

Coherent states of alternating current

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(Received 2 December 1996)

We study counting statistics of electric current pumped by pulses of an external field. The fluctuations depend on the pulse shape, and can be minimized by choosing the pulse shape properly. For an optimal pulse shape, the fluctuations are reduced to the dc level, i.e., they do not depend on the duty cycle of the signal. We develop an approach that allows us to calculate all counting statistics for various driving fields, optimal and nonoptimal. The statistics depend in an interesting way on the analytic structure of the field time dependence, and display an analogy with coherent states and instantons. [S0163-1829(97)04735-8]

I. INTRODUCTION

One of the fundamental problems of “single electronics” is that of quantized charge transfer monitored by an external field.¹ A real device such as electronic pump or turnstile is operated by a periodically alternating field that drives electric current at a rate of one electron per cycle.² Due to a microscopic scale of the currently studied systems, the issue of current fluctuations, classical and quantum, becomes very important. Ideally, in order to reduce current fluctuations well below the level of the ac signal, the pump should transfer charge adiabatically.³ For a perfectly operating pump, the cycle duration T is much longer than \hbar/E_c , where E_c is charging energy. If the adiabatic limit $E_c T \gg \hbar$ is achieved, the current fluctuations are exponentially small.

In practice, the adiabatic parameter can be of the order of 1, which makes the analysis very difficult, since one has to consider many effects simultaneously: cotunneling,⁴ finite relaxation rates,⁵ quantum fluctuation of charge,⁶ etc. In this paper we consider the problem in the nonadiabatic limit $E_c \ll \hbar/T$, where the charging energy can be ignored. With the usual assumption that the interaction of the Fermi-liquid quasiparticles vanishes near the Fermi surface, the pump can be described by single-particle scattering amplitudes periodically varying in time. In this case the only effective mechanism of the current fluctuation suppression is due to Fermi statistics, which makes electron transmission events correlated in the time domain. The effect of Fermi statistics on the dc current fluctuations has been studied.⁷ We shall extend this theory to the ac current.

From the point of view of the pump quality, the extreme nonadiabatic limit we consider is the least-efficiency mode. So, it gives an upper estimate of the fluctuations at low temperature, and provides a reference for understanding real systems where nonadiabatic effects are strong. Actually, for the noninteracting fermions one can develop a complete theory

that gives not just the mean square of current fluctuations, but *all* statistics of transmitted charge.¹¹ Although the free fermion counting statistics problem captures only part of the physics relevant for operation of a real device, it is interesting enough and nontrivial by itself.

It turns out that the fluctuations strongly depend on the pulse shape of the driving ac signal. The character of this dependence resembles the H theorem. It was shown recently by two of us that at fixed average current the fluctuations level is bounded from below by the fluctuations of the dc current with the same mean value.^{8,9} Moreover, there exist “optimal” ac signals for which the minimum is reached, and the fluctuations remain on the dc level (see Fig. 1). It is interesting that for optimal signals the fluctuations are independent on the relative pulse width τ/T (the signal duty cycle), no matter how sharp the pulses are.

The case is appealing to an analogy with coherent states

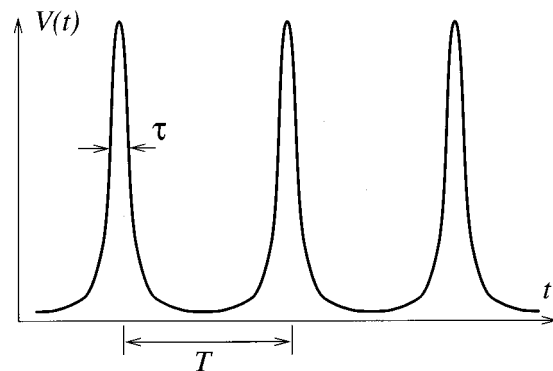


FIG. 1. Voltage time dependence [Eqs. (45), (46), and (30)] that makes current fluctuations minimal. Shown is the simplest solution: periodic sequence of Lorentzian peaks of the area h/e each. For the optimal pulse shape the fluctuations do not depend on the pulse width τ .

that minimize quantum-mechanical uncertainty. Similarly, the optimal ac signals drive current in such a way that quantum noise is reduced to the dc minimum level. Coherent states are known to possess an interesting analytic structure,¹⁰ and one encounters a similar situation in this problem. We study analytic structure of the problem, and show that it is related to the group of modular transformations. The analytic structure facilitates the study of counting statistics. We develop a method discussed in our previous work,¹¹ and use it to calculate complete counting statistics for various interesting examples of the driving signal.

To be specific, we study a one-dimensional model of a Fermi gas of electrons transferred through a potential barrier by an external ac field. Electrons are incident from the left and the right reservoirs that supply zero-temperature Fermi distribution. In the ac field, the scattering becomes nonelastic, and electrons with different energies interfere due to their Fermi statistics. We treat the problem as a multichannel scattering problem with the scattering amplitudes given by the Fourier components of the external field, and present a general formalism that gives counting statistics.

We compute the distribution exactly for a particular class of external fields, which includes the optimal signals. For these fields the law of time dependence is a periodic analytic function of time given by a rational functions of the ‘‘circular’’ variable $z = e^{i\Omega t}$. We find exact statistics of the charge transfer, and show that it displays interesting features explained by the analytic character of the time-dependence law. In particular, we shall discuss Lorentzian pulses of voltage with quantized flux: $c \int V(t) dt = n\Phi_0$, where n is an integer, and $\Phi_0 = hc/e$ is the flux quantum. We find that the pulses represent a quantum analog of the classical picture of n independent attempts to transmit electrons through the scatterer. We arrive at this result by means of the method of Ref. 12 combined with a special treatment that allows one to reduce the problem in an infinite-dimensional space to a finite-dimensional problem (cf. Ref. 11).

The paper is organized as follows. In Secs. II and III we describe the model and state the mathematical problem to solve. We introduce a generating function of the charge-transfer statistics and express it as the determinant of a scattering operator in an infinite-dimensional space. For computing the determinant it is important to find physically meaningful regularization. The regularization problem is treated in Sec. IV. We find that the regularization is sensitive to gauge transformation of the electromagnetic field, and choose the gauge so that the regularization becomes simple. In Sec. V we address the question of noise minimization and recall that the signals that give minimal noise have particular analytic structure: rational functions of the circular variable $e^{i\Omega t}$ analytic inside the unit disk.⁹ We show how this result follows from our expression for the generating function. Next, in Sec. VI we treat the case of a rational time dependence of the external field for which we reduce the problem to finding the determinant of a finite-dimensional matrix, and compute the probability distribution explicitly. Section VII contains the discussion of the symmetries of the problem. We observe that the system possesses a symmetry group of conformal transformations isomorphic to $PSL(2, \mathbf{R})$. Section VIII treats several interesting examples of the driving signal that reveal interesting features of the counting statistics. In

Sec. IX we summarize our discussion. In order to make the readers’ burden less heavy, some technical details are moved from the main body of the paper to Appendixes A and D. For completeness, we review relevant results of Refs. 9 and 12 in Appendixes B and C. In Appendix E we recall gauge transformations that relate the ac flux problem to the ac bias problem, which is essential for the issue of the determinant regularization.

II. MODEL

We consider the following model of a microscopic contact. One-dimensional ideal Fermi gas scatters off a potential barrier $U(x)$, so that the Hamiltonian of the system is

$$H = \Psi^\dagger(x) \left[\frac{1}{2} \left(-i \frac{\partial}{\partial x} - \frac{e}{c} a(x,t) \right)^2 + e v(x,t) + U(x) \right] \Psi(x), \quad (1)$$

where $\Psi^\dagger(x)$ and $\Psi(x)$ are the canonical operators of electrons, $U(x)$ is the scattering potential, and $a(x,t)$ and $v(x,t)$ are the external electromagnetic field vector and scalar potentials, respectively. We shall treat electrons as spinless and noninteracting. The scattering potential $U(x)$ will be described by the matrix of scattering amplitudes

$$\mathbf{A} = \begin{pmatrix} A_{LL} & A_{RL} \\ A_{LR} & A_{RR} \end{pmatrix}, \quad (2)$$

so that the scattering states have the standard asymptotic form:

$$\begin{aligned} \Psi_{L,k} &= \begin{cases} e^{ikx} + A_{LL} e^{-ikx}, & x \rightarrow -\infty \\ A_{LR} e^{ikx}, & x \rightarrow +\infty; \end{cases} \\ \Psi_{R,k} &= \begin{cases} A_{RL} e^{-ikx}, & x \rightarrow -\infty \\ e^{-ikx} + A_{RR} e^{ikx}, & x \rightarrow +\infty. \end{cases} \end{aligned} \quad (3)$$

We would like to study the response of the system to a time-dependent external field, electric or magnetic. Let us recall that by a gauge transformation one can go from a problem with the electric potential $v(x,t)$ to that with the vector potential $a(x,t)$ and vice versa (see Appendix E). In the limit of *instant scattering*, which we shall assume from now on, the external field may be taken into account by introducing a time-dependent phase in the scattering amplitudes:

$$\mathbf{A} \rightarrow \tilde{\mathbf{A}}(t) = \begin{pmatrix} A_{LL} & A_{RL} e^{-i\varphi(t)} \\ A_{LR} e^{i\varphi(t)} & A_{RR} \end{pmatrix} \quad (4)$$

with

$$\begin{aligned} \varphi(t) &= e \int_{-\infty}^t [v(x = +\infty, t') - v(x = -\infty, t')] dt' \\ &+ \frac{e}{c} \int_{-\infty}^{+\infty} a(x,t) dx. \end{aligned} \quad (5)$$

Instant scattering means that the external fields $a(x,t)$ and $v(x,t)$ (and thus $e^{i\varphi(t)}$ as well) vary slowly in comparison with the scattering time $\tau_{sc} \sim \hbar |\partial A_{\alpha\beta} / \partial E|$, where $A_{\alpha\beta}$ are

the scattering amplitudes and E is the energy of electrons. The physical meaning of τ_{sc} is the time that the particle spends inside the scatterer. Technically, neglecting τ_{sc} means that the scattering amplitudes $A_{\alpha\beta}$ do not depend on the energy.

III. STATING THE PROBLEM

Now, after the model is defined, we are ready to state the problem of the counting statistics of electron transmission. Namely, we are interested in finding the probabilities of a given charge transfer over a fixed time interval. We assume that the counting time $T^{(0)}$ is much longer than the period T of the external field. Let us observe the system under the action of the external field during a long time interval $T^{(0)}$. The field from now on is treated only as the phase factor $e^{i\varphi(t)}$ in Eq. (4). Let us denote the probabilities of transmitting exactly n electrons during this time by $P_n^{(0)}$. The probabilities can be conveniently combined into a generating function

$$\chi^{(0)}(\lambda) = \sum_{n=-\infty}^{+\infty} P_n^{(0)} e^{i\lambda n}. \quad (6)$$

This function of the auxiliary parameter λ will be the main quantity we shall work with. It encodes all the information about the statistics of the charge transfer. The moments of the distribution are given by the coefficients of the Taylor expansion of $\chi^{(0)}(\lambda)$ around $\lambda=0$. In particular, the average charge transfer is

$$\langle n \rangle = -i \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} \chi^{(0)}(\lambda), \quad (7)$$

and the dispersion

$$\langle \langle n^2 \rangle \rangle = \langle n^2 \rangle - \langle n \rangle^2 = - \left. \frac{\partial^2}{\partial \lambda^2} \right|_{\lambda=0} \ln \chi^{(0)}(\lambda). \quad (8)$$

Notice that even if we apply no external field there exist equilibrium fluctuations. At zero temperature $\langle \langle n^2 \rangle \rangle$ is growing with time as $\ln T^{(0)}$.¹³ The zero-temperature equilibrium fluctuations can be neglected if the external field $e^{i\varphi(t)}$ is periodic in time: in this case the field-induced fluctuations grow linearly with $T^{(0)}$ and dominate over the equilibrium noise.¹⁴ In this paper we disregard the equilibrium noise and study only the effect of the periodic field, and only briefly consider the nonperiodic case and the equilibrium noise in Appendix D.

There exists a useful description of the field-induced statistics in terms of the quantities that do not depend on the counting time. Let us introduce

$$P_n = \{\text{probability of transmitting } n \text{ electrons per cycle}\}. \quad (9)$$

The advantage of P_n is that the total generating function $\chi^{(0)}(\lambda)$ for a long time interval $T^{(0)} \gg T$ can be written as an exponent:

$$\chi^{(0)}(\lambda) \sim [\chi(\lambda)]^{T^{(0)}/T} \quad \text{as } T^{(0)} \rightarrow \infty, \quad (10)$$

where

$$\chi(\lambda) = \sum_{n=-\infty}^{+\infty} P_n e^{i\lambda n} \quad (11)$$

is the generating function per one cycle of the field.

This factorization can be explained by recalling that the zero-temperature equilibrium fluctuations are essentially due to the counting beginning and ending. It is completely analogous to the fluctuations of the number of particles in one-dimensional ideal Fermi gas (here the role of the dimension is taken by the time). One can regard the field-induced fluctuations as ‘‘extensive in the time domain,’’ and the equilibrium fluctuations as ‘‘boundary effects.’’ As usual, to treat the ‘‘bulk’’ effects separately from ‘‘boundary’’ ones, we adopt periodic boundary conditions on the evolution of the system (1) in the time domain. Below we argue that this assumption leads to the factorization (10).

In general, periodic external fields $a(x,t)$ and $v(x,t)$ correspond to a quasiperiodic phase $\varphi(t)$: $\varphi(t+T) = \varphi(t) + \Delta\varphi$. First, consider the case when $\Delta\varphi$ is a multiple of 2π (periodic phase factor $e^{i\varphi(t)}$) and then we can easily extend our results to arbitrary $\Delta\varphi$. If we observe the system during a large number M of field cycles and impose boundary conditions periodic in time, it selects a discrete set of energies out of the continuum spectrum of the Hamiltonian. The energy spacing between the levels is inversely proportional to the observation time: $\Delta E = h/MT = \hbar\Omega/M$ ($\Omega = 2\pi/T$ is the circular frequency of the external field). At the same time, scattering is possible only between energy levels separated by a multiple of $\hbar\Omega = h/T$. Thus, since the energy is conserved modulo $\hbar\Omega$, we have M noninterfering copies of the scattering problem with discrete energy levels spaced by $\hbar\Omega$. Since $M = T^{(0)}/T$, this proves our formula (10).

In the case of a quasiperiodic phase factor, the generating function of the statistics per cycle is constructed in the following way. One has to find the smallest positive ϵ for which the phase factor $e^{i\varphi(t)+i\epsilon t}$ is periodic. Then $\chi(\lambda) = \chi_1^{1-\epsilon}(\lambda) \chi_2^\epsilon(\lambda)$, where χ_1 and χ_2 are found for the periodic phase factors $e^{i\varphi(t)+i\epsilon t}$ and $e^{i\varphi(t)+i(\epsilon-2\pi)t}$. For the proof and discussion of the quasiperiodic case we refer to our previous paper.¹¹ In this paper only periodic phase factors appear.

The above argument shows that the field-driven contribution $\chi(\lambda)$ can be extracted by closing the time axis into a circle of period T , thus quantizing the energy with the quantum $\hbar\Omega = h/T$. The problem becomes a multichannel scattering problem, where the channels represent the discrete energy levels. For such a problem $\chi(\lambda)$ can be quite generally expressed in terms of scattering amplitudes.¹² In this method the function $\chi(\lambda)$ depends on a vector argument $\lambda = (\lambda_1, \dots, \lambda_m)$, m being the number of channels. The Fourier transform of $\chi(\lambda)$ gives the probabilities of scattering between channels. If \mathbf{A} is the matrix of the multichannel scattering amplitudes, \mathbf{n} is the occupation number operator (diagonal in the energy representation), $\Lambda = \text{Diag}(e^{i\lambda_1}, \dots, e^{i\lambda_m})$, $\mathbf{1}$ is the unit matrix, then

$$\chi(\lambda) = \det[\mathbf{1} + \mathbf{n}(\mathbf{S} - \mathbf{1})], \quad (12)$$

where

$$\mathbf{S} = \mathbf{A}^\dagger \Lambda^\dagger \mathbf{A} \Lambda, \quad (13)$$

and “det” means the determinant of an $m \times m$ matrix. The proof of this formula is reviewed in Appendix A.

Although derived under the implicit assumption of a finite number of channels m , Eq. (12) will further be used for infinite-dimensional matrices. This will require a careful definition of determinant in the infinite-dimensional case, which will be done in the next section.

In our treatment of the periodic problem the operators \mathbf{n} , \mathbf{A} , Λ , etc. act in the linear space

$$H = V \oplus V \quad (14)$$

of left and right states with discrete energies. Formally, V is the \mathbf{C}^∞ space of the states of the discrete spectrum, and by Fourier transformation it can be treated as the space of periodic functions of time, with the period T . Two copies of V correspond to the left and right channels. Further, the computations will be performed in the basis (α, m) , where $\alpha \in \{R, L\}$ specifies the side, $m \in \mathbf{Z}$ labels the Fourier harmonics. To be specific about notation, we shall sometimes write operators \mathbf{F} acting in H in the form

$$\mathbf{F} = \begin{pmatrix} F_{LL} & F_{RL} \\ F_{LR} & F_{RR} \end{pmatrix}, \quad (15)$$

where $F_{\alpha\beta}$ are operators acting in $V \cong \mathbf{C}^\infty$.

In the application to our problem, the scattering matrix is time dependent:

$$\tilde{\mathbf{A}} = \begin{pmatrix} B & A^*[f(z)]^* \\ Af(z) & -B^* \end{pmatrix}, \quad (16)$$

where A and B are the transmission and reflection amplitudes for the potential barrier $U(x)$, $|A|^2 + |B|^2 = 1$; $z = \exp(2\pi it/T)$ is the variable on the unit circle; the phase factor $f(z) = \exp[i\varphi(t)]$. We choose the formal variables λ_i to count only total charge transfer, regardless of the energy of transmitted electrons:

$$\lambda_{L,m} = 0, \quad \lambda_{R,m} = \lambda. \quad (17)$$

Then from Eq. (13) we get

$$\begin{aligned} \tilde{\mathbf{S}} &= \tilde{\mathbf{A}}^\dagger \Lambda^\dagger \tilde{\mathbf{A}} \Lambda \\ &= \begin{pmatrix} |A|^2(e^{-i\lambda} - 1) + 1 & A^*B^*(e^{i\lambda} - 1)[f(z)]^* \\ -AB(e^{-i\lambda} - 1)f(z) & |A|^2(e^{i\lambda} - 1) + 1 \end{pmatrix}, \end{aligned} \quad (18)$$

where the functions $f(z)$ and $[f(z)]^* = 1/f(z)$ should be understood as operators acting in V by multiplication:

$$\langle m | f(z) | n \rangle = \oint \frac{dz}{2\pi i} \frac{f(z)}{z^{m-n+1}}. \quad (19)$$

We consider the system at zero temperature, thus

$$\mathbf{n} | \alpha, m \rangle = \begin{cases} | \alpha, m \rangle, & m \leq 0 \\ 0, & m > 0. \end{cases} \quad (20)$$

By that, formally, all operators in Eqs. (12) and (13) are specified, and we can proceed with the calculation.

IV. REGULARIZATION OF THE DETERMINANT

Now, the problem is to find the determinant (12) of an infinite matrix. Because of the infinite dimensionality, this determinant needs to be understood properly. Eventually, to compute the determinant, we are going just to cut a finite submatrix of $\mathbf{1} + \mathbf{n}(\tilde{\mathbf{S}} - \mathbf{1})$ at some positive and negative energies way off the Fermi level, and to take its determinant. However, one has to be careful and to make sure that the infinite parts of the matrix thrown away do not matter. Physically, the reason is that all incident states with very high energies are empty, and remain so after scattering off an alternating field. Similarly, all states with very low energies are deep in the Fermi sea, and remain doubly occupied throughout the scattering. Thus, in order that our regularization procedure has no effect on the determinant, we have to assure that the unitarity of scattering is preserved when a submatrix is cut. Then the infinite submatrices that we remove will contain only the contribution of “Fermi vacuum” states, and will not affect the counting statistics, and thus the determinant.

More formally, we have an infinite number of states below the Fermi level in the left and the right reservoirs to be put in a one-to-one correspondence. Shifting states in one reservoir with respect to the states in the other one can be achieved by a gauge transformations different for the left and the right reservoirs. The key observation is that the matrix $\mathbf{1} + \mathbf{n}(\tilde{\mathbf{S}} - \mathbf{1})$ is not invariant under a gauge transformation. (The transformation rule of the particle number operator \mathbf{n} is reviewed in Appendix E, and the matrices \mathbf{A} and $\tilde{\mathbf{S}}$ transform the same way as the density matrix.) Of course, since the determinant of $\mathbf{1} + \mathbf{n}(\tilde{\mathbf{S}} - \mathbf{1})$ gives counting statistics, the determinant regularization must depend on the gauge transformation in such a way that the regularized determinant is gauge invariant.

In order to clarify the relation between the regularization and the gauge transformations, let us consider a system with no barrier. Then $A = 1$, $B = 0$, and the scattering is only forward, no backward. Then the matrix $\tilde{\mathbf{S}}$ given by Eq. (18) becomes diagonal and time independent:

$$\tilde{\mathbf{S}} = \begin{pmatrix} e^{-i\lambda} & 0 \\ 0 & e^{i\lambda} \end{pmatrix}. \quad (21)$$

In this case, one could try to compute the determinant of $\mathbf{1} + \mathbf{n}(\tilde{\mathbf{S}} - \mathbf{1})$ by using “naive” regularization, i.e., by simply cutting all columns and rows above some large positive and below some large negative energy. Then $\chi(\lambda) = \det(\tilde{\mathbf{S}}) = 1$: no transport for any $\varphi(t)$, which is unreasonable. The problem becomes even more striking if one thinks of a gauge transformation. Under a gauge transformation the $\tilde{\mathbf{S}}$ given by Eq. (21) does not change, and \mathbf{n} is transformed according to rule (E9) (see Appendix E). For example, if the gauge phase $\phi(t) = n\Omega t$, in the energy representation one gets

$$\mathbf{n}'_L(E) = \mathbf{n}_L(E), \quad \mathbf{n}'_R(E) = \mathbf{n}_R(E - n\hbar\Omega). \quad (22)$$

Now, by using the naive regularization one finds

$$\chi(\lambda) = e^{in\lambda}, \quad (23)$$

which means that the result is not gauge invariant.

At this point one can conclude that the correct regularization must change together with the gauge. For the general scattering problem (16) we proceed in the following way. We choose a gauge transformation so that the scattering matrix \mathbf{A} becomes time independent. Then $\tilde{\mathbf{S}}$ is also time independent, and all phase factors are shifted to \mathbf{n} . We argue that after such gauge transformation one can use the naive regularization. The transformation is chosen so that all dependence on the ac field is moved to the occupation of the incoming one-, two-, or many-particle states. The advantage of going to a purely elastic scattering is that all outgoing channels that enter the scattering unitarity relation will have equal energies. Thus, while removing from $\mathbf{1} + \mathbf{n}(\mathbf{S} - \mathbf{1})$ all states with energies below some large negative energy we preserve the unitarity of scattering. In other words, the states that interfere at the scattering are either both included in the truncated matrix, or both are removed. Therefore, for our gauge transformation the naive regularization is meaningful. Moreover, it is clear that such transformation is unique, unless $\tilde{\mathbf{S}}$ is diagonal.¹⁵

On the basis of this consideration, we argue that the determinant (12) must be understood in the following way:

$$\begin{aligned} \chi(\lambda) &= \det[\mathbf{1} + \mathbf{n}(\tilde{\mathbf{S}} - \mathbf{1})] = \det \left[\mathbf{1} + \mathbf{n} \begin{pmatrix} [f(z)]^* & 0 \\ 0 & 1 \end{pmatrix} \right. \\ &\quad \left. \times (\mathbf{S} - \mathbf{1}) \begin{pmatrix} f(z) & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= \det \left[\mathbf{1} + \begin{pmatrix} f(z) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{n} \begin{pmatrix} [f(z)]^* & 0 \\ 0 & 1 \end{pmatrix} (\mathbf{S} - \mathbf{1}) \right] \\ &= \det[\mathbf{1} + \tilde{\mathbf{n}}(\mathbf{S} - \mathbf{1})], \end{aligned} \quad (24)$$

where

$$\mathbf{S} = \begin{pmatrix} |A|^2(e^{-i\lambda} - 1) + 1 & A^*B^*(e^{i\lambda} - 1) \\ -AB(e^{-i\lambda} - 1) & |A|^2(e^{i\lambda} - 1) + 1 \end{pmatrix} \quad (25)$$

is the time-independent scattering matrix, and

$$\tilde{\mathbf{n}} = \begin{pmatrix} f(z) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{n} \begin{pmatrix} [f(z)]^* & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

is the time-dependent occupation number operator. In the last line of Eq. (24) we can assign an unambiguous meaning to the determinant (cf. Ref. 11). Namely, we note that \mathbf{S} is unitary and has unit determinant. Also, $\tilde{\mathbf{n}}$ tends to 0 at high energies and to 1 at low energies. Therefore, the matrix $[\mathbf{1} + \tilde{\mathbf{n}}(\mathbf{S} - \mathbf{1})]$ behaves at infinity like a block-diagonal matrix, with 2×2 blocks each having a unit determinant. We define the determinant in Eq. (24) by cutting the matrix along one of these blocks at infinity. In such a way the determinant is well defined and depends essentially on the matrix entries around the Fermi level. Below we shall be able to compute it explicitly for a special choice of $f(z)$.

V. NOISE MINIMIZATION

It turns out that the analytic structure of $f(z)$ plays an important role in the current fluctuations. In particular, the functions $f(z)$ which minimize the noise $\langle\langle n^2 \rangle\rangle$ for a given

average charge transfer $\langle n \rangle$ belong to the class of rational functions. The problem of noise minimization has been considered in detail in Ref. 9, and here we briefly review the result. The rate of charge transfer $\langle n \rangle$ is simply proportional to the phase gain per period:

$$\langle n \rangle = |A|^2 \frac{\Delta\varphi}{2\pi}, \quad \Delta\varphi = \arg f(z) \Big|_{t=0}^{t=T}, \quad (27)$$

while the noise $\langle\langle n^2 \rangle\rangle$ depends on the whole function $f(z)$:

$$\langle\langle n^2 \rangle\rangle = 2|A|^2|B|^2 \oint \oint \frac{dz_1 dz_2}{(2\pi i)^2} \frac{f(z_1)f^*(z_2) - 1}{(z_1 - z_2)^2}. \quad (28)$$

For completeness, both expressions are derived in Appendix B.

The variational problem arises of finding the function $f(z)$, which defines a map of a fixed degree $N = \Delta\varphi/2\pi$ of the unit circle into itself and minimizes the noise functional (28). In Appendix C we review the proof of Ref. 9 that optimal $f(z)$ is analytic either inside or outside the unit circle $|z| = 1$. In other words, its Laurent expansion $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$ contains either only nonpositive or only non-negative powers. Such a function can be written as

$$f(z) = \prod_{i=1}^N \frac{z - a_i}{1 - a_i^* z}, \quad (29)$$

where either all $|a_i| > 1$, or all $|a_i| < 1$. The corresponding time dependence of the phase is

$$\varphi(t) = \sum_{i=1}^N \tan^{-1} \left(\frac{(1 - |a_i|^2) \sin \Omega(t - t_i)}{(1 + |a_i|^2) \cos \Omega(t - t_i) - 2|a_i|} \right) + \phi_0, \quad (30)$$

where $t_i = \arg a_i / \Omega$, $\phi_0 = \sum_i \arg a_i$. Thus the optimal phase time dependence is a sum of N ‘‘elementary excitations,’’ or ‘‘kinks,’’ each corresponding to a 2π phase change of the scattering amplitude per one cycle of the signal.

For any of such functions the mean square fluctuation of the transmitted charge $\langle\langle (en)^2 \rangle\rangle$ is equal to $e^2 |AB|^2 N$ per cycle. It is remarkable that the noise does not depend on relative displacement of the kinks in the time domain, nor on their durations. The degeneracy is described by $2N - 1$ real parameters.

It is interesting that for the time dependence (29) all the probabilities P_n can be computed and admit a simple interpretation of N noninterfering attempts of electrons to pass through the barrier (see Examples 1 and 2 in Sec. VIII). In fact, the class of functions to which our method applies is broader: it includes all rational functions (29) regardless of the location of the poles. (The phase time dependence then has the form (30) with arbitrary signs of different terms.) We show that for any such $f(z)$ the generating function $\chi(\lambda)$ can be expressed as the determinant of a *finite* matrix, and that it yields only a *finite* number of nonzero probabilities P_n .

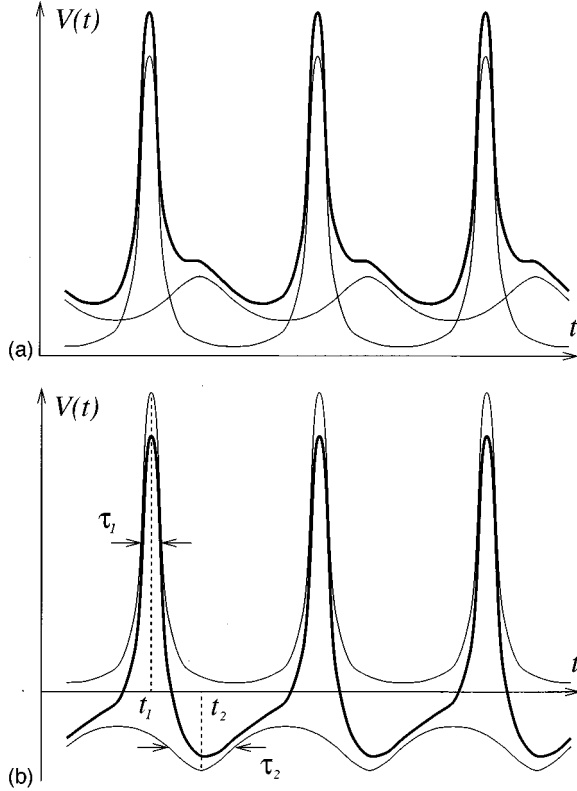


FIG. 2. Voltage optimal time dependence corresponding to a pair of 2π kinks of $\varphi(t) = (e/\hbar) \int_{-\infty}^t V(t') dt'$. For the pulses of equal sign (a) the counting statistics do not depend on the relative position of the pulses $t_{1,2}$ and on their duration $\tau_{1,2}$ [see Example 2, Eq. (39)]. For the pulses of opposite sign (b) average current is zero and other statistics show nontrivial interference of the kinks [see Example 3, Eqs. (40) and (41)].

We see that $\chi(\lambda)$ contains N equal factors, each corresponding to one of the factors of $f(z)$. This means that each ‘‘elementary excitation’’ in Eq. (30) corresponds to one attempt to pass the barrier with the one-particle outcome probabilities. Of course, the actual scattering state in this case, as well as in Example 1, is a many-particle coherent state. One could expect this result when all $a_i = \infty$ (this again corresponds to constant voltage). However, here we get the same expression for the superposition of *any* set of ‘‘elementary excitations’’ (Example 1) of the same polarization with no dependence on the values of a_i . It is a surprising result. Of course, in this example different sets of $\{a_i\}$ cannot be transformed into each other by the symmetry group G , therefore we cannot explain this invariance merely by the $PSL(2, \mathbf{R})$ symmetry discussed previously. At present we look at this invariance as at a miracle and admit this to be a consequence of a broader group of symmetries.

We shall also list the answers for two other examples of $f(z)$ that demonstrate the interference of excitations with opposite polarizations.

Example 3. $N=2$. $|a_1| > 1$, $|a_2| < 1$. Then one has

$$(p_{ij}) = \frac{1}{a_1^* - a_2^*} \begin{pmatrix} a_1^* & -1 \\ a_1^* a_2^* & -a_2^* \end{pmatrix}, \quad (40)$$

$$(q_{ij}) = \frac{1}{a_1 - a_2} \begin{pmatrix} a_1 & -a_1 a_2 \\ 1 & -a_2 \end{pmatrix},$$

$$\chi(\lambda) = 1 - 2F + F(e^{i\lambda} + e^{-i\lambda}),$$

where

$$F = |A|^2 |B|^2 \frac{|1 - a_1^* a_2|^2}{|a_1 - a_2|^2}. \quad (41)$$

Note that in this example the probabilities of the electron transfer in both directions are equal, and thus the average charge flux is zero, although for arbitrary parameters a_1 and a_2 the signal can be asymmetric [see Fig. 2(b)]. In this case $\chi(\lambda)$ cannot be represented as a product of independent contributions of the two elementary excitations, but exhibit their interference. To make this clear, let us show what happens if one tries to factor the generating function:

$$\chi(\lambda) = (u + w e^{i\lambda})(u + w e^{-i\lambda}), \quad (42)$$

where the ‘‘probabilities’’ $u, w = \frac{1}{2}(1 \pm \sqrt{1 - 4F})$, so in this case there is no natural relation between the factors of $\chi(\lambda)$ and of $f(z)$.

Example 4. $N > 1$. $a_1 = a$, $a_i = b$ for $i > 1$. $|a| > 1$, $|b| < 1$. Computations can be most easily performed for the case $b = 0$ (which corresponds to a constant voltage applied against one elementary excitation). Then by using the $PSL(2, \mathbf{R})$ symmetry one can extend the result to arbitrary b :

$$\chi(\lambda) = [1 - 2F + F(e^{i\lambda} + e^{-i\lambda})] (|A|^2 e^{i\lambda} + |B|^2)^{N-2}, \quad (43)$$

where

$$F = |A|^2 |B|^2 \frac{|1 - a^* b|^2}{|a - b|^2}. \quad (44)$$

As in Example 3, there is an interference of the kinks of opposite sign. It is interesting that the factors of $\chi(\lambda)$ can be interpreted by saying that one positive and one negative kink interfere and form a ‘‘neutral’’ system with which other kinks do not interfere. The degree of interference is measured by $\Lambda = |1 - a^* b|/|a - b|$, and varies from 0 to 1 depending on how much the kinks overlap in time, or by how much their durations differ. For example, $\Lambda \rightarrow 1$ if the kinks almost not overlap, and then $\chi(\lambda)$ factors into separate contributions of independent kinks: the ‘‘probabilities’’ u and v in Eq. (42) become just the one-particle probabilities $|A|^2$ and $|B|^2$.

IX. DISCUSSION

To summarize, we found the probability distribution for the charge transfer under the action of a periodic external field described by a rational function of $z = e^{i\Omega t}$. According to our previous remarks about the equivalence between magnetic and electric fields (also, see Appendix E), we can treat the field $f(z)$ of form (29) as the alternating voltage $V(t)$:

$$V(t) = -\frac{i}{e} \frac{d}{dt} \ln f(e^{i\Omega t}) = \frac{\Omega z}{e} \sum_{i=1}^N \frac{1 - |a_i|^2}{1 + |a_i|^2 - (a_i^* z + a_i z^{-1})}$$

$$= \frac{\Omega}{e} \sum_{i=1}^N \frac{1 - \rho_i^2}{1 + \rho_i^2 - 2\rho_i \cos(\Omega t - \varphi_i)}, \quad (45)$$

where $a_i = \rho_i e^{i\varphi_i}$.

This expression represents $V(t)$ as a sum of elementary excitations. Each elementary excitation alone represents one attempt of an electron to pass the barrier, with the probability to pass given by the transmission coefficient of the barrier. We found that if the excitations are of the same sign, then these attempts do not interfere. The general formula (36) holds for any $V(t)$ composed of elementary excitations of arbitrary signs. It predicts a nontrivial interference of the excitations of opposite signs.

Each elementary excitation in Eq. (45) can be written as a sum of Lorentzian pulses:

$$V_k(t) = \frac{\Omega}{e} \frac{1 - \rho_k^2}{1 + \rho_k^2 - 2\rho_k \cos(\Omega t - \varphi_k)}$$

$$= \sum_{m=-\infty}^{\infty} \frac{\Omega}{e} \frac{2\tau_k}{(t - t_k - mT)^2 + \tau_k^2}, \quad (46)$$

with the width $\tau_k = \Omega^{-1} \ln(\rho_k^{-1})$, centered at $t_k = \Omega^{-1} \varphi_k$. Each of the pulses carries flux proportional to its area. The flux is quantized:

$$c \int_{-\infty}^{\infty} \frac{\Omega}{e} \frac{2\tau_k dt}{(t - t_k - mT)^2 + \tau_k^2} = \frac{hc}{e} = \Phi_0. \quad (47)$$

The flux quantization is just another way to say that each elementary excitation corresponds to a 2π phase shift.

Using representation (46) our results can be to some extent translated to the nonperiodic case. The limiting form of the excitation as the period $T \rightarrow \infty$ ($\Omega \rightarrow 0$) is

$$V(t) = \frac{2\hbar}{e} \frac{\tau}{(t - t_0)^2 + \tau^2}, \quad (48)$$

a single Lorentzian pulse with the area $\int V(t) dt = h/e$. If many such pulses all of the same sign are generated continuously at a finite rate so that average current dominates over the equilibrium fluctuations, then the calculation will tell that each pulse corresponds to a ‘‘one-electron-like’’ attempt to pass the barrier, and the distribution of outcomes is binomial (exactly as for constant voltage). The interference of the field-driven current with the equilibrium noise will be discussed elsewhere.

X. CONCLUSION

We studied quantum counting statistics of an ac current driven by pulses of external field. There are special pulse configurations that create many-particle coherent scattering states in which the quantum noise is reduced to a dc minimum. The analytic structure of such states is studied, and put in connection with the modular symmetry group of the problem. A general method to calculate counting statistics is presented and applied to the coherent states and to other states ‘‘naturally related’’ to them. The counting statistics are

found to be binomial and ‘‘generalized binomial,’’ respectively.

ACKNOWLEDGMENTS

The research of L.S.L. was partly supported by the Alfred Sloan Foundation. Research at the L. D. Landau Institute was supported by the International Science Foundation Grant No. M9M000.

APPENDIX A: EXPRESSION FOR THE GENERATING FUNCTION

In this Appendix we review the proof of Eq. (12) from Ref. 12. First we express the determinant (12) as

$$\chi(\lambda) = \det(\mathbf{1} - \mathbf{n} + \mathbf{nS}) = \sum_{\{i_1, \dots, i_k\}} S_{i_1 \dots i_k}^{i_1 \dots i_k} \prod_{i \neq i_\alpha} (1 - n_i) \prod_{i=i_\alpha} n_i, \quad (A1)$$

where the summation is over all subsets of channels $\{i_1, \dots, i_k\}$. Here $S_{i_1 \dots i_k}^{i_1 \dots i_k}$ denotes the determinant of the submatrix of \mathbf{S} formed by the entries in the rows $\{i_1, \dots, i_k\}$ and in the columns $\{j_1, \dots, j_k\}$. From Eq. (13) it follows that

$$S_{i_1 \dots i_k}^{i_1 \dots i_k} = \sum_{\{j_1, \dots, j_k\}} e^{i(\lambda_{j_1} + \dots + \lambda_{j_k} - \lambda_{i_1} - \dots - \lambda_{i_k})} |A_{i_1 \dots i_k}^{j_1 \dots j_k}|^2, \quad (A2)$$

where the determinants

$$A_{i_1 \dots i_k}^{j_1 \dots j_k} = \sum_P \varepsilon(P) A_{i_1}^{j_{P(1)}} \dots A_{i_k}^{j_{P(k)}} \quad (A3)$$

are n -particle scattering amplitudes. Note that

$$P_{i_1 \dots i_k | j_1 \dots j_k} = \prod_{i \neq i_\alpha} (1 - n_i) \prod_{i=i_\alpha} n_i |A_{i_1 \dots i_k}^{j_1 \dots j_k}|^2 \quad (A4)$$

are the probabilities of the many-electron scattering from the channels i_1, \dots, i_k to the channels j_1, \dots, j_k . This proves that

$$\chi(\lambda) = \sum_{\substack{\{i_1, \dots, i_k\} \\ \{j_1, \dots, j_k\}}} P_{i_1 \dots i_k | j_1 \dots j_k} e^{i(\lambda_{j_1} + \dots + \lambda_{j_k} - \lambda_{i_1} - \dots - \lambda_{i_k})} \quad (A5)$$

is the generating function for the probability distribution $P_{i_1 \dots i_k | j_1 \dots j_k}$ of the charge transfer.

APPENDIX B: EXPRESSIONS FOR $\langle N \rangle$ AND $\langle \langle N^2 \rangle \rangle$

From Eq. (24) we can derive expressions for the total charge transfer and its dispersion in terms of the external field $e^{i\varphi(t)}$. We shall perform the calculations for the periodic field $e^{i\varphi(t)} = f(e^{i\Omega t})$. One can write

$$\langle n \rangle = -i \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} \chi(\lambda) = -i \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} \ln \chi(\lambda)$$

$$= -i \text{Tr} \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} [\mathbf{1} + \tilde{\mathbf{n}}(\mathbf{S} - \mathbf{1})]$$

$$\begin{aligned}
&= -i \operatorname{Tr} \left[\tilde{\mathbf{n}} \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} (\mathbf{S} - \mathbf{1}) \right] \\
&= \operatorname{Tr} \left[\tilde{\mathbf{n}} \begin{pmatrix} -|A|^2 & -A^*B^* \\ -AB & |A|^2 \end{pmatrix} \right]. \tag{B1}
\end{aligned}$$

$$\tilde{\mathbf{n}}(z_1, z_2) = z_2 \begin{pmatrix} \frac{f(z_1)f^*(z_2)}{z_2(1+\delta) - z_1} & 0 \\ 0 & \frac{1}{z_2(1+\delta) - z_1} \end{pmatrix}. \tag{B3}$$

Therefore,

In the time representation ($z = e^{i\Omega t}$),

$$\mathbf{n}(z_1, z_2) = \sum_{n \leq 0} z_1^{-n} z_2^n e^{\delta n} = z_2 \frac{1}{z_2(1+\delta) - z_1}, \tag{B2}$$

$$\begin{aligned}
\langle n \rangle &= |A|^2 \oint \left(\lim_{z_2 \rightarrow z_1} \frac{f(z_1)f^*(z_2) - 1}{z_1 - z_2} \right) \frac{dz_1}{2\pi i} \\
&= -|A|^2 \oint \frac{dz}{2\pi i} f(z) \partial f^*(z) \\
&= |A|^2 \oint \frac{d\varphi(t)}{2\pi} = |A|^2 \frac{\Delta\varphi}{2\pi}, \tag{B4}
\end{aligned}$$

where δ is infinitesimal positive. Thus one gets

where $\Delta\varphi$ is the phase change per period.
Similarly,

$$\begin{aligned}
\langle \langle n^2 \rangle \rangle &= - \frac{\partial^2}{\partial \lambda^2} \Big|_{\lambda=0} \ln \chi(\lambda) = - \operatorname{Tr} \frac{\partial^2}{\partial \lambda^2} \Big|_{\lambda=0} \ln [\mathbf{1} + \tilde{\mathbf{n}}(\mathbf{S} - \mathbf{1})] \\
&= - \operatorname{Tr} \left[\left(\tilde{\mathbf{n}} \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} (\mathbf{S} - \mathbf{1}) \right)^2 - \tilde{\mathbf{n}} \frac{\partial^2}{\partial \lambda^2} \Big|_{\lambda=0} (\mathbf{S} - \mathbf{1}) \right] \\
&= \operatorname{Tr} \left[\tilde{\mathbf{n}} \begin{pmatrix} -|A|^2 & -A^*B^* \\ -AB & |A|^2 \end{pmatrix} \tilde{\mathbf{n}} \begin{pmatrix} -|A|^2 & -A^*B^* \\ -AB & |A|^2 \end{pmatrix} - \tilde{\mathbf{n}} \begin{pmatrix} |A|^2 & -A^*B^* \\ AB & |A|^2 \end{pmatrix} \right] \\
&= \oint \frac{dz_1}{2\pi i} \lim_{z_2 \rightarrow z_1} \left[\oint \frac{dz_3}{2\pi i} \left(|A|^4 \frac{f(z_1)f^*(z_2) + 1}{[z_1 - z_3(1+\delta)][z_3 - z_2(1+\delta)]} \right. \right. \\
&\quad \left. \left. + |A|^2 |B|^2 \frac{f(z_1)f^*(z_3) + f(z_3)f^*(z_2)}{[z_1 - z_3(1+\delta)][z_3 - z_2(1+\delta)]} - \frac{2|A|^2}{z_1 - z_2(1+\delta)} \right) \right] \\
&= 2|A|^2 |B|^2 \oint \oint \frac{dz_1 dz_2}{(2\pi i)^2} \frac{f(z_1)f^*(z_2) - 1}{(z_1 - z_2)^2}. \tag{B5}
\end{aligned}$$

The last line shows that $\langle \langle n^2 \rangle \rangle$ depends in a nontrivial way on the whole function $f(z)$, unlike $\langle n \rangle$, which depends only on the total phase shift $\Delta\varphi$ per period. This recovers the result of Ref. 9 for a periodically varying field.

APPENDIX C: VARIATIONAL PROBLEM

In this Appendix we review the proof of Ref. 9 that the variational problem of minimizing $\langle \langle n^2 \rangle \rangle$ for a fixed value of $\langle n \rangle$ is equivalent to the analyticity of $f(z)$ either inside or outside the unit circle.

We decompose $f(z)$ into a sum of $f_+(z)$ and $f_-(z)$, which are analytic inside and outside the unit circle, respectively,

$$f_+(z) = \sum_{n=0}^{\infty} a_n^+ z^n, \tag{C1}$$

$$f_-(z) = \sum_{n=0}^{\infty} a_n^- z^{-n}.$$

Then, by the Cauchy theorem,

$$\oint \frac{dz}{2\pi i} f_+(z) \partial f_-^*(z) = \oint \frac{dz}{2\pi i} f_-(z) \partial f_+^*(z) = 0. \tag{C2}$$

Therefore,

$$\begin{aligned}
\langle n \rangle &= -|A|^2 \oint \frac{dz}{2\pi i} f(z) \partial f^*(z) \\
&= -|A|^2 \oint \frac{dz}{2\pi i} [f_+(z) \partial f_+^*(z) + f_-(z) \partial f_-^*(z)] \\
&= |A|^2 \sum_n n (|a_n^+|^2 - |a_n^-|^2), \tag{C3}
\end{aligned}$$

and

$$\begin{aligned}
\langle\langle n^2 \rangle\rangle &= 2|A|^2|B|^2 \oint \oint \frac{dz_1 dz_2}{(2\pi i)^2} \frac{f(z_1)f^*(z_2) - 1}{(z_1 - z_2)^2} \\
&= |A|^2|B|^2 \oint \frac{dz}{2\pi i} [\partial f_+(z) - \partial f_-(z)] f^*(z) \\
&= |A|^2|B|^2 \oint \frac{dz}{2\pi i} [\partial f_+(z) f_+^*(z) - \partial f_-(z) f_-^*(z)] \\
&= |A|^2|B|^2 \sum_n n(|a_n^+|^2 + |a_n^-|^2). \tag{C4}
\end{aligned}$$

By comparing the two expressions we see that at fixed $\langle n \rangle$ the fluctuation $\langle\langle n^2 \rangle\rangle$ is minimal when f_+ or f_- vanishes: then $\langle\langle n^2 \rangle\rangle_{\min} = |B|^2 \langle n \rangle$.

APPENDIX D: NONPERIODIC SIGNAL

Here we make a remark on how formulas (12) and (13) can be applied to the case of a nonperiodic field. Actually, expression (12) for the multichannel characteristic function

$$\begin{aligned}
\langle\langle n^2 \rangle\rangle &= -\text{Tr} \left. \frac{\partial^2}{\partial \lambda^2} \right|_{\lambda=0} \ln[\mathbf{1} + \tilde{\mathbf{n}}(\mathbf{S} - \mathbf{1})] \\
&= \int_0^{T^{(0)}} \frac{dt_1}{2\pi i} \lim_{t_2 \rightarrow t_1} \left[2|A|^4 \int_0^{T^{(0)}} \frac{dt_3}{2\pi i} \frac{1}{(t_1 - t_3 + i0)(t_3 - t_2 + i0)} - \frac{2|A|^2}{t_1 - t_2 + i0} \right. \\
&\quad \left. + 2|A|^2|B|^2 \int_0^{T^{(0)}} \frac{dt_3}{2\pi i} \frac{1}{(t_1 - t_3 + i0)(t_3 - t_2 + i0)} \right] \\
&= 2|A|^2 \int_0^{T^{(0)}} \frac{dt_1}{2\pi i} \lim_{t_2 \rightarrow t_1} \left[\int_0^{T^{(0)}} \frac{dt_3}{2\pi i} \frac{1}{(t_1 - t_3 + i0)(t_3 - t_2 + i0)} - \frac{1}{t_1 - t_2 + i0} \right] \\
&= -2|A|^2 \int_0^{T^{(0)}} \frac{dt}{2\pi i} \int_{t > T^{(0)}}^{t < 0} \frac{dt_3}{2\pi i} \frac{1}{(t - t_3)^2} = \frac{|A|^2}{2\pi^2} \int_0^{T^{(0)}} dt \left[\frac{1}{T^{(0)} - t} + \frac{1}{t} \right] = \frac{|A|^2}{\pi^2} \ln \frac{T^{(0)}}{\tau_{\text{sc}}}, \tag{D3}
\end{aligned}$$

where τ_{sc} is the ultraviolet cutoff set by a characteristic scattering time of the system $\hbar \partial A(E)/\partial E$.

It is straightforward to check that the fluctuations given by the last line of Eq. (D3) agree with the Nyquist equilibrium noise spectrum $S_\omega = (e^2/h)G(\omega)|\omega|$. Indeed,

$$\begin{aligned}
\langle\langle n^2 \rangle\rangle &= \int_0^{T^{(0)}} dt \int_0^{T^{(0)}} dt' \langle\langle j(t)j(t') \rangle\rangle \\
&= \int \frac{d\omega}{2\pi} \frac{|1 - e^{i\omega t}|^2}{\omega^2} S_\omega \\
&= \frac{|A|^2}{\pi^2} \ln \frac{T^{(0)}}{\tau_{\text{sc}}}, \tag{D4}
\end{aligned}$$

where $|A|^2 = |A(E_F)|^2 = G(\omega=0)$, and \hbar/τ_{sc} is the ultraviolet frequency cutoff.

$\chi(\lambda)$ is quite general and can be applied to the nonperiodic case as well. Suppose the charge is measured during a finite time $T^{(0)}$. This can be taken into account by making the parameter λ of the generating function time dependent, and by ‘‘turning it on’’ only for the time interval of the measurement. If the external field is encoded into a (nonperiodic) phase factor $f(t)$, then the distribution is given by expressions (24) and (13), where

$$\Lambda = \Lambda(t) = \begin{pmatrix} e^{i\lambda(t)} & 0 \\ 0 & 1 \end{pmatrix}, \tag{D1}$$

$$\lambda(t) = \begin{cases} \lambda, & 0 < t < T^{(0)} \\ 0, & t < 0 \text{ or } t > T^{(0)}. \end{cases} \tag{D2}$$

The operator $\mathbf{S}(t)$ becomes unity outside the observation time interval, and this makes the determinant (12) well defined. We postpone a general discussion of the nonperiodic case to elsewhere. Here we only demonstrate that this approach gives the correct answer for the equilibrium noise $\langle\langle n^2 \rangle\rangle$ in the absence of external field.

By the argument of Appendix B,

APPENDIX E: GAUGE TRANSFORMATIONS

In this Appendix we show how by gauge transformations one can switch between the problem with an electric potential and that with a vector potential. Recall that the gauge transformation

$$\Psi = \tilde{\Psi} e^{-i\phi(t,x)} \tag{E1}$$

changes the vector potential $a(x,t)$ and the electric potential $v(x,t)$ as

$$\tilde{a} = a - \frac{c}{e} \frac{\partial \phi(t,x)}{\partial x},$$

$$\tilde{v} = v + \frac{1}{e} \frac{\partial \phi(t,x)}{\partial t}. \tag{E2}$$

Assume that we apply a time-dependent magnetic field with the vector potential $a(x, t)$ localized around the scatterer. For example, this can be realized as a varying magnetic flux threading a conducting loop with the contact. Then, by the gauge transformation (E1) with

$$\phi(t, x) = \frac{e}{c} \int_{-\infty}^x a(t, x) dx \quad (\text{E3})$$

we can turn to the problem with the zero vector potential and the electric potential

$$v(t, x) = \frac{1}{e} \frac{\partial \phi(t, x)}{\partial t}. \quad (\text{E4})$$

In this case, the gauge phase shift across the scatterer is

$$\varphi(t) = \phi(t, x) \Big|_{x=-\infty}^{x=+\infty} = \frac{e}{c} \int_{-\infty}^{+\infty} a(t, x) dx, \quad (\text{E5})$$

and it can be viewed as the time-dependent voltage

$$V(t) = \frac{1}{e} \frac{\partial \varphi(t)}{\partial t} \quad (\text{E6})$$

applied to the contact.

Thus we recalled the familiar gauge transformation for the one-particle problem. When looking at a many-body problem, one also has to transform the density matrix and Green's functions according to rule (E1). Let us consider the occupation number operator of a reservoir:

$$\hat{\mathbf{n}}(\omega) = \sum_k \delta(\omega - E_k) \hat{\psi}_k^\dagger \hat{\psi}_k, \quad (\text{E7})$$

where the summation is over all reservoir states. In the determinant we have $\mathbf{n} = \langle \hat{\mathbf{n}} \rangle$ averaged over actual distribution. In the time representation

$$\mathbf{n}(t, t') = \sum_k e^{iE_k(t-t')} \langle \hat{\psi}_k^\dagger \hat{\psi}_k \rangle = \sum_k e^{iE_k(t-t')} n_F(E_k), \quad (\text{E8})$$

where $n_F(E)$ is Fermi distribution. Under the gauge transformation (E1) with the gauge phase (E3) the occupation number operator transforms as

$$\begin{aligned} \tilde{\mathbf{n}}_L(t, t') &= \mathbf{n}_L(t, t'), \\ \tilde{\mathbf{n}}_R(t, t') &= \mathbf{n}_R(t, t') e^{i[\varphi(t) - \varphi(t')]}, \end{aligned} \quad (\text{E9})$$

for the left and the right reservoirs, respectively.

We assume that the scattering is instant, then the alternating field effect on the scattering states is entirely determined by the phase shift $\varphi(t)$. This means that the ac magnetic field can be introduced simply by adding the phase in the scattering amplitudes (4), while in the problem with the electric field we have the energies of occupied states in one reservoir to be shifted with respect to those in the other one. By virtue of gauge invariance, these two formulations are obviously equivalent, and we make use of this in the discussion of the determinant regularization [see Eq. (24)].

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- ¹³The form factor $\langle\langle n_q n_{-q} \rangle\rangle$ of the equilibrium density-density correlator in a Fermi gas at $T=0$ is given by $|q|/2p_F$ at $|q| < 2p_F$, 1 at $|q| > 2p_F$. The logarithmic dependence of the fluctuation of the number of particles in a finite interval is derived by looking at a double integral of the form factor, very similar to expression (D4) for the Nyquist noise.
- ¹⁴At finite temperature, equilibrium fluctuations grow linearly in time: $\langle\langle n^2 \rangle\rangle \sim T^{(0)}$. However, for a temperature much less than the mean voltage the effect on the counting statistics will be small.
- ¹⁵Diagonal \mathbf{S} means that the scattering is purely forward, and thus the left and the right states neither mix nor interfere. In this case there is an ambiguity in the regularization: the truncation energies for the left and the right channels can be chosen differently. However, the ambiguity is absent if there is a backscattering, no matter how weak.