built families of models depending on some continuous parameter, for each universality class. The idea is, essentially, to minimize or suppress the amplitude of the dominant confluent correction to scaling in Eq. (6) by taking advantage of its nonuniversality, namely of its continuous dependence on the parameter entering in the Hamiltonian. Using our LCE computation, it is now possible to implement easily this procedure for any N and on two different lattices, thereby corroborating its reliability. These developments of our study and a detailed analysis of other features of the series including estimates of the universal ratios of the leading confluent amplitudes for χ and ξ^2 will be presented elsewhere.[24]

B. Biased analysis

We have also analyzed our series by various *biased* methods, in particular by using properly designed first-order inhomogeneous DA's in which both $\beta_c(N)$ and the correction to scaling exponent $\theta(N)$ have been fixed, or by second-order inhomogeneous DA's in which only $\theta(N)$ (Refs. 7,59) has been fixed.

In order to provide the additional information needed in these approaches, for $N\leq 3$, we have assumed that the exponents $\theta(N)$ take the values predicted by the FD perturbative RG (Ref. 26)

$$\theta(0)=0.470(25),\quad \theta(1)=0.498(20),\quad \theta(2)=0.522(18),\quad \theta(3)=0.550(16). \tag{8}$$

For $N>3$, the first FD perturbative estimates at six-loop order have only been obtained very recently by Sokolov[60] and have been kindly communicated to us before publication

$$\theta(4)=0.578(10),\quad \theta(6)=0.626(10),\quad \theta(8)=0.670(10),\quad \theta(10)=0.707(10),\quad \theta(12)=0.737(10). \tag{9}$$

We have therefore been able to revise the biased analysis presented in the first preprint version of this paper, where, for lack of a better choice, we had used values for $\theta(N)$ ($N>3$) [larger than those in Eq. (9)], obtained by a reasonable, but unwarranted interpolation method. In some cases, in particular for $N=0$ in Refs. 4,47,61 and for $N=1$ in Refs. 7,62, numerical work has suggested slightly different confluent exponents, which might be more accurate or, to some extent, also provide an ''effective'' description of higher confluent corrections. We shall return later to this point, but we should mention that, at the level of precision of the following calculations, the precise values of the exponents $\theta(N)$ do not matter too much, since we have observed that our biased estimates of the leading critical exponents remain practically stable (within their errors) under variations of the confluent exponents up to $\simeq 5-10\%$. Of course other quantities, such as the confluent amplitudes, are very sensitive to the values of $\theta(N)$.

Let us now sketch the first biased method. If an accurate estimate for β_c is available, we can formulate a quite simple procedure, biased with both β_c and θ. From the asymptotic formula (6) we get

$$\gamma(N,\beta)\equiv[\beta_c(N)-\beta]\frac{d\ln\chi(N,\beta)}{d\beta}=\gamma(N)+\rho(N)[\beta_c(N)-\beta]^{\theta(N)}+o[(\beta_c(N)-\beta)^{\theta(N)}] \tag{10}$$

where $\rho(N)=-\theta(N)a_\chi(N)$.

We can approximate the quantity $\gamma(N,\beta)$ by the particular class of first-order inhomogeneous DA's defined as the solutions of the equation

$$Q_m(\beta)\left[(\beta_c(N)-\beta)\frac{dy}{d\beta}+\theta(N)y(\beta)\right]+R_n(\beta)=0. \tag{11}$$

Here $Q_m(\beta)$ and $R_n(\beta)$ are polynomials of degrees m and n, respectively, calculated, as usual, from the known series expansion of $\gamma(N,\beta)$. As a result the exponent γ is simply estimated as

$$\gamma(N)_{m,n}=\frac{-R_n(\beta_c(N))}{\theta(N)Q_m(\beta_c(N))}$$

and the amplitude $\rho(N)$ of the subleading term in Eq. (10) is given by the formula

$$\rho(N)_{m,n}=\frac{y(0)-\gamma(N)_{m,n}}{\beta_c(N)^{\theta(N)}}-\int_0^{\beta_c(N)}\frac{D(t)_{m,n}dt}{[\beta_c(N)-t]^{1+\theta(N)}} \tag{12}$$

where

$$D(t)_{m,n}=\frac{R_n(t)}{Q_m(t)}-\frac{R_n(\beta_c(N))}{Q_m(\beta_c(N))}.$$

We consider only almost diagonal approximants, namely those with $|m-n|\leq 4$. This procedure, which might be seen as the simplest, although certainly not the most general, DA extension of the biased PA method introduced in Refs. 63,64, works accurately on model series having the analytic structure expected for $\gamma(N,\beta)$ in the vicinity of $\beta_c(N)$. A similar procedure can be applied to $\xi^2(N,\beta)/\beta$ in order to compute the exponent $\nu(N)$. In order to give an idea of the results, let us for example, consider the $N=1$ sc lattice series. Assuming $\beta_c=0.2\,216\,544(3)$ as suggested in Ref. 49 and $\theta=0.498(20)$, we estimate $\gamma=1.2388(10)$ and $\nu=0.6315(8)$. With the same value of θ, in the bcc lattice case, assuming $\beta_c=0.157\,373(2)$ as suggested in Ref. 7, we obtain $\gamma=1.2384(6)$ and $\nu=0.6308(5)$. In general, when the value of β_c is not known accurately, it would be more appropriate to present the results in the form of a linear relationship between the critical exponent and β_c, at a given fixed value of θ. We can, however, use this method also to determine the value of β_c by fixing only θ and looking for the (generally small) range of values of β_c for which the uncertainty of the exponent γ is minimal. The estimates obtained in this way are in general completely consistent with those from the analogous biased PA procedure of Refs. 63,64. The values of β_c so obtained are also, in most cases, consistent with those obtained from the unbiased improved ratio methods of Ref. 38 after extrapolating the sequences of results linearly in $1/r^{1+\theta(N)}$ (where r is the number of series coefficients used), and with the estimates obtained by similarly extrapolating the results from unbiased DA's. Analogous considerations apply to our exponent estimates, which, however, have to be compared to the results obtained from improved ratio methods by extrapolating linearly in $1/r^{\theta(N)}$.

A second method, modeled after Refs. 7,59, which is biased only with the value of the confluent exponent, may be described as follows: for each value of N, we have considered the approximants derived from inhomogeneous second-order differential equations with the structure

$$[\beta_c(N)-\beta]^2[\beta_c(N)+\beta]Q_l(\beta)\frac{d^2y}{d\beta^2}+[\beta_c(N)-\beta]P_m(\beta)\frac{dy}{d\beta}+R_n(\beta)y(\beta)+T_s(\beta)=0,$$

where $Q_l(\beta), P_m(\beta), R_n(\beta)$, and $T_s(\beta)$ are polynomials of degrees l,m,n, and s in the variable β. By this choice the DA's are biased to be singular at $\beta=\beta_c(N)$ and at $\beta=-\beta_c(N)$. We restrict ourselves to almost diagonal DA's (namely those with $l+3\simeq m+1\simeq n$ and $s\leqslant 4$), which use at least 19 series coefficients. For each DA, we adjust $\beta_c(N)$ in a small range around the values indicated by the unbiased analysis of the previous section until the correction to scaling exponent $\theta(N)$ reaches precisely the central value indicated in Eqs. (8),(9). The corresponding inverse critical temperature and exponents are then taken as the best estimates of these quantities. It should be noticed that, within this approach, the values of β_c for χ and ξ^2 cannot be forced to be equal, but the differences are generally not much larger than the errors. In this approach (like in the previous one) the evaluation of the errors is based, as usual, on the spread among the approximant values and includes a generous allowance for the uncertainty in the biased value of $\theta(N)$. Let us finally mention that we also have preliminarily tested the reliability of this procedure on various model series built in such a way to reproduce the main expected features of χ and ξ^2.

An important remark concerns the values of the amplitudes of the confluent corrections in χ and ξ^2, which can be obtained from Eq. (12). We have observed that, in the bcc lattice case, they are negative for $N\lesssim 2$, positive and increasing for $N>2$. A quite similar behavior of these amplitudes is observed in the sc lattice case, where the change in sign occurs for $N\simeq 3$. For $2<N<4$, the amplitudes are relatively small for both lattices, but they become important for $N>4$. This remark, which will be illustrated in full detail elsewhere,[24] is also consistent with the fact that our biased estimates of the critical exponents are increasing functions of the confluent exponents for $N\lesssim 2$, while they are decreasing functions for $N\gtrsim 4$. For $2\lesssim N\lesssim 4$ some approximants give increasing functions and others give decreasing functions. Completely consistent indications on the behavior of the confluent amplitudes can also be inferred from the features of the improved ratio plots,[38] or by studying, as function of N, the difference among the unbiased DA estimates of β_c obtained from χ, μ_2 or ξ^2, as suggested in Ref. 39. In the $N=1$ case, our remark agrees with the results of earlier studies[7,39,65,41] on the sc, bcc, and fcc lattices, which established that the sign of the leading confluent amplitudes in χ and ξ^2 is negative.

Some features of the biased procedures we have adopted, may appear questionable or may call for further improvement. We have always tried to minimize the possible defects by forming our final estimates with the, generally quite compatible, indications coming from all the various available biased methods and not only from the two biased DA procedures we have described above. Moreover we have indicated very conservative error bars. It is however reasonable to expect that the accuracy of the biased estimates depends not only on whether the present series are long enough to provide the essential information on the subdominant singularities, but also on whether most of the corrections to scaling can actually be well described by the first nonanalytic term in the asymptotic formula (6). We believe, however, that the final results are at least consistent and suggestive. First of all, our analysis confirms that the total (statistical + systematical) errors of the previous unbiased approach are larger than we have indicated. For $N\leqslant 1$ the biased exponent estimates differ only slightly from the unbiased ones and in such a way to improve the agreement with the most accurate FD perturbative estimates, with stochastic simulations and with experimental results. For $N=2,3$ the agreement with the RG, in the FD perturbative approach, is, perhaps, less convincing. For $N\geqslant 4$ our biased exponent estimates are systematically larger (up to $\simeq 4\%$) than the FD six-loop perturbative values.[31] This discrepancy is parallel to and, in our opinion, is strictly related to the fast increase of the confluent amplitudes of χ and ξ^2 in the same range of N. Therefore the problem seems to be of a purely numerical nature. Of course, we cannot prove that, in these circumstances, our biased analyses can allow for the confluent singularities better than the present Padé-Borel resummation techniques do within the FD perturbative approach. We can perhaps only argue that, at least for large N, our results might be more accurate by comparing them with the results of the $1/N$ expansion of the exponents. Indeed for $N\gtrsim 14$, the $1/N$ estimates, which seem to be rather well converged and therefore reasonably accurate, seem to approach our biased bcc estimates faster than the FD values. Let us finally add that these numerical problems with the perturbative estimates of the exponents might also be partly due to possible residual imprecisions in the values of the renormalized coupling constants entering in the FD perturbative calculation.[30] We shall return to this point when new estimates of the coupling constant will be available from HT series.[24]

C. Final comments

Let us now finally add some comments on the results of the series analysis which are presented in the tables. In the $N=0$ case, the SAW model, we have not attempted either to report in Table II or to cite in the references even only a representative sample of the large amount of numerical work accumulated over the years, which fortunately has been reviewed recently in the very extensive and valuable new treatise[66] and in Ref. 47 devoted to a high precision stochastic study.

On the sc lattice, our series for χ and μ_2 are not yet longer than those of Ref. 4 and of Ref. 9, respectively, but we have taken advantage of the two additional published expansion coefficients of χ in order to test the stability of our biased estimates. We recall that the previous HT analysis by unbiased DA's of the sc lattice series to order β^{21} performed in Ref. 9 produced the estimates $\beta_c^{sc}=0.213\,496(4)$, $\gamma=1.161(1)$, and $\nu=0.592(4)$, which are all completely consistent with the results of the later analysis of Ref. 4 and with our own unbiased analysis, but slightly larger than our biased estimates. Very recently the significantly lower estimates $\beta_c^{sc}=0.213\,492(1)$ and $\gamma=1.1575(6)$ have been obtained by a stochastic method sampling SAW's which extend up to 4×10^4 steps.[48] Using this estimate of β_c^{sc} in the first DA method introduced in the previous section, we get $\gamma=1.1589(8)$; in order to obtain the lower value $\gamma\simeq1.1575$ a value $\beta_c^{sc}\simeq0.213\,488$ would be required, which is somewhat below the range suggested by the presently available HT series on the sc lattice, even allowing for the confluent corrections.

In the bcc lattice case we have computed five new coefficients for χ and μ_2 beyond those reported in Ref. 9. This makes it worth computing new estimates for the exponents. We recall that the analysis in Ref. 9 of the $O(\beta^{16})$ bcc lattice series available until now yielded the values $\beta_c^{bcc}=0.153\,137(10)$, $\gamma=1.162(2)$, and $\nu=0.592(2)$, which are less precise, but compatible with our new unbiased estimates and somewhat larger than our corresponding biased estimates. On the other hand, our biased estimate of γ for the bcc lattice agrees more closely with the new stochastic estimate.[48] We should also stress that, for both the sc and the bcc lattices, our biased estimates of ν have now come very close to the RG estimates and to the experimental value $\nu=0.586(4)$ reported in Ref. 67.

The value of $\theta(0)$ is still controversial.[61] For example, the study[4] of the long HT series available on the sc lattices suggests $\theta(0)\simeq1$, while an extensive Monte Carlo study[47] on the same lattice rather indicates an effective exponent $\theta(0)\simeq0.56(3)$. Assuming this last value in our computation instead of the one in Eq. (8), would only shift the biased estimates of γ and ν, in the bcc lattice case, from 1.1582(8) to 1.1587(8) and from 0.5879(6) to 0.5883(6), respectively. Similarly in the sc lattice case the estimate of γ (using all 23 available coefficients) would change from 1.1594(8) to 1.1597(8) and that of ν from 0.5878(6) to 0.5894(8).

A final remark is that from our extended series for χ on the bcc lattice we can derive[68] the new rigorous and stricter inequality $\beta_c^{bcc}(0)\geqslant\exp(-(1/21)\ln a_{21}(0))\simeq0.148\,582$, which slightly improves the previous bound obtained from $a_{16}(0)$ and quoted in Ref. 66.

In the $N=1$ case, the Ising spin-1/2 model, the relevant numerical studies are even more numerous than for the SAW model, so that we can only address the reader to the recent extensive review which in Ref. 49 complements stochastic computations of unprecedented accuracy on the sc lattice. On this lattice, we have extended by two terms the χ series and by six terms the μ_2 series[11] so that it is worthwhile at least to update the estimate of ν. In order to compute the exponents reported in Table II from the sc series, we have assumed $\theta(1)=0.498(20)$ and simply taken the extremely accurate value $\beta_c^{sc}(1)=0.221\,654\,4(3)$ obtained in Ref. 49 and therefore indicated within parentheses. Our bcc lattice series for χ and μ_2 are not yet longer than the series of Refs. 6,7. In this case, we have assumed the same value of $\theta(1)$ and taken $\beta_c^{bcc}(1)=0.157\,373(2)$ from Ref. 7 (similarly indicating it within parentheses). Also for the Ising model, values of the confluent exponent slightly larger than the one we have assumed, such as $\theta(1)=0.52(3)$ in Ref. 7 and $\theta(1)=0.54(3)$ in Refs. 39,62 have been reported. Taking the largest of these values in our biased computation would only change, in the bcc lattice case, the central estimate of γ from 1.2384 to 1.2387 and would shift that of ν from 0.6308 to 0.6311. In the sc lattice case the central value of γ would be shifted from 1.2388 to 1.2392 and that of ν from 0.6315 to 0.6321.

In the $N=2$ case, the XY model, we have computed four more terms of the χ and μ_2 series in the sc lattice case. Notice that the last two coefficients of the previously published $O(\beta^{17})$ series contained tiny numerical errors (inconsequential for the analysis in Ref. 12) which are corrected by our computation. In the case of the bcc lattice, our extension of the series for χ and μ_2 amounts to nine terms and gives a greater significance to the new exponent estimates.

In the $N=3$ case, the classical Heisenberg model, we have extended by seven terms the series for χ and μ_2 on the sc lattice. In the case of the bcc lattice we have extended the series by ten terms.

For all $N>3$, only series up to $O(\beta^9)$ were available until now on the bcc lattice and therefore our extension amounts to 12 terms. On the sc lattice we have computed seven additional series coefficients for χ and μ_2. Let us finally note that, on the sc lattice for $N=2$,3, and 4, the estimates of β_c indicated by the simulation of Ref. 54 are only slightly smaller than ours. Using these values in our biased DA method, we would obtain $\gamma\simeq1.322$, $\gamma\simeq1.402$, and $\gamma\simeq1.476$, for $N=2$,3 and 4, respectively. For the exponent ν the agreement would be somewhat closer.

IV. CONCLUSIONS

In conclusion, we have produced HT expansions through order β^{21} of the susceptibility and of the second correlation moment for the classical N-vector model with general N, on the sc and the bcc lattices. This rich material has been conveniently tabulated in the Appendixes in order to offer an easy opportunity for further study.

As a first application of our results, we have updated the direct estimates of the critical parameters of the N-vector model with a considerable improvement in accuracy over previous analyses and, for all values of N, we have confirmed a generally good agreement with the most precise calculations by current approximate RG methods.

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APPENDIX A: THE SUSCEPTIBILITY ON THE sc LATTICE

The HT expansion coefficients of the susceptibility $\chi(N,\beta)=1+\Sigma_{r=1}{}^{\infty}a_r(N)\beta^r$ on the sc lattice are

$$a_1(N)=6/N,$$

$$a_2(N)=30/N^2,$$

$$a_3(N)=(300+144N)/[N^3(2+N)],$$

$$a_4(N)=(1452+666N)/[N^4(2+N)],$$

$$a_5(N)=(28272+19116N+3024N^2)/[N^5(8+6N+N^2)],$$

$$a_6(N)=(270816+310296N+114312N^2+13476N^3)/[N^6(2+N)^2(4+N)].$$

For the coefficients which follow, it is typographically more convenient to set $a_r(N)=P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$

$$P_7(N)=15626880+27729888N+19020144N^2+6271680N^3+989304N^4+59328N^5,$$

$$Q_7(N)=N^7(2+N)^3(4+N)(6+N),$$

$$P_8(N)=74489472+129538464N+87007488N^2+28118784N^3+4362672N^4+258354N^5,$$

$$Q_8(N)=N^8(2+N)^3(4+N)(6+N),$$

$$P_9(N)=2847568896+5193056640N+3778300896N^2+1399800432N^3+278225208N^4$$

$$+28031388N^5+1115856N^6,$$

$$Q_9(N)=N^9(2+N)^3(4+N)(6+N)(8+N),$$

$$P_{10}(N)=108255780864+274456495104N+296103049728N^2+177344398656N^3$$

$$+64445676384N^4+14541739920N^5+1987352808N^6+150093888N^7+4784508N^8,$$

$$Q_{10}(N)=N^{10}(2+N)^4(4+N)^2(6+N)(8+N),$$

$$P_{11}(N)=41222946816000+137827824353280N+204681013383168N^2+178043163816960N^3$$

$$+100715199380736N^4+38875384707840N^5+10440501635712N^6+1949080239936N^7$$

$$+247560454704N^8+20342269008N^9+971521080N^{10}+20393856N^{11},$$

$$Q_{11}(N)=N^{11}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{12}(N)=195470369095680+645668315578368N+947108563906560N^2+813739409599488N^3$$

$$+454747923263232N^4+173480151209088N^5+46077593061312N^6+8514940815264N^7$$

$$+1071698638656N^8+87360454320N^9+4143560904N^{10}+86473548N^{11},$$

$$Q_{12}(N)=N^{12}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{13}(N)=11135576427724800+37234096307896320N+55589350635700224N^2$$

$$+48958061664927744N^3+28310386659250176N^4+11317910548800768N^5$$

$$+3205796898092544N^6+647734280214336N^7+92551274785344N^8+9107063617056N^9$$

$$+584874211968N^{10}+21978467784N^{11}+365034816N^{12},$$

$$Q_{13}(N)=N^{13}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)(12+N),$$

$$P_{14}(N)=632609126775521280+2510657938044223488N+4524841591275257856N^2$$

$$+4905292095121784832N^3+3571783932314320896N^4+1847293475548572672N^5$$

$$+699626042228688384N^6+197061627772250880N^7+41467574110306944N^8$$

$$+6484097497687296N^9+741110387409216N^{10}+59988858732000N^{11}$$

$$+3247646112432N^{12}+105152764464N^{13}+1534827960N^{14},$$

$$Q_{14}(N)=N^{14}(2+N)^6(4+N)^3(6+N)^2(8+N)(10+N)(12+N),$$

$$P_{15}(N)=50358104883694927 8720+234835971250434814 7712N+504810797135170004 5824N^2$$

$$+663966053684500915 8144N^3+598167699580621592 9856N^4+391559881864330181 8368N^5$$

$$+192792215327900581 4784N^6+72911069365969234 3296N^7+21436455395684401 7664N^8$$

$$+4926274936382501 5296N^9+884573804115079910 4N^{10}+123346421081745638 4N^{11}$$

$$+131857500849437952N^{12}+10572674738185920N^{13}+613643418709152N^{14}$$

$$+24276406641408N^{15}+584083191744N^{16}+6431000832N^{17},$$

$$Q_{15}(N)=N^{15}(2+N)^7(4+N)^3(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{16}(N)=9524275145197517537280+46390115403005090070528N$$

$$+104731459930689707704320N^2+145566739232360789704704N^3$$

$$+139543768553865728753664N^4+97954888922330792558592N^5$$

$$+52173686760085255741440N^6+21557196989806394449920N^7$$

$$+7003643461540643672064N^8+1802237141876529444864N^9$$

$$+368115588315892769280N^{10}+59520402393193492992N^{11}+7558378513523449344N^{12}$$

$$+743196152207439744N^{13}+55303243308153696N^{14}+3002949947640192N^{15}$$

$$+111961279505712N^{16}+2555753709864N^{17}+26862228450N^{18},$$

$$Q_{16}(N)=N^{16}(2+N)^7(4+N)^4(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{17}(N)=288412479481381405065 2160+1482194883372929139710 3616N$$

$$+3553753504200903981268 9920N^2+5284139077644012382637 2608N^3$$

$$+5463642577255422804099 0720N^4+4175056649259152907121 4592N^5$$

$$+2446163025717060072151 4496N^6+1125117874877120801616 6912N^7$$

$$+412547582309640843944 7552N^8+121758136422257433987 0720N^9$$

$$+290758470689165111359488N^{10}+56259347598450772319232N^{11}$$

$$+8801134575778630883328N^{12}+1106537969172391515648N^{13}+110648412346432039680N^{14}$$

$$+8658383202979328640N^{15}+517381911369218976N^{16}+22735098262416528N^{17}$$

$$+690603553455288N^{18}+12920283439884N^{19}+111891970512N^{20},$$

$$Q_{17}(N)=N^{17}(2+N)^{7}(4+N)^{5}(6+N)^{3}(8+N)(10+N)(12+N)(14+N)(16+N),$$

$$P_{18}(N)=217976346922358527041208320+1247004670417928347642232832N$$

$$+3348540874518147385786368000N^{2}+5613148994349631127955701760N^{3}$$

$$+6590072055351489100479725568N^{4}+5763083478483804675832283136N^{5}$$

$$+3897888461121422479771828224N^{6}+2089768604256621305408520192N^{7}$$

$$+902981161808515241031892992N^{8}+318007770855729282389164032N^{9}$$

$$+91939617800907817008390144N^{10}+21907825707932552015228928N^{11}$$

$$+4307465840589562836633600N^{12}+697723692261228206825472N^{13}$$

$$+92695690771498523876352N^{14}+10023320035153698276864N^{15}$$

$$+871855745423393822976N^{16}+59964276974514814272N^{17}$$

$$+3179801136982484640N^{18}+125125211925357552N^{19}$$

$$+3432931683486312N^{20}+58487382066960N^{21}+464902263372N^{22},$$

$$Q_{18}(N)=N^{18}(2+N)^{8}(4+N)^{5}(6+N)^{3}(8+N)^{2}(10+N)(12+N)(14+N)(16+N),$$

$$P_{19}(N)=296700450340637062963616808960+1886082398092366330185904029696N$$

$$+5662136011415980732650891509760N^{2}+10681540470863672081684625358848N^{3}$$

$$+14214840138131391287200208388096N^{4}+14201748680271458288227420471296N^{5}$$

$$+11068758854452700350702902312960N^{6}+6903939385191827596931349086208N^{7}$$

$$+3507783440365393696965572689920N^{8}+1470131125450387324690374328320N^{9}$$

$$+512755817352656248120720687104N^{10}+149730951505094108586863493120N^{11}$$

$$+36742627293190488952004542464N^{12}+7589622993183617076162772992N^{13}$$

$$+1319393921104232186958508032N^{14}+192629775320452465799491584N^{15}$$

$$+23522039474265335230021632N^{16}+2386849075382586753748992N^{17}$$

$$+199402197646963855587072N^{18}+13537162170429884901888N^{19}$$

$$+733321598582129423232N^{20}+30882006188871353280N^{21}+972389216182878192N^{22}$$

$$+21497810677868592N^{23}+297078927653016N^{24}+1927243551744N^{25},$$

$$Q_{19}(N)=N^{19}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{20}(N)=175047833885469594895484190720+1083189374486362174077787963392N$$

$$+3159030383965627131669377974272N^{2}+5776642474848647236393604481024N^{3}$$

$$+7433523901503483650562519465984N^{4}+7162141555142551648679554449408N^{5}$$

$$+5367431463160610290238163517440N^{6}+3208587985930191848963878944768N^{7}$$

$$+1556775419045816406280559394816N^{8}+620539477835398125879179280384N^{9}$$

$$+204913470821461592404243218432N^{10}+56361425370072560355526164480N^{11}$$

$$+1295071825027009024629343027 2N^{12}+2487952293159639433357836288N^{13}$$

$$+39906151750689117249391411 2N^{14}+532525635765740669145968 64N^{15}$$

$$+5876566435741758607537152N^{16}+531472096161644127687936N^{17}$$

$$+3889058475058157858880 0N^{18}+226129900790867672352 0N^{19}$$

$$+10179820104994368048 0N^{20}+3412713259442732352N^{21}$$

$$+80011930329961584N^{22}+1167962591785224N^{23}+7972767768996N^{24},$$

$$Q_{20}(N)=N^{20}(2+N)^9(4+N)^5(6+N)^3(8+N)^2(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{21}(N)=132252402294127946914523932262400+8355801331462872195974745921945 60N$$

$$+249511715848580947412844962591539 2N^2+468631583850956844321300387830169 6N^3$$

$$+621588628629090834298174001538662 4N^4+619763975540655304734975898052198 4N^5$$

$$+482795158484527028944783554065203 2N^6+301513467105698409566027517001728 0N^7$$

$$+153702642171544596416031796022476 8N^8+647862574304847494504716000296960N^9$$

$$+227890072652375863061977808240640N^{10}+673324108361806851872063778324 48N^{11}$$

$$+167814058786845289694635653857 28N^{12}+353638219753935775043840208076 8N^{13}$$

$$+63049725309000777252378569932 8N^{14}+9500236113590903635676540928 0N^{15}$$

$$+120639718501796434590364385 28N^{16}+128498211140525594481208320 0N^{17}$$

$$+11400649783873466896033075 2N^{18}+834367695783159165062784 0N^{19}$$

$$+497004712530114950587392N^{20}+23652814810720445171904N^{21}$$

$$+875972314784652462144N^{22}+24275875061786457696N^{23}$$

$$+472677189689416800N^{24}+5755922489682504N^{25}+32919877239552N^{26},$$

$$Q_{21}(N)=N^{21}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N)(20+N).$$

Let us now evaluate these general expressions for a few particularly interesting values of N. For $N=0$, χ is also called the SAW chain generating function, and $a_n(0)$ is the number of n-step SAW's with fixed origin. We have (in terms of the variable $\widetilde{\beta}=\beta/N$)

$$\chi(0,\widetilde{\beta})=1+6\widetilde{\beta}+30\widetilde{\beta}^{2}+150\widetilde{\beta}^{3}+726\widetilde{\beta}^{4}+3534\widetilde{\beta}^{5}+16926\widetilde{\beta}^{6}+81390\widetilde{\beta}^{7}+387966\widetilde{\beta}^{8}+1853886\widetilde{\beta}^{9}$$

$$+8809878\widetilde{\beta}^{10}+41934150\widetilde{\beta}^{11}+198842742\widetilde{\beta}^{12}+943974510\widetilde{\beta}^{13}+4468911678\widetilde{\beta}^{14}+21175146054\widetilde{\beta}^{15}$$

$$+100121875974\widetilde{\beta}^{16}+473730252102\widetilde{\beta}^{17}+2237723684094\widetilde{\beta}^{18}+10576033219614\widetilde{\beta}^{19}$$

$$+49917327838734\widetilde{\beta}^{20}+235710090502158\widetilde{\beta}^{21}+\cdots.$$

For $N=1$ (the spin-1/2 Ising model), we have

$$\chi(1,\beta)=1+6\beta+30\beta^2+148\beta^3+706\beta^4+16804/5\beta^5+47260/3\beta^6+7744136/105\beta^7$$

$$+35975026/105\beta^8+1502899924/945\beta^9+6942884236/945\beta^{10}+1763022244376/51975\beta^{11}$$

$$+24340522634492/155925\beta^{12}+1455564288731288/2027025\beta^{13}+9352060224330104/2837835\beta^{14}$$

$$+32178562616325440 32/212837625\beta^{15}+147368373316136488 66/212837625\beta^{16}$$

$$+1147018945297343112964/3618239625\beta^{17}+9439914937963264249708/6512831325\beta^{18}$$

$$+409937614736484543765605 6/618718975875\beta^{19}+93592219478518291774477772/3093594879375\beta^{20}$$

$$+897280352706410994409924 1768/64965492466875\beta^{21}+\cdots.$$

For $N=2$ (the XY model) we have

$$\chi(2,\beta)=1+3\beta+15/2\beta^2+147/8\beta^3+87/2\beta^4+3275/32\beta^5+30343/128\beta^6+560073/1024\beta^7$$

$$+1282053/1024\beta^8+58548583/20480\beta^9+159649417/24576\beta^{10}+21728194031/1474560\beta^{11}$$

$$+24562301663/737280\beta^{12}+138033442711/1835008\beta^{13}+12437279658753/73400320\beta^{14}$$

$$+4030272758976071/10569646080\beta^{15}+27159922666646281/31708938240\beta^{16}$$

$$+2194618901469187717/1141521776640\beta^{17}+49190911821730863239/11415217766400\beta^{18}$$

$$+734579849432186905291/76101451776000\beta^{19}+21914994837738716597/1014686023680\beta^{20}$$

$$+970370638325356435542409/20090783268864000\beta^{21}+\cdots.$$

For $N=3$ (the Heisenberg classical model), we have

$$\chi(3,\beta)=1+2\beta+10/3\beta^2+244/45\beta^3+230/27\beta^4+37612/2835\beta^5+864788/42525\beta^6$$

$$+19773464/637875\beta^7+89686514/1913625\beta^8+25478812/360855\beta^9+140348301868/1326142125\beta^{10}$$

$$+477383158731608/3016973334375\beta^{11}+426768736125964/1810184000625\beta^{12}$$

$$+28560817226680664/81458280028125\beta^{13}+775988604270248/1491909890625\beta^{14}$$

$$+16004552656617124832/20771861407171875\beta^{15}+35495085198042760759 4/311577921107578125\beta^{16}$$

$$+146412835281395509631267 6/870237133653465703125\beta^{17}$$

$$+106876465586445485837641782 8/430767381158465523046875\beta^{18}$$

$$+25981409369018879755093315714 4/710766178911468113027343 75\beta^{19}$$

$$+10427316181032515873697224369 2/19384532152130948537109375\beta^{20}$$

$$+68456433541059284889589959461 36/865462347262787643509765625\beta^{21}+\cdots.$$

APPENDIX B: THE SECOND CORRELATION MOMENT ON THE sc LATTICE

The HT expansion coefficients of the second correlation moment $\mu_2(N,\beta)=\Sigma_{r=1}^{\infty}s_r(N)\beta^r$ on the sc lattice are

$$s_1(N)=6/N,$$

$$s_2(N)=72/N^2,$$

$$s_3(N)=(1164+576N)/[N^3(2+N)],$$

$$s_4(N)=(8064+3888N)/[N^4(2+N)],$$

$$s_5(N)=(204528+146124N+23760N^2)/[N^5(8+6N+N^2)].$$

For the coefficients which follow, it is typographically more convenient to set $s_r(N)=P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$

$$P_6(N)=2456448+2929248N+1122528N^2+136080N^3,$$

$$Q_6(N)=N^6(2+N)^2(4+N),$$

$$P_7(N)=170289792+312319968N+221383920N^2+75264960N^3+12170904N^4+743616N^5,$$

$$Q_7(N)=N^8(2+N)^3(4+N)(6+N),$$

$$P_8(N)=956823552+1728273408N+1206868608N^2+404721600N^3+64724352N^4+3921696N^5,$$

$$Q_8(N)=N^8(2+N)^3(4+N)(6+N),$$

$$P_9(N)=42088707072+80027621760N+60705869280N^2+23384805744N^3+4800525624N^4$$

$$+495870684N^5+20110032N^6,$$

$$Q_9(N)=N^9(2+N)^3(4+N)(6+N)(8+N),$$

$$P_{10}(N)=1820564029440+4769433145344N+5321186764800N^2+3295384254720N^3$$

$$+1236516916608N^4+287290885440N^5+40275314208N^6+3107523840N^7+100804176N^8,$$

$$Q_{10}(N)=N^{10}(2+N)^4(4+N)^2(6+N)(8+N),$$

$$P_{11}(N)=776696905728000+2669258856775680N+4076555067445248N^2$$

$$+3647307112190976N^3+2121429738632448N^4+841182276069120N^5$$

$$+231709799815296N^6+44274759493440N^7+5741865229488N^8$$

$$+480498609168N^9+23310434136N^{10}+495849600N^{11},$$

$$Q_{11}(N)=N^{11}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{12}(N)=4095660376719360+13934832721330176N+21070456869617664N^2$$

$$+18668449163280384N^3+10756490095534080N^4+4227254991461376N^5$$

$$+1154847354120192N^6+219020166457344N^7+28215891209472N^8$$

$$+2347567763904N^9+113324238144N^{10}+2400521184N^{11},$$

$$Q_{12}(N)=N^{12}(2+N)^5(4+N)^3(6+N)(8+N)(10+N),$$

$$P_{13}(N)=256698768799825920+885687727893184512N+1365396429834780672N^2$$

$$+1242082941562257408N^3+741684670798319616N^4+305895801259154688N^5$$

$$+89237579848134144N^6+18527188676264256N^7+2712711410020416N^8$$

$$+272727691809312N^9+17842958870016N^{10}+681157898376N^{11}+11463937344N^{12},$$

$$Q_{13}(N)=N^{13}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)(12+N),$$

$$P_{14}(N)=1595975930512146432O+6512631337482662707 2N+12077640977003996774 4N^2$$

$$+1347994931094387425 28N^3+1010759615353832079 36N^4+5382104930871633100 8N^5$$

$$+2097209199484755763 2N^6+607056095666467123 2N^7+131064019523388057 6N^8$$

$$+209854631867283456N^9+24507398880206592N^{10}+2022286243464576N^{11}$$

$$+111357392644800N^{12}+3659422931136N^{13}+54103051872N^{14},$$

$$Q_{14}(N)=N^{14}(2+N)^6(4+N)^3(6+N)^2(8+N)(10+N)(12+N),$$

$$P_{15}(N)=1379463097170410864640 0+659612569629186588672 00N$$

$$+1454823051541650824232 96N^2+196426790908008971894784N^3$$

$$+181712188501778813288448N^4+122149149808092940001280N^5$$

$$+61746019751467960037376N^6+23960432106776991375360N^7$$

$$+7221833041396399896576N^8+1699364135273723386368N^9$$

$$+311994055471124022528N^{10}+44409455416113821952N^{11}$$

$$+4837668848932793088N^{12}+394573762100880576N^{13}$$

$$+23254673371516320N^{14}+932602461181824N^{15}+22709826213888N^{16}+252699284736N^{17},$$

$$Q_{15}(N)=N^{15}(2+N)^7(4+N)^3(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{16}(N)=282174155802421116272640+1409136434527339234197504N$$

$$+3263688499481715696205824N^2+4655957078252245556920320N^3$$

$$+4582618987923856995385344N^4+3303221971610187595776000N^5$$

$$+1806414639535332256776192N^6+766021764539003646738432N^7$$

$$+255248344593156938809344N^8+67302471660197147738112N^9$$

$$+14069190944214259089408N^{10}+2324969745522621573120N^{11}$$

$$+301290841669925501952N^{12}+30183470541493269504N^{13}$$

$$+2284617842207913984N^{14}+125980350725428992N^{15}$$

$$+4762443604812672N^{16}+110061663058176N^{17}+1169505190080N^{18},$$

$$Q_{16}(N)=N^{16}(2+N)^7(4+N)^4(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{17}(N)=918574729817508309211545 60+48389916726342506927200 6656N$$

$$+118986817194558672605282304 0N^2+181517394119644868450136883 2N^3$$

$$+1926067302758962518697181184N^4+1510557041183984864999768064N^5$$

$$+908241734818700901130567680N^6+428565795442223083117608960N^7$$

$$+161124373007167836401491968N^8+48721486461929019612254208N^9$$

$$+11908799319497649472468992N^{10}+2355824328081975826799616N^{11}$$

$$+376301863299979913115648N^{12}+48239787542820598777344N^{13}$$

$$+4911231909191337855744N^{14}+390699515456311012992N^{15}$$

$$+23699390491011237792N^{16}+1055645514024821136N^{17}$$

$$+32459860654884408N^{18}+613932222869772N^{19}+5368457021904N^{20},$$

$$Q_{17}(N)=N^{17}(2+N)^7(4+N)^5(6+N)^3(8+N)(10+N)(12+N)(14+N)(16+N),$$

$$P_{18}(N)=74405686785054623558900121 60+43556714841735485934551433216N$$

$$+119745506575262677827342827520N^2+205600123176421699517732093952N^3$$

$$+247329442844425438300263677952N^4+221673049003323839621659361280N^5$$

$$+153674002155144625516386975744N^6+844405594580712231893320335 36N^7$$

$$+373855808001450801813588541 44N^8+134848124361096416924154593 28N^9$$

$$+399043313727921800117521612 8N^{10}+972476515921418968813977600N^{11}$$

$$+195365912565036970555809792N^{12}+32298750507708450873901056N^{13}$$

$$+4374436773849815715926016N^{14}+481604346770073389316096N^{15}$$

$$+42597083415881650775040N^{16}+2975234030918604431616N^{17}$$

$$+160016859063956698752N^{18}+6378334212423145920N^{19}$$

$$+177053920216273824N^{20}+3048514523067840N^{21}+24463062529488N^{22},$$

$$Q_{18}(N)=N^{18}(2+N)^8(4+N)^5(6+N)^3(8+N)^2(10+N)(12+N)(14+N)(16+N),$$

$$P_{19}(N)=108026729043490241481560555520 00+70163690014890044522149466603520N$$

$$+2153172338295465472073885936517 12N^2+4153998825783186246900734471700 48N^3$$

$$+5655443688595886007733698334556 16N^4+5782018645534055315071950944993 28N^5$$

$$+4612373955476346290381340876472 32N^6+2944639333740819472279779347005 44N^7$$

$$+1531226912419872709983763326566 40N^8+6566481120910692811809575141376 0N^9$$

$$+2342553036195241878583423185715 2N^{10}+6992940825868245995024634347520N^{11}$$

$$+1753046568431902647835251179520N^{12}+369630925832778786734031814656N^{13}$$

$$+65531446138803785751245647872N^{14}+9747432364485234622732087296N^{15}$$

$$+1211351100137737820129992704N^{16}+124959028114518620762345472N^{17}$$

$$+10600636042455571892260608N^{18}+729964109544372672407040N^{19}$$

$$+40064614626652387587456N^{20}+1707651659115705954240N^{21}+54364356083417641584N^{22}$$

$$+1214012987890747632N^{23}+16929898750843704N^{24}+110737848870144N^{25},$$

$$Q_{19}(N)=N^{19}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{20}(N)=542533038948081453192730745241 60+3502103883860461315664464796712 96N$$

$$+1068117279557885577337430058467328N^2+2048046462803855050848737955938304N^3$$

$$+2771353871328191245609935795388416N^4+2816339118396208118571451803500544N^5$$

$$+2233308998837890298190379125571584N^6+1417505561364781878449152154664960N^7$$

$$+732927543097038243001328908369920N^8+312574995769798442844907773100032N^9$$

$$+110915545592238742983402344939520N^{10}+32940864293524930651585012039680N^{11}$$

$$+8217496120240756722296489508864N^{12}+1724609018206108184412122775552N^{13}$$

$$+304409891806384221588352598016N^{14}+45092002493947222162981109760N^{15}$$

$$+5582064073159548784709222400N^{16}+573750523480464178256412672N^{17}$$

$$+48510059643505700194836480N^{18}+3330097276976649985453056N^{19}$$

$$+1822549653719055025704 96N^{20}+7747894986731613064704N^{21}$$

$$+2460723543779283217 92N^{22}+548313316898104 4160N^{23}$$

$$+76313940451961664N^{24}+498277338207840N^{25},$$

$$Q_{20}(N)=N^{20}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{21}(N)=5430787293066006887355252945715200+35108759514076334960218328247828480N$$

$$+107329504696473088472731061852307456N^2+206477214573138515932854809354305536N^3$$

$$+280632187188928125901899221692317696N^4+286811495776956909710765518158299136N^5$$

$$+229066873791452857318720674006564864N^6+146680248330932745127688632944033792N^7$$

$$+76662722498288753620483761550393344N^8+33122820337959947874239120850026496N^9$$

$$+11938327300343325580622654112792576N^{10}+3612243766907364223803656339914752N^{11}$$

$$+921317991792033678079649178451968N^{12}+198518394502579844827717921947648N^{13}$$

$$+36154647910306278999540056408064N^{14}+5558926665913863471127518437376N^{15}$$

$$+719493683774301127434614464512N^{16}+78018551293996681395111180288N^{17}$$

$$+7038314076238455658170184704N^{18}+523128632061598945510086912N^{19}$$

$$+31608578395426246066990080N^{20}+1524110853677163900394176N^{21}$$

$$+57125321213021562328128N^{22}+1600502063799965143392N^{23}$$

$$+31474107720549665760N^{24}+386725681457093256N^{25}+2229799326628608N^{26},$$

$$Q_{21}(N)=N^{21}(2+N)^9(4+N)^5(6+N)^3(8+N)^3(10+N)(12+N)(14+N)(16+N)(18+N)(20+N).$$

In particular, for $N=0$, μ_2 is also called the mean square end-to-end distance generating function and $s_n(0)$ is the sum of the squares of the end to end distance of all n-step SAW's. We have (in terms of the variable $\widetilde{\beta}=\beta/N$)

$$\mu_2(0,\widetilde{\beta})=6\widetilde{\beta}+72\widetilde{\beta}^2+582\widetilde{\beta}^3+4032\widetilde{\beta}^4+25566\widetilde{\beta}^5+153528\widetilde{\beta}^6+886926\widetilde{\beta}^7+4983456\widetilde{\beta}^8$$

$$+27401502\widetilde{\beta}^9+148157880\widetilde{\beta}^{10}+790096950\widetilde{\beta}^{11}+4166321184\widetilde{\beta}^{12}+21760624254\widetilde{\beta}^{13}$$

$$+112743796632\widetilde{\beta}^{14}+580052260230\widetilde{\beta}^{15}+2966294589312\widetilde{\beta}^{16}+15087996161382\widetilde{\beta}^{17}+76384144381272\widetilde{\beta}^{18}$$

$$+385066579325550\widetilde{\beta}^{19}+1933885653380544\widetilde{\beta}^{20}+9679153967272734\widetilde{\beta}^{21}+\cdots.$$

For $N=1$ (the Ising model), we have

$$\mu_2(1,\beta)=6\beta+72\beta^2+580\beta^3+3984\beta^4+124804/5\beta^5+738256/5\beta^6+17603848/21\beta^7$$

$$+161679008/35\beta^8+23502612244/945\beta^9+207084686896/1575\beta^{10}+7118350956184/10395\beta^{11}$$

$$+183105409684576/51975\beta^{12}+36357148765588568/2027025\beta^{13}+428105990895623072/4729725\beta^{14}$$

$$+3858980041412441056/8513505\beta^{15}+159972662342929823168/70945875\beta^{16}$$

$$+4034695345833341222592 4/3618239625\beta^{17}+59548022003549908858417 6/10854718875\beta^{18}$$

$$+3324157983299356572437 2216/123743795175\beta^{19}+13508192226020541830758 67296/1031198293125\beta^{20}$$

$$+41342000424913856999443104 4328/64965492466875\beta^{21}+\cdots.$$

For $N=2$ (the XY model), we have

$$\begin{aligned}\mu_2(2,\beta)=\,&3\beta+18\beta^2+579/8\beta^3+495/2\beta^4+24659/32\beta^5+72363/32\beta^6+6513321/1024\beta^7+4440007/256\beta^8\\&+945324463/20480\beta^9+3702230609/30720\beta^{10}+91347937699/294912\beta^{11}+19312589941/24576\beta^{12}\\&+18091435549531/9175040\beta^{13}+809472644281601/165150720\beta^{14}+25546108054005659/2113929216\beta^{15}\\&+234520789250333527/7927234560\beta^{16}+821560001814126454 69/1141521776640\beta^{17}\\&+16562862207604581055 3/951268147200\beta^{18}+3189592434500928270 6091/76101451776000\beta^{19}\\&+818960508913574446756 7/8153726976000\beta^{20}+963242021882589103915 1597/4018156653772800\beta^{21}+\cdots.\end{aligned}$$

For $N=3$ (the Heisenberg model), we have

$$\begin{aligned}\mu_2(3,\widetilde{\beta})=\,&2\beta+8\beta^2+964/45\beta^3+2192/45\beta^4+57116/567\beta^5+8340368/42525\beta^6+33324872/91125\beta^7\\&+1263947744/1913625\beta^8+73478278372/63149625\beta^9+13325285538064/6630710625\beta^{10}\\&+3434294378983784/1005657778125\beta^{11}+5181988210150198 4/9050920003125\beta^{12}\\&+773005999283909656/81458280028125\beta^{13}+19031243835736702816/1221874200421875\beta^{14}\\&+225650609227937809568/8902226317359375\beta^{15}+2679122602589277257843 84/65431363432591406 25\beta^{16}\\&+171544906778131647970688684/26107114009603971093 75\beta^{17}\\&+45158335170568649028207863344/43076738115846552304687 5\beta^{18}\\&+198749733493286846805507467 92/11945650065738959882812 5\beta^{19}\\&+6158281572675994404515451988 16/23431852052026421308593 75\beta^{20}\\&+467851964714541441817299887365304/1131758454112876149205078125\beta^{21}+\cdots.\end{aligned}$$

APPENDIX C: THE SUSCEPTIBILITY ON THE bcc LATTICE

The HT expansion coefficients of the susceptibility $\chi(N,\beta)=1+\Sigma_{r=1}^{\infty}a_r(N)\beta^r$ on the bcc lattice are

$$a_1(N)=8/N,$$

$$a_2(N)=56/N^2,$$

$$a_3(N)=(784+384N)/[N^3(2+N)],$$

$$a_4(N)=(5296+2536N)/[N^4(2+N)],$$

$$a_5(N)=(143680+101904N+16512N^2)/[N^5(2+N)(4+N)],$$

$$a_6(N)=(1920896+2286048N+873664N^2+105664N^3)/[N^6(2+N)^2(4+N)].$$

For the coefficients which follow, it is typographically more convenient to set $a_r(N)=P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_7(N)=154526208+283153792N+200425920N^2+68016448N^3+10978624N^4+669696N^5,$$

$$Q_7(N)=N^7(2+N)^3(4+N)(6+N),$$

$$P_8(N)=1027530240+1858845568N+1298441728N^2+435042496N^3+69464512N^4+4201832N^5,$$

$$Q_8(N)=N^8(2+N)^3(4+N)(6+N),$$

$$P_9(N)=54762516480+104445905408N+79347803264N^2+30563781696N^3+6267722464N^4+646524208N^5$$

$$+26183808N^{6},$$

$$Q_{9}(N)=N^{9}(2+N)^{3}(4+N)(6+N)(8+N),$$

$$P_{10}(N)=2903544987648+7640375627776N+8552587753472N^{2}+5307832448256N^{3}$$

$$+1993598086784N^{4}+463207567296N^{5}+64896018528N^{6}+5002101120N^{7}+162073408N^{8},$$

$$Q_{10}(N)=N^{10}(2+N)^{4}(4+N)^{2}(6+N)(8+N),$$

$$P_{11}(N)=1541455439462400+5322818812313600N+8161728409911296N^{2}+7325494948007936N^{3}$$

$$+4270699263198208N^{4}+1695943901147136N^{5}+467519893018112N^{6}+89347859497728N^{7}$$

$$+11583894692416N^{8}+968797835648N^{9}+46962844416N^{10}+998129664N^{11},$$

$$Q_{11}(N)=N^{11}(2+N)^{5}(4+N)^{3}(6+N)(8+N)(10+N),$$

$$P_{12}(N)=10192817348935680+34916640297648128N+53104852074643456N^{2}$$

$$+47274357345341440N^{3}+27337281183867904N^{4}+10770444302738944N^{5}$$

$$+2946841983489280N^{6}+559252490775936N^{7}+72048621004928N^{8}+5991833735936N^{9}$$

$$+289036804288N^{10}+6117389760N^{11},$$

$$Q_{12}(N)=N^{12}(2+N)^{5}(4+N)^{3}(6+N)(8+N)(10+N),$$

$$P_{13}(N)=809552875974819840+2816953771808784384N+4374585258210099200N^{2}$$

$$+4003673280064782336N^{3}+2402076288529055744N^{4}+994132403202585600N^{5}$$

$$+290679924510640128N^{6}+60428742766729984N^{7}+8852518124203776N^{8}+889976271834368N^{9}$$

$$+58202792120192N^{10}+2220562578336N^{11}+37346353152N^{12},$$

$$Q_{13}(N)=N^{13}(2+N)^{5}(4+N)^{3}(6+N)(8+N)(10+N)(12+N),$$

$$P_{14}(N)=64130271858288230400+264066083110335283200N+493695724862466686976N^{2}$$

$$+554938627488783794176N^{3}+418612627132132098048N^{4}+223996317536493072384N^{5}$$

$$+87615497305685485568N^{6}+25431728633300134912N^{7}+5501047485284784640N^{8}$$

$$+881788286375843328N^{9}+103029908100346368N^{10}+8502206688736640N^{11}$$

$$+468050982316992N^{12}+15373889753312N^{13}+227164816896N^{14},$$

$$Q_{14}(N)=N^{14}(2+N)^{6}(4+N)^{3}(6+N)^{2}(8+N)(10+N)(12+N),$$

$$P_{15}(N)=71173493905168867000320+343461539954772003520512N+763948140990101106720768N^{2}$$

$$+1039364375188561578688512N^{3}+968023439264944191176704N^{4}+654531480238143388811264N^{5}$$

$$+332495160909431971856384N^{6}+129542765956143879962624N^{7}+39168342655704020885504N^{8}$$

$$+9238537902576806709248N^{9}+1698999588160390636544N^{10}+242102822226209927168N^{11}$$

$$+26389588821248815616N^{12}+2152963627739608576N^{13}+126885369593601792N^{14}$$

$$+5087525640871680N^{15}+123844692209568N^{16}+1377490599936N^{17},$$

$$Q_{15}(N)=N^{15}(2+N)^{7}(4+N)^{3}(6+N)^{3}(8+N)(10+N)(12+N)(14+N),$$

$$P_{16}(N)=1876979130303246489354240+9471577214194059747262464N$$

$$+22149665431198847345885184N^2+31877715547591801734955008N^3$$

$$+31624026815500069881511936N^4+22953839008339482680098816N^5$$

$$+12627923848247621505482752N^6+5381959679798671698755584N^7$$

$$+1800741309759264778862592N^8+476364400204668565323776N^9$$

$$+99830450585322119526400N^{10}+16527352075788357199872N^{11}$$

$$+2144441239708699601920N^{12}+214997737032367408640N^{13}$$

$$+16279875671710226560N^{14}+897816852778841984N^{15}$$

$$+33936752050575616N^{16}+784094093888896N^{17}+8328848160360N^{18},$$

$$Q_{16}(N)=N^{16}(2+N)^7(4+N)^4(6+N)^3(8+N)(10+N)(12+N)(14+N),$$

$$P_{17}(N)=79243552920293577607024640+4222140895430347509133737984N$$

$$+10491979115544189205512978432N^2+16161534656668214692379885568N^3$$

$$+17299978054698585966078066688N^4+13674498471920703095531634688N^5$$

$$+8278686728088313567471206400N^6+3929642002798861992101675008N^7$$

$$+1484824967962458751520964608N^8+450855902965522928390258688N^9$$

$$+110571134100164644485947392N^{10}+21931112763464023289729024N^{11}$$

$$+3510129107788965101195264N^{12}+450635203075167731752960N^{13}$$

$$+45924661947564596022272N^{14}+3655727493573478880768N^{15}$$

$$+221828258031520815488N^{16}+9882110327533350848N^{17}$$

$$+303849712222003168N^{18}+5745966508168464N^{19}+50233057515648N^{20},$$

$$Q_{17}(N)=N^{17}(2+N)^7(4+N)^5(6+N)^3(8+N)(10+N)(12+N)(14+N)(16+N),$$

$$P_{18}(N)=83507745279393776631734599680+494606778876844544338100748288N$$

$$+1374842423776694140867719462912N^2+2384980581491184335810061139968N^3$$

$$+2896424821652315126561511571456N^4+2618571127287142216719610675200N^5$$

$$+1829555799925870815015077937152N^6+1012311427843216219812641374208N^7$$

$$+450932244560352558960811180032N^8+163505366538443953961368354816N^9$$

$$+48600320751180887259008499712N^{10}+11887881202450793297985355776N^{11}$$

$$+2395419597277559569157611520N^{12}+396969980659736571298029568N^{13}$$

$$+53863833205174458351316992N^{14}+5938346973499025558951936N^{15}$$

$$+525755744030523229924352N^{16}+36746279798448300236032N^{17}$$

$$+1977118373223141598336N^{18}+78824256302074652224N^{19}+2188144033077897056N^{20}$$

$$+37672654614424384N^{21}+302260146702400N^{22},$$

$$Q_{18}(N)=N^{18}(2+N)^{8}(4+N)^{5}(6+N)^{3}(8+N)^{2}(10+N)(12+N)(14+N)*(16+N),$$

$$P_{19}(N)=15847423886835421278233042092032 0+10415498089673044257666937619742 72N$$

$$+32324791973304528297593025268285 44N^{2}+63028833203928968991438368428523 52N^{3}$$

$$+86668605705224200290528373789163 52N^{4}+89430527483086423061039381657681 92N^{5}$$

$$+71947789536467936164968137794519 04N^{6}+46289048354967512844715615458426 88N^{7}$$

$$+24238274559763232728272413392896 00N^{8}+10458644071060984900121432088903 68N^{9}$$

$$+37513228388799505011571231634227 2N^{10}+11251009884812083629625950365286 4N^{11}$$

$$+28318048555752265350478248542208N^{12}+59909975271001723502917983928 32N^{13}$$

$$+10650952757097847529822879580 16N^{14}+15878509397369087786985511321 6N^{15}$$

$$+19768336145696922659741351936N^{16}+20420895297119535572970045 44N^{17}$$

$$+17341964644773830104224051 2N^{18}+11950974338192075640468480N^{19}$$

$$+65629080842923315142041 6N^{20}+27982455698879378963968N^{21}$$

$$+89101840315109384659 2N^{22}+19898986423803471360N^{23}$$

$$+277497323823185728N^{24}+18149687218544 64N^{25},$$

$$Q_{19}(N)=N^{19}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{20}(N)=10429100166934279912568778443980 80+68223717800325057016782848608174 08N$$

$$+21073470343840989502583247187476 480N^{2}+40894747767325105591045502923702 272N^{3}$$

$$+55963797112693608459475043305390 080N^{4}+57470452437084820328323451635892 224N^{5}$$

$$+46014452801325563363308302206238 720N^{6}+29463649767822099804353478764003 328N^{7}$$

$$+15355522275752808702230364501835 776N^{8}+65951699771476424876996258368061 44N^{9}$$

$$+23548648399553326597484052545536 00N^{10}+70316472413395029512118902626713 6N^{11}$$

$$+17622816701598725898507741213491 2N^{12}+37130297734661044363738865172480N^{13}$$

$$+65752641364023183969193811968 00N^{14}+97659401393383643446320832512 0N^{15}$$

$$+12115536523719895885192495923 2N^{16}+12474040011944675687769305088N^{17}$$

$$+10560478841307910961655040 00N^{18}+72566286207194184977521152N^{19}$$

$$+39743531117758563275159 04N^{20}+16903814509794935109568 0N^{21}$$

$$+53703384861193962918 40N^{22}+11968667714142852659 2N^{23}$$

$$+16659165840940309 76N^{24}+10877235273306176N^{25},$$

$$Q_{20}(N)=N^{20}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N),$$

$$P_{21}(N)=13731792945816913886577519797207 0400+90083562130745725210634036315160 5760N$$

$$+27926617868275076732800703779905 20832N^{2}+54439831702863275400345337956201 92256N^{3}$$

$$+74916752028419979473924895177399 33696N^{4}+77457729352955055289660172494627 14368N^{5}$$

$$+6252736599652178166896979174683049984N^{6}+4043154246053532940260091980981207040N^{7}$$

$$+2131912063458287138157702593130790912N^{8}+928421393379773859641523395754983424N^{9}$$

$$+336975218565454742726998375191805952N^{10}+102585473522764282202717854632247296N^{11}$$

$$+26303281521902958783715401210003456N^{12}+5693171830798150117800423350009856N^{13}$$

$$+1040783657174864257650177310588928N^{14}+160527761470226824593523378225152N^{15}$$

$$+20830550424448301030960205561856N^{16}+2263442229921435778953046802432N^{17}$$

$$+204527463480120802884033564672N^{18}+15221071176361614573823071232N^{19}$$

$$+920584951315678219503968256N^{20}+44421119188140270602770176N^{21}$$

$$+1665819737158517381045760N^{22}+46688727974444664155264N^{23}$$

$$+918355227096954088704N^{24}+11285460028384964704N^{25}+65074192439519232N^{26},$$

$$Q_{21}(N)=N^{21}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N)(20+N).$$

In particular, for $N=0$, we have (in terms of the variable $\widetilde{\beta}=\beta/N$)

$$\chi(0,\widetilde{\beta})=1+8\widetilde{\beta}+56\widetilde{\beta}^{2}+392\widetilde{\beta}^{3}+2648\widetilde{\beta}^{4}+17960\widetilde{\beta}^{5}+120056\widetilde{\beta}^{6}+804824\widetilde{\beta}^{7}+5351720\widetilde{\beta}^{8}+35652680\widetilde{\beta}^{9}$$

$$+236291096\widetilde{\beta}^{10}+1568049560\widetilde{\beta}^{11}+10368669992\widetilde{\beta}^{12}+68626647608\widetilde{\beta}^{13}+453032542040\widetilde{\beta}^{14}$$

$$+2992783648424\widetilde{\beta}^{15}+19731335857592\widetilde{\beta}^{16}+130161040083608\widetilde{\beta}^{17}+857282278813256\widetilde{\beta}^{18}$$

$$+5648892048530888\widetilde{\beta}^{19}+37175039569217672\widetilde{\beta}^{20}+244738250638121768\widetilde{\beta}^{21}+\cdots.$$

For $N=1$ we have

$$\chi(1,\beta)=1+8\beta+56\beta^{2}+1168/3\beta^{3}+7832/3\beta^{4}+262096/15\beta^{5}+5186272/45\beta^{6}+239256896/315\beta^{7}$$

$$+223501256/45\beta^{8}+92020145776/2835\beta^{9}+2992356296032/14175\beta^{10}+213982290222656/155925\beta^{11}$$

$$+594224833766272/66825\beta^{12}+350264122225879712/6081075\beta^{13}+205770248144874592/552825\beta^{14}$$

$$+1535420477610758667232/638512875\beta^{15}+9901906655322559820312/638512875\beta^{16}$$

$$+1085334591165630782296336/10854718875\beta^{17}+628299591085147558675552/97594875\beta^{18}$$

$$+769253342779387633797929 1776/1856156927625\beta^{19}+35332545614795796034757603392/1325826376875\beta^{20}$$

$$+256870871802430351480251 7007264/14992036723125\beta^{21}+\cdots.$$

For $N=2$ we have

$$\chi(2,\beta)=1+4\beta+14\beta^{2}+97/2\beta^{3}+162\beta^{4}+12923/24\beta^{5}+169265/96\beta^{6}+4421401/768\beta^{7}$$

$$+14321503/768\beta^{8}+308660121/5120\beta^{9}+17872634681/92160\beta^{10}+688956690877/1105920\beta^{11}$$

$$+1102888778273/552960\beta^{12}+395053233560309/61931520\beta^{13}+1440484062875779/70778880\beta^{14}$$

$$+73472774375429137/1132462080\beta^{15}+4909723847974444627/23781703680\beta^{16}$$

$$+56205545883506594 4253/856141332480\beta^{17}+178478528365850967288 01/8561413324800\beta^{18}$$

$$+1132877749890754784251337/171228266496000\beta^{19}+3591366948964760556413069/171228266496000\beta^{20}$$

$$+476869250093007743453 27827/717527973888000\beta^{21}+\cdots.$$

For $N=3$ we have

$$\chi(3,\beta)=1+8/3\beta+56/9\beta^2+1936/135\beta^3+12904/405\beta^4+119600/1701\beta^5+2784992/18225\beta^6$$

$$+632918848/1913625\beta^7+4075984504/5740875\beta^8+287925718448/189448875\beta^9$$

$$+64384719769312/19892131875\beta^{10}+186782368415874752/27152760009375\beta^{11}$$

$$+1186773786369487616/81458280028125\beta^{12}+7528780320376815776/244374840084375\beta^{13}$$

$$+732954612970918048/11278838773125\beta^{14}+1279725705891488185900048/9347337633227343 75\beta^{15}$$

$$+807291210775528531339816/2804201289968203125\beta^{16}$$

$$+473662226546810949208118161 6/78321342028811913281 25\beta^{17}$$

$$+1639367056527449858924222363488/1292302143475396569140625\beta^{18}$$

$$+566937383305125856734568614018688/2132298536734404339082031 25\beta^{19}$$

$$+508646907020827390705273581508288/9138422300290304310351 5625\beta^{20}$$

$$+51396113238442986195968809467955836 8/441385797104021698189980468 75\beta^{21}+\cdots.$$

APPENDIX D: THE SECOND CORRELATION MOMENT ON THE bcc LATTICE

The HT expansion coefficients of the second correlation moment $\mu_2(N,\beta)=\Sigma_{r=1}^{\infty}s_r(N)\beta^r$ on the bcc lattice are

$$s_1(N)=8/N,$$

$$s_2(N)=128/N^2,$$

$$s_3(N)=1408/N^3+16/[N^3(2+N)],$$

$$s_4(N)=(27136+13312N)/[N^4(2+N)],$$

$$s_5(N)=(954688+697872N+114816N^2)/[N^5(8+6N+N^2)].$$

For the coefficients which follow, it is typographically more convenient to set $a_r(N)=P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_6(N)=15931904+19410560N+7587328N^2+932864N^3,$$

$$Q_6(N)=[N^6(2+N)^2(4+N)],$$

$$P_7(N)=1535992320+2872374656N+2074427840N^2+717218112N^3+117562688N^4+7256064N^5,$$

$$Q_7(N)=[N^8(2+N)^3(4+N)(6+N)],$$

$$P_8(N)=12010328064+22254990336N+15929639936N^2+5462666496N^3$$

$$+889419008N^4+54616064N^5,$$

$$Q_8(N)=[N^8(2+N)^3(4+N)(6+N)],$$

$$P_9(N)=735485374464+1441044494848N+1124713670784N^2+444350745664N$$

$$+93091815392N^4+9765033776N^5+400525440N^6,$$

$$Q_9(N)=[N^9(2+N)^3(4+N)(6+N)(8+N)],$$

$$P_{10}(N)=44302912585728+119138720923648N+136354854002688N^2+86513824402432N^3$$

$$+33191607212544N^4+7864232156928N^5+1120983318912N^6+87686077056N^7+2876013568N^8,$$

$$Q_{10}(N)=[N^{10}(2+N)^4(4+N)^2(6+N)(8+N)],$$

$$P_{11}(N)=26325669351260160+92611036866871296N+144717226646847488N^2+132382989581848576N^3$$

$$+78641764731102208N^4+31802834512539648N^5+8919341723426304N^6+1731942609659648N^7$$

$$+227802476687936N^8+19296321355392N^9+945810964480N^{10}+20292747264N^{11},$$

$$Q_{11}(N)=[N^{11}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)],$$

$$P_{12}(N)=193386300215132160+675993638329122816N+1049662711446241280N^2$$

$$+954246907202404352N^3+563468529471660032N^4+226569298925752320N^5+63205870761504768N^6$$

$$+12213762418126848N^7+1599512832179712N^8+134973057538304N^9+6593975832320N^{10}$$

$$+141081387008N^{11},$$

$$Q_{12}(N)=[N^{12}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)],$$

$$P_{13}(N)=1688683158187278336O+60044398899547078656N+95334004765199040512N^2$$

$$+89227042853281333248N^3+54738880404711260160N^4+23151147398852015104N^5$$

$$+6910462866337355776N^6+1464416704532198144N^7+218298047999270656N^8$$

$$+22288526391915264N^9+1477424654034560N^{10}+57022884609696N^{11}+968470831104N^{12},$$

$$Q_{13}(N)=[N^{13}(2+N)^5(4+N)^3(6+N)(8+N)(10+N)(12+N)],$$

$$P_{14}(N)=14629096949906040422240+61418308931407199960064N+117142804157150621859840N^2$$

$$+134382670892062526668800N^3+103473261856908446269440N^4+56510659350432076431360N^5$$

$$+22550970063839308103680N^6+6673257056881003847680N^7+1470080460198478499840N^8$$

$$+239688359659249909760N^9+28445556328187105280N^{10}+2380642532688686080N^{11}$$

$$+132709235520130560N^{12}+4407442956627200N^{13}+65753994608640N^{14},$$

$$Q_{14}(N)=[N^{14}(2+N)^6(4+N)^3(6+N)^2(8+N)(10+N)(12+N)],$$

$$P_{15}(N)=176197169806846935957504O+865445759902492538594918 4N+196023806628083665238753280N^2$$

$$+271678145099893029218549760N^3+257818670196642743724277760N^4+177634097256833195662704640N^5$$

$$+91936506230576027046543360N^6+36481190339750720730726400N^7+11227853398010300282388480N^8$$

$$+2693617894499304678707200N^9+503371632303463031695360N^{10}+72810311504938267607040N^{11}$$

$$+8046750456321150161920N^{12}+664810419427924659200N^{13}+39629787799435527680N^{14}$$

$$+1605287423818685440N^{15}+39433747105604160N^{16}+442141632430080N^{17},$$

$$Q_{15}(N)=[N^{15}(2+N)^7(4+N)^3(6+N)^3(8+N)(10+N)(12+N)(14+N)],$$

$$P_{16}(N)=5022647482966076351840256O+25798962043569351580817817 6N+614398722349307968861765632N^2$$

$$+900807113938252399141453824N^3+910606900238029022858051584N^4+673571444329263447919296512N^5$$

$$+377614269751899053565935616N^6+163960919908619742555930624N^7+55866301681257199442722816N^8$$

$$+15040925019264750417149952N^9+3205553808422296083021824N^{10}+539209477484097660960768N^{11}$$

$$+7101444450801988884480 0N^{12}+7219013824735894138880N^{13}+5536338095273164963 84N^{14}$$

$$+308886165185146705 92N^{15}+117988075355141094 4N^{16}+27518591175817472N^{17}$$

$$+294773303869440N^{18},$$

$$Q_{16}(N)=[N^{16}(2+N)^{7}(4+N)^{4}(6+N)^{3}(8+N)(10+N)(12+N)(14+N)],$$

$$P_{17}(N)=2278649063055102281930 2440960+1235762298969848961434816 34816N$$

$$+3126919473129342204353886 28992N^{2}+4906065546212512963997120 59392N^{3}$$

$$+5350307564094469993700261 88800N^{4}+4308895146088716797189721 94816N^{5}$$

$$+2657767387655322465366719 85664N^{6}+1285065318327797923147111 46496N^{7}$$

$$+4944404278022307928871950 7456N^{8}+1528016675076371836283940 0448N^{9}$$

$$+3811610006534643901029376 000N^{10}+7683781861828171445715968 00N^{11}$$

$$+1248854804449672531549921 28N^{12}+1626600983621925011704217 6N^{13}$$

$$+1680110485364468843396096N^{14}+135412027062997840729600N^{15}$$

$$+8310816170333932730752N^{16}+374090262518662420928N^{17}+11610577757113435360N^{18}$$

$$+221417841624576528N^{19}+1950304510449792N^{20},$$

$$Q_{17}(N)=[N^{17}(2+N)^{7}(4+N)^{5}(6+N)^{3}(8+N)(10+N)(12+N)(14+N)(16+N)],$$

$$P_{18}(N)=2572388703341675057536774963200+1549073617006033929837157023744 0N$$

$$+4379667062356295788434354811 6992N^{2}+7730356795189544913836488 5762048N^{3}$$

$$+9554829904808922280938928970 1376N^{4}+8793316367779543943184352 5427200N^{5}$$

$$+6254591539021699972226418632 2944N^{6}+3523061393449152136862439 6763136N^{7}$$

$$+1597367224107331721444306452 4800N^{8}+5893763588513416006604717 555712N^{9}$$

$$+1781938328420907284005997641 728N^{10}+4431238156578234811507342 70464N^{11}$$

$$+9071877291903006064239709388 8N^{12}+1526357036394260200955972 8128N^{13}$$

$$+2101040498701978693594644480N^{14}+234786621711017546808623104N^{15}$$

$$+21051290105729759459909632N^{16}+1488687398802313602573312N^{17}$$

$$+80970359711362371464704N^{18}+3260391296463194638592N^{19}+91332593076687166336N^{20}$$

$$+1585455937972251264N^{21}+12815704729451520N^{22},$$

$$Q_{18}(N)=[N^{18}(2+N)^{8}(4+N)^{5}(6+N)^{3}(8+N)^{2}(10+N)(12+N)(14+N)(16+N)],$$

$$P_{19}(N)=520527886328068273962166203187 2000+3474918369399669411887604528316 4160N$$

$$+1095822729488015255568802624422 54336N^{2}+2171838448478769090366839879620 36224N^{3}$$

$$+3036372186117635323082385813311 32416N^{4}+3186232017823421399997863005058 49856N^{5}$$

$$+2607161918344636753680060551759 13472N^{6}+1706124666468661232583514993459 20000N^{7}$$

$$+9086528224465878779011357274328 2688N^{8}+3987250619812935240555788360273 1008N^{9}$$

$$+1454044849801465540434028811701452 8N^{10}+44323003123600254769275375003893 76N^{11}$$

$$+11333193025472272112456803761520 64N^{12}+24344891547781158691682624785612 8N^{13}$$

$$+4391891835713313614023746976153 6N^{14}+663948063655003410430044802252 8N^{15}$$

$$+83760766632578723245823513395 2N^{16}+87611473125485396587006050304N^{17}$$

$$+7527667957445546882492902400N^{18}+524442821114449492810753024N^{19}$$

$$+29092566833509407917197312N^{20}+1252063771597395462270464N^{21}+40211990416209732975168N^{22}$$

$$+905135613785367281408N^{23}+12713207844348260928N^{24}+83694110414622720N^{25},$$

$$Q_{19}(N)=[N^{19}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)*(16+N)(18+N)],$$

$$P_{20}(N)=364364216507000739048781032456192 00+242252329259029149292981685059584000N$$

$$+760836819365998429440807117225197568N^{2}+150178565206532890541552198408051097 6N^{3}$$

$$+2091085636115266937041225621778202624N^{4}+2185466580009965172298815007619022848N^{5}$$

$$+1781169327233971975994097103934586880N^{6}+1161037087306635416601822682842923008N^{7}$$

$$+615979827257321748283543072421183488N^{8}+269288340608252471172568439091888128N^{9}$$

$$+97846798796078803878757697178304512N^{10}+29722035105904835928282451265191936N^{11}$$

$$+7574320184022150171504972296355840N^{12}+1621838229539819475164579420700672N^{13}$$

$$+291695540794512596709162178707456N^{14}+43970729279749087591513089572864N^{15}$$

$$+5532179700179033378083069755392N^{16}+577189034334496978087219724288N^{17}$$

$$+49475950475711816867430146048N^{18}+3439398747885949967063019520N^{19}$$

$$+190409861888427839784282112N^{20}+8179501446499459821179904N^{21}+262250369249432974358016N^{22}$$

$$+5893840222977683906816N^{23}+82665562945029918464N^{24}+543508305591050240N^{25},$$

$$Q_{20}(N)=[N^{20}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N)],$$

$$P_{21}(N)=5083700620556224094232614007943987200$$

$$+33914134117517361086489637271822663680N+106960011653231339336630081141317042176N^{2}$$

$$+212220505329746193452005977369732762828 8N^{3}+297302260175368795310942213050101399552N^{4}$$

$$+313025237995276078238138495911642267648N^{5}+257371094070321967954895479267682942976N^{6}$$

$$+169521259776501752389863842493350543360N^{7}+91050364189862954461189550890408738816N^{8}$$

$$+40384289133597449631576437145429082112N^{9}+14925093338861197964526532185281265664N^{10}$$

$$+4624921344106584117426507259006156800N^{11}+1206502137461352640448773722200342528N^{12}$$

$$+265539261406447751129975093315960832N^{13}+49329783084003642191028020556726272N^{14}$$

$$+7726071603449474489033002220912640N^{15}+1017253236852422389810623529254912N^{16}$$

$$+112062128851825470502997024882688N^{17}+10257273964318892809997578702848N^{18}$$

$$+772575678470520098133370336256N^{19}+47249856261231012352546619392N^{20}$$

$$+2303551580831028822666004224N^{21}+8720656306667688660501043 2N^{22}$$

$$+24654718903159177212079 36N^{23}+4888048398104696035328 0N^{24}$$

$$+605018274981696275296N^{25}+35114712307995217 92N^{26},$$

$$Q_{21}(N)=[N^{21}(2+N)^{9}(4+N)^{5}(6+N)^{3}(8+N)^{3}(10+N)(12+N)(14+N)(16+N)(18+N)(20+N)],$$

In particular, for $N=0$ we have (in terms of the variable $\widetilde{\beta}=\beta/N$)

$$\mu_2(0,\widetilde{\beta})=8\widetilde{\beta}+128\widetilde{\beta}^{2}+1416\widetilde{\beta}^{3}+13568\widetilde{\beta}^{4}+119336\widetilde{\beta}^{5}+995744\widetilde{\beta}^{6}+7999960\widetilde{\beta}^{7}$$

$$+62553792\widetilde{\beta}^{8}+478831624\widetilde{\beta}^{9}+3605380256\widetilde{\beta}^{10}+26779855704\widetilde{\beta}^{11}+196722717504\widetilde{\beta}^{12}$$

$$+1431514450232\widetilde{\beta}^{13}+10334365950624\widetilde{\beta}^{14}+74089380717928\widetilde{\beta}^{15}+527994918966848\widetilde{\beta}^{16}$$

$$+3742781855466712\widetilde{\beta}^{17}+26407888779849440\widetilde{\beta}^{18}+185544720650769800\widetilde{\beta}^{19}$$

$$+1298794138462713280\widetilde{\beta}^{20}+9060550224957149224\widetilde{\beta}^{21}+\cdots.$$

For $N=1$ we have

$$\mu_2(1,\beta)=8\beta+128\beta^{2}+4240/3\beta^{3}+40448/3\beta^{4}+1767376/15\beta^{5}+43862656/45\beta^{6}$$

$$+488322112/63\beta^{7}+18867219968/315\beta^{8}+1282950553456/2835\beta^{9}+47619744077056/14175\beta^{10}$$

$$+766490179148608/31185\beta^{11}+83121960882597376/467775\beta^{12}+7733317943660749472/6081075\beta^{13}$$

$$+384090193673880221312/42567525\beta^{14}+8114589875096001385568/127702575\beta^{15}$$

$$+283873458550660118340608/638512875\beta^{16}+2582349138528384648119632/834978375\beta^{17}$$

$$+2091109673217277557956275456/97692469875\beta^{18}+995381021756635176086923648/6749661555\beta^{19}$$

$$+939094200965970473008545313945 6/9280784638125\beta^{20}$$

$$+1037028767968617333539369974859 84/14992036723125\beta^{21}+\cdots.$$

For $N=2$ we have

$$\mu_2(2,\beta)=4\beta+32\beta^{2}+353/2\beta^{3}+840\beta^{4}+87803/24\beta^{5}+361583/24\beta^{6}+45760537/768\beta^{7}$$

$$+10981361/48\beta^{8}+13182440203/15360\beta^{9}+72831253657/23040\beta^{10}+12683037443293/1105920\beta^{11}$$

$$+284019102193/6912\beta^{12}+9021584611263989/61931520\beta^{13}+12677663451078517/24772608\beta^{14}$$

$$+1414035206576526155 9/7927234560\beta^{15}+305934957463202891 9/495452160\beta^{16}$$

$$+181894032587921447576 29/856141332480\beta^{17}+15561939557899235869069 3/2140353331200\beta^{18}$$

$$+42399131891951527511159177/171228266496000\beta^{19}$$

$$+17971503390357918425668499/21403533312000\beta^{20}$$

$$+4273293188426542164055548742 3/15068087451648000\beta^{21}+\cdots.$$

For $N=3$ we have

$$\mu_2(3,\beta)=8/3\beta+128/9\beta^{2}+784/15\beta^{3}+67072/405\beta^{4}+4081648/8505\beta^{5}+167636864/127575\beta^{6}$$

$$+944026304/273375\beta^{7}+16849951744/1913625\beta^{8}+4153759481008/189448875\beta^{9}$$

$$+1065794492624896/19892131875\beta^{10}+116677223724748652 8/9050920003125\beta^{11}$$

$$+\,8314233519972990976/27152760009375\beta^{12}+1757996758934716965 44/244374840084375\beta^{13}$$

$$+\,409170117445661176448/244374840084375\beta^{14}+361305927020036448 3884384/934733763322734375\beta^{15}$$

$$+\,173915360520409186670373376/19629409029777421875\beta^{16}$$

$$+\,31611058478034436738314658288/1566426840576238265625\beta^{17}$$

$$+\,84383259864054068055963337861 12/1846145919250566527343 75\beta^{18}$$

$$+\,2196394046423223252344967160626 1888/2132298536734404339082031 25\beta^{19}$$

$$+\,1478205935399679818528444670174 40256/6396895610203213017246093 75\beta^{20}$$

$$+\,2071607179815945880528993224873 734816/4012598155491106347181640625\beta^{21}+\cdots.$$

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