Analytic periodic solutions for domain-wall motion

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We apply multiple time scales to the study of the domain-wall motion for some anisotropic magnetics with uniaxial magnetic anisotropy. Analytic periodic solutions were obtained. We also investigated the stability of the periodic solutions. The periodic windows found are in good agreement with experimental and numerical results. [S0163-1829(97)06925-7]

I. INTRODUCTION

In the last few years the investigation of micromagnetics has increased in importance. This is because of miniaturization and high-density data storage. Recently, some studies[1–5] addressed the problem of domain-wall motion. The present study develops analytical methods and provides an analytical solution of that problem.

II. MODEL

The starting point is the Landau-Lifshitz[6] and Gilbert[7] equation of the magnetic spin system dynamics,

$$\frac{d\mathbf{I}}{dt}=-\gamma(\mathbf{I}\times\mathbf{H})+\frac{\alpha}{I_s}\left(\mathbf{I}\times\frac{d\mathbf{I}}{dt}\right), \tag{1}$$

where $\mathbf{I}$ is the magnetization vector, I_s is the saturation magnetization, $\mathbf{H}$ is the external magnetic field, t is the time, γ is the gyromagnetic constant, and α is the Gilbert damping coefficient.

Following Sloczewski[8] we study the behavior of the anisotropic magnetics with uniaxial magnetic anisotropy in the z-axis direction. The Bloch wall plane is infinite and parallel to the zy plane. An external magnetic field is applied in the direction of the z axis, and the domain wall moves in the direction of the x axis. It is also assumed that the angle measured from the y axis in the yz plane (the precession angle) is small. These requirements are satisfied for samples with the values of material parameters indicated in Table I and with the amplitude of the magnetic external field smaller than about a few hundred A/m. Having all of this in mind, using Eq. (1), the equation of the Bloch wall motion is

$$\frac{2\mu_0(1+\alpha^2)}{\gamma^2\Delta}\frac{d^2x}{dt^2}+\frac{8\pi\mu_0 I_s\alpha}{|\gamma|\Delta}\frac{dx}{dt}=2I_sH, \tag{2}$$

where x is the coordinate of the domain wall, μ_0 is the permeability of the vacuum, and Δ is the parameter of the width of magnetic domain wall. The second term on the left-hand side of Eq. (2) is due to damping.

A major modification was made[4,5] in order to consider new features. Eddy current damping is considered and an energy[9] associated with restoring forces. This term is assumed in the form of sine wave with wavelength l due to the particular form of the internal stress considered. Equation (2) has the final form

$$\frac{2\mu_0(1+\alpha^2)}{\gamma^2\Delta}\frac{d^2x}{dt^2}+\left(\frac{8\pi\mu_0 I_s\alpha}{|\gamma|\Delta}+\frac{16dI_s^2}{\pi^3\rho}\right)\frac{dx}{dt}+\frac{2\delta H_c I_s}{l}\sin\left(\frac{2\pi x}{l}\right)=2I_sH\cos\omega t, \tag{3}$$

where d is the thickness of the magnetic material and ρ is the electrical resistivity of the magnetic material. We will make the usual notation[4,5] M, K, R, and B for the corresponding coefficients of Eq. (3) in this order.

III. ANALYTIC SOLUTION

Equation (3) is equivalent to the nonauthonoumous system of equations

$$\frac{du_1}{dt}=u_2,$$

$$\frac{du_2}{dt}=-\frac{K}{M}u_2-\frac{R}{M}\sin\frac{2\pi u_1}{l}+\frac{B}{M}\cos\omega t. \tag{4}$$

To find the periodic solutions we use the averaging method proposed by Krylov and Bogoliubov.[10] It is assumed that the periodic solution can be described by the equations

$$u_1=a(t)\cos[\omega t+\varphi(t)],$$

$$u_2=-\omega a(t)\sin[\omega t+\varphi(t)]. \tag{5}$$

Substituting Eqs. (5) into Eqs. (4), one obtains the differential equations of the amplitude and phase:

TABLE I. Material parameters and coefficients.

Saturation magnetization	$I_s=0.46$ (W b/m^2)
Coercive force	$H_c=0.29$ (A/m)
Wavelength of the internal stress	$l=1$ (μm)
Frequency	$f=2$ (MHz)
Mass of domain wall	$M=1.66\times10^{-9}$ (kg/m^2)
Damping coefficient	$K=7.8\times10^{-2}$ (kg/s m^2)
Restitution coefficient	$R=0.2668$ (kg/ms^2)

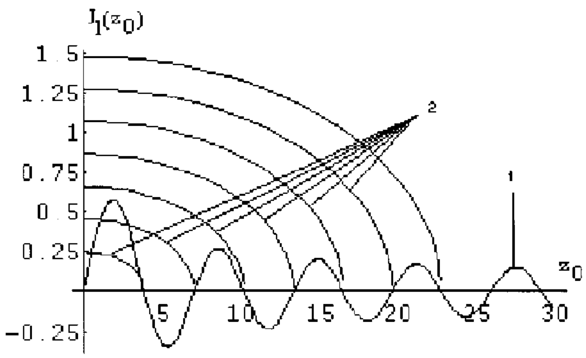

FIG. 1. Graph of the Bessel function of the first kind, labeled 1, and the graph of the right-hand side of Eq. (9a), labeled 2, for some particular values of normalized steady-state variable z_0.

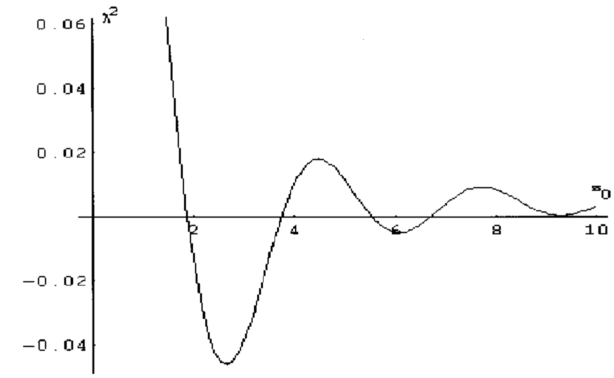

FIG. 2. Change of sign of the function indicating the change of the stability of the periodic solution.

$$\frac{da}{dt}=-\frac{K}{M}\,a\,\sin^2[\omega t+\varphi(t)]+\frac{R}{M\omega}\sin\left(\frac{2\pi}{l}\,a\,\cos[\omega t+\varphi(t)]\right)\sin[\omega t+\varphi(t)]-\frac{B}{M\omega}\sin[\omega t+\varphi(t)]\cos\omega t,$$

$$\frac{d\varphi}{dt}=\frac{K}{M}\sin[\omega t+\varphi(t)]\cos[\omega t+\varphi(t)]-\frac{R}{M\omega a}\sin\left(\frac{2\pi}{l}\,a\,\cos[\omega t+\varphi(t)]\right)\cos[\omega t+\varphi(t)]-\frac{B}{M\omega a}\cos[\omega t+\varphi(t)]\cos\omega t. \tag{6}$$

Averaging Eqs. (6) over a period of the external magnetic field results in

$$\frac{da}{dt}=-\frac{K}{2M}\,a-\frac{B}{2M\omega}\sin\varphi,$$

$$\frac{d\varphi}{dt}=-\frac{R}{M\omega a}\,J_1\left(\frac{2\pi a}{l}\right)+\frac{B}{2M\omega a}\cos\varphi, \tag{7}$$

where $J_1(x)$ is the Bessel function of the first kind (see the Appendix). The steady-state solutions of Eqs. (7) are

TABLE II. Zeros of the Bessel function of the first kind and corresponding values of the external magnetic field.

x_0	H (A/m)	B (kg/ms^2)
3.83171	0.14277	0.13334
7.01559	0.26139	0.24414
10.17350	0.37905	0.35404
13.3237	0.49643	0.46367
16.4706	0.61368	0.67318
19.6159	0.73087	0.68263
22.7601	0.84802	0.79205
25.9037	0.96515	0.90144
29.0468	1.08226	1.01083

$$\sin\varphi_0=-\frac{K\omega}{B}\,a_0,$$

$$\cos\varphi_0=\frac{2R}{B}\,J_1\left(\frac{2\pi a_0}{l}\right), \tag{8}$$

or in a more useful form

$$J_1(z_0)=\pm\beta_0\sqrt{\alpha_0 H^2-z_0^2},$$

$$\cos\varphi_0=\frac{2R}{B}\,J_1(z_0), \tag{9}$$

where a_0 and φ_0 are the steady-state amplitude and phase, $\beta_0=Kfl/2R$, $\alpha_0=2I_s/Kfl$, and f is the frequency of the applied external magnetic field. The usual parameters for the model are given in the Table I.

Instead of solving the transcendental equation (8a), we prefer to extract as much information as possible from a qualitative approach. To do that, we have represented in the same picture the graphs of the left-hand side (the Bessel function of the first kind) and the right-hand side of Eq. (9a) in Fig. 1. Due to the symmetry of the solution, we consider here only the positive domain. Table II indicates the zeros of the Bessel function and corresponding values of the magnetic field and of the B coefficient.

According to Fig. 1 for z_0 less than the first zero of the Bessel function there is only one intersection point of the two graphs and therefore only one allowed steady state. Increasing z_0, it is possible to find three or two or only one steady state and so on. Then we may conclude that one bifurcation point is located near $B=0.244\,14$ (see Table II and Fig. 2), another bifurcation point is near $B=0.354\,04$, the next bifurcation point is around $B=0.463\,67$, and so on. Our

TABLE III. Stability range of the periodic solution in the linear approximation.

x_0	$H_{\text{bifurcation}}$ (A/m)	$B=2HI_s$ (kg/ms^2)
1.87466	0.34472	0.31714
3.73682	0.14312	0.13167
5.54368	0.28735	0.26436
6.70043	0.25940	0.23864

theoretical study is in good agreement with previously reported results.[1,4,5] As examples, Refs. 4 and 5 reported bifurcation and period-doubling scenarios for first transition to chaotic motion for $B=0.255$. The next bifurcation point is, according to Refs. 4 and 5, at $B=0.389$. Previous studies numerically demonstrated that periodic motion is allowed, and here we obtain analytical expressions of the periodic solution.

This formulation also enables us to derive some important conclusions about the stability of the periodic solution. The traditional way to study the stability of the periodic solution is based on the monodromy matrix.[11] Here we will try to avoid the complicated numerical evaluations required by this method. To this purpose let us observe that the Jacobian matrix of Eqs. (7) is

$$J=\begin{pmatrix} -\dfrac{K}{2M} & -\dfrac{R}{M\omega}J_1\left(\dfrac{2\pi a_0}{l}\right) \\ -\dfrac{R}{Mfa_0 l}J_1'\left(\dfrac{2\pi a_0}{l}\right) & \dfrac{K}{2M} \end{pmatrix}, \tag{10}$$

where $J_1'(x)$ is the first derivative of the Bessel function of the first kind. From Eq. (10) it is now possible to obtain the eigenvalues

$$\lambda_{1,2}=\pm\frac{R}{Mfl}\sqrt{\left(\frac{Kfl}{2R}\right)^2+\frac{J_1(z_0)J_1'(z_0)}{z_0}}. \tag{11}$$

The graph of the function under the radical sign is shown in Fig. 2. Table III indicates the zeros of the function in Fig. 2.

IV. CONCLUSIONS

For the problem of domain-wall motion in an external magnetic field, we found analytic expressions of periodic solutions. The predictions of our theory are compared with new numerical and existing numerical data and with experiment. The agreement is quit satisfactory.

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APPENDIX

In this appendix we shall briefly discuss the averaging procedure that enables us to obtain the relations (7). It is straightforward to show that

$$\overline{\sin^2(\omega t+\varphi)}=\frac{1}{2},\quad \overline{\sin(\omega t+\varphi)\cos\omega t}=\frac{\sin\varphi}{2},\quad \overline{\cos(\omega t+\varphi)\cos\omega t}=\frac{\cos\varphi}{2}. \tag{A1}$$

Let us define

$$A_1=\overline{\sin(\omega t+\varphi)\sin\left(\frac{2\pi x}{l}\right)}, \tag{A2a}$$

$$A_2=\overline{\cos(\omega t+\varphi)\sin\left(\frac{2\pi x}{l}\right)}. \tag{A2b}$$

It is convenient for further evaluations for us to remember that

$$\begin{aligned}\sin[z\cos(\theta+\varphi)]&=\sum_{p=0}^{\infty}(-1)^p\frac{z^{2p+1}}{(2p+1)!}\cos^{2p+1}(\theta+\varphi)\\ &=\sum_{p=0}^{\infty}\sum_{k=0}^{2p+1}(-1)^p\frac{z^{2p+1}}{(2p+1)!}C_{2p+1}^k(\cos\theta\cos\varphi)^k(-\sin\theta\sin\varphi)^{2p+1-k}\\ &=\sum_{p=0}^{\infty}\sum_{k=0}^{2p+1}(-1)^{p-k+1}\frac{z^{2p+1}}{(2p+1)!}C_{2p+1}^k\cos^k\theta\cos^k\varphi\sin^{2p+1-k}\theta\sin^{2p+1-k}\varphi.\end{aligned} \tag{A3}$$

If one introduces Eq. (A3) into Eq. (A2a), one obtains

$$A_1=\frac{1}{2\pi}\sum_{p=0}^{\infty}\sum_{k=0}^{2p+1}(-1)^{p+1-k}\frac{z^{2p+1}}{(2p+1)!}C_{2p+1}^k\cos^k\varphi\sin^{2p+2-k}\varphi\int_{-\pi}^{\pi}\cos^{k+1}u\sin^{2p+1-k}u\,du, \tag{A4}$$

where $\theta=u+\pi$. It is known that

$$\int_{-\pi}^{\pi}\cos^k u\sin^{2p+2-k}u\,du=\begin{cases}2\displaystyle\int_0^{\pi}\cos^k u\sin^{2p+2-k}u\,du & \text{if } k \text{ is even,}\\ 0, & \text{otherwise,}\end{cases}$$

$$\int_{-\pi}^{\pi} \cos^{k+1}u \ \sin^{2p+1-k}u \ du = \begin{cases} 2\int_0^{\pi} \cos^{k+1}u \ \sin^{2p+1-k}u \ du & \text{if } k \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\int_0^{\pi} \cos^{2p}u \ \sin^{2n}u \ du = B\left(n+\frac{1}{2}, p+\frac{1}{2}\right), \tag{A5}$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the B Euler function. In order to find explicit form of the coefficient A_1, we have to insert expressions (A5) into formula (A4). This gives

$$A_1=\left[-\frac{\Gamma(p-l+\frac{3}{2})\Gamma(l+\frac{1}{2})}{\Gamma(p+2)}+\frac{p-l+\frac{1}{2}}{l+\frac{1}{2}}\frac{\Gamma(p-l+\frac{1}{2})\Gamma(l+\frac{3}{2})}{\Gamma(p+2)}\right]\frac{1}{\pi}\sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^p\frac{z^{2p+1}}{(2p+1)!}C_{2p+1}^{2l}\cos^{2l+1}\varphi\ \sin^{2p+1-2l}\varphi$$
$$=0.$$

Similarly, we have

$$A_2=\frac{1}{\pi}\sum_{p=0}^{\infty}\sum_{l=0}^{p}(-1)^p\frac{z^{2p+1}}{(2p+1)!}C_{2p+1}^{2l}\cos^{2l}\varphi\ \sin^{2p-2l}\varphi\ \frac{\Gamma(p-l+\frac{3}{2})\Gamma(l+\frac{1}{2})}{\Gamma(p+2)}=\sum_{p=0}^{\infty}(-1)^p\left(\frac{z}{2}\right)^{2p+1}\frac{1}{p!(p+1)!}=J_1(z), \tag{A6}$$

where we used the definition of the Bessel function of the first kind,

$$J_n(z)=\sum_{p=0}^{\infty}(-1)^p\left(\frac{z}{2}\right)^{2p+n}\frac{1}{p!(p+1)!}$$

and

$$\Gamma(p+\tfrac{1}{2})=\frac{(2p)!\sqrt{\pi}}{p!2^{2p}}.$$

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