Quantum Hall Skyrmions with higher topological charge

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We have investigated quantum Hall Skyrmions at filling factor $\nu=1$ carrying more than one unit of topological, and hence electric, charge. Using a combination of analytic and numerical methods we find the counterintuitive result that when the Zeeman energy is tuned to values much smaller than the interaction energy $[g\mu_B B/(e^2/\epsilon\ell)<8.4\times10^{-5}]$, the creation energy of a charge two Skyrmion becomes less than twice the creation energy of a charge one Skyrmion, i.e., Skyrmions bind in pairs. The doubly charged Skyrmions are stable to further accretion of charge and exhibit a 10% larger spin per unit charge than charge one Skyrmions which would, in principle, signal this pairing. [S0163-1829(97)06335-2]

Ferromagnetic quantum Hall (QH) systems have quasiparticle excitations, ''Skyrmions,'' that involve a texturing of the spins.[1] There is now strong experimental evidence that the lowest energy charged excitations at filling factor $\nu=1$ are Skyrmions.[2–5]

A smooth texturing of the spins can be described by an effective nonlinear σ model [see Eq. (1) below] where the spin is represented by a unit vector $\mathbf{n}(\mathbf{r})$. Skyrmions are topological excitations characterized by an integer topological charge $Z=\int d\mathbf{r} q(\mathbf{r})$, where $q=\mathbf{n}\cdot(\partial_x\mathbf{n}\times\partial_y\mathbf{n})/4\pi$ is the topological (Pontryagin) density; Z is the winding number of the mapping $\mathbf{n}(\mathbf{r})$ from the compactified plane (S_2) to the target space of the σ model (S_2). A characteristic feature of QH Skyrmions is that the topological density is proportional to the deviation of the electron density ρ from its uniform value $\overline{\rho}$: $q=\nu(\rho-\overline{\rho})$. Thus, at $\nu=1$, Skyrmions with topological charge Z carry electric charge $-Z|e|$.

Up till now only Skyrmions with charge one have been considered for the natural reason that one would expect Coulombically repelling Skyrmions to disfavor charge aggregation. In this paper we study Skyrmions at $\nu=1$ with higher charge and find that this expectation is violated for extremely small values of the Zeeman energy where the Skyrmions are large objects. More precisely, we find that a charge two Skyrmion has lower energy than a pair of charge one Skyrmions for $\tilde{g}<\tilde{g}_c=8.4\times10^{-5}$, where $\tilde{g}\equiv g\mu_B B/(e^2/\epsilon\ell)$ is the dimensionless Zeeman energy. This implies that the interaction between two Skyrmions, which must be repulsive at long distances on account of their electric charges, has an attractive core in a region where their identities are no longer distinct. We also find that Skyrmions with charge three and higher are energetically disfavored at all values of $\tilde{g}$.

We begin with variational estimates of the Skyrmion energies for various topological charges at small $\tilde{g}$ that were the motivation for this work. We then study the Skyrmions numerically, both in the effective σ model by integrating the equations of motion,[6] and more microscopically using a Hartree-Fock scheme.[7] These establish the result quoted above and determine $\tilde{g}_c$. Finally, we discuss the prospects for observing this effect in experiments.

EFFECTIVE σ MODEL

The long-wavelength physics of the spin degrees of freedom in ferromagnetic QH states is described by the effective Lagrangian density[1]

$$\mathcal{L}_{\text{eff}}=\tfrac{1}{2}\overline{\rho}\mathcal{A}(\mathbf{n})\cdot\partial_t\mathbf{n}-\tfrac{1}{2}\rho^s(\nabla\mathbf{n})^2+\tfrac{1}{2}g\overline{\rho}\mu_B\mathbf{n}\cdot\mathbf{B} -\frac{e^2}{2\epsilon}\int d^2r'\,\frac{q(\mathbf{r})q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1}$$

Here $\mathcal{A}$ is the vector potential of a unit magnetic monopole, ρ^s is the spin stiffness [$\rho^s=e^2/(16\sqrt{2\pi}\epsilon\ell)$ for $\nu=1$], ϵ is the dielectric constant and ℓ the magnetic length.

The first two terms in $\mathcal{L}_{\text{eff}}$ are the leading terms in the effective action for any ferromagnet and on account of the scale invariance of the gradient term lead, *prima facie*, to charge Z Skyrmions of arbitrary size λ and energy $E_Z=4\pi|Z|\rho^s$.[8]

The Zeeman and Coulomb terms in $\mathcal{L}_{\text{eff}}$ break the scale invariance and their competition sets the size and energy of the Skyrmions and modifies the detailed form of their profiles; these now depend on the dimensionless ratio $\tilde{g}=(g\mu_B B)/(e^2/\epsilon\ell^2)$ of the Zeeman energy to the Coulomb energy. An ansatz for a Skyrmion with topological charge Z, which leads to a rotationally invariant topological density, is

$$n_x=\sqrt{1-f^2(r)}\cos(Z\theta),\quad n_y=\sqrt{1-f^2(r)}\sin(Z\theta), \quad n_z=f(r), \tag{2}$$

where f obeys the boundary conditions $f(0)=-1, f(\infty)=1$. The topological density is $q=(Z/4\pi r)df/dr$. The Skyrmions in the scale-invariant σ model are given by $f(r)=[(r/\lambda)^{2Z}-4]/[(r/\lambda)^{2Z}+4]$. For $Z>1$, their topological density has a hollow core.

Substituting Eq. (2) in the action (1) and minimizing leads to a nonlinear, nonlocal integro-differential equation for f. Below, we discuss results obtained by numerically integrating this equation using a relaxational technique.[6] But first it is instructive to consider approximate solutions at small $\tilde{g}$; the details are discussed elsewhere.[6] For $Z>1$ we take the solution to be of the form of the scale-invariant σ model solution with an optimized scale parameter λ. This yields

$$\lambda = 0.780\ell\tilde{g}^{-1/3} \quad \text{and}$$

$$E = \frac{e^2}{\epsilon\ell}\left[2\sqrt{\frac{\pi}{32}} + 2.86\tilde{g}^{1/3}\right] \quad (Z=2),$$

$$\lambda = 1.29\ell\tilde{g}^{-1/3} \quad \text{and}$$

$$E = \frac{e^2}{\epsilon\ell}\left[3\sqrt{\frac{\pi}{32}} + 4.82\tilde{g}^{1/3}\right] \quad (Z=3),$$

$$\lambda = 1.75\ell\tilde{g}^{-1/3} \quad \text{and}$$

$$E = \frac{e^2}{\epsilon\ell}\left[4\sqrt{\frac{\pi}{32}} + 7.21\tilde{g}^{1/3}\right] \quad (Z=4), \tag{3}$$

for the lowest three values of Z. Note that the energy per unit charge is the same for the $\tilde{g}$ independent piece as is appropriate for solutions of the scale-invariant problem[8] while for the $\tilde{g}$ dependent piece it increases monotonically with Z. Hence, there is no binding between this set of Skyrmions.

However, this procedure runs into trouble with the $Z=1$ Skyrmion, where the scale-invariant solution yields a Zeeman energy that diverges logarithmically with system size for any λ. This can be fixed by matching the scale-invariant solution onto the exact asymptotic solution in the outer region, which decays exponentially with a length $\propto \ell\tilde{g}^{-1/2}$. As a consequence we get logarithmically modified expressions,[1,9]

$$\lambda = 0.558\ell(\tilde{g}|\ln\tilde{g}|)^{-1/3} \quad \text{and}$$

$$E = \frac{e^2}{\epsilon\ell}\left[\sqrt{\frac{\pi}{32}} + 0.622(\tilde{g}|\ln\tilde{g}|)^{1/3}\right] \quad (Z=1), \tag{4}$$

for the core scale parameter and energy. It is evident that the presence of the logarithm by itself implies that charge one Skyrmions will pair bind for small enough $\tilde{g}$ and that the binding energy will vanish as $\tilde{g}$ vanishes. Using the above expressions one finds a critical value of $\tilde{g}_c \approx 5.3\times10^{-6}$ for this to happen. That such a critical value has to exist can be inferred by a separate microscopic computation of the pair-binding energy at large $\tilde{g}$, where the Skyrmion size is $O(\ell)$ and they reduce to polarized quasiparticles, which shows the abscence of binding. However, the quantitative issue is rather delicate. Even at small $\tilde{g}$ the result (4) is expected to have logarithmically subdominant corrections and is known numerically to be not terribly accurate in estimating the *difference* $E(\tilde{g})-E(0)$, even for $\tilde{g}$ as small as 10^{-6}.[6] Also, the asymptotic validity of our small $\tilde{g}$ expressions is not on completely rigorous footing at present. So it follows that while the pair binding should be present only at small $\tilde{g}$, a reliable estimate of the critical $\tilde{g}_c$ requires the more accurate computations of the Skyrmion energies reported below.

HARTREE FOCK

We complement the effective action calculations by the Hartree-Fock method introduced by Fertig *et al.*[7] For Skyrmions with topological charge Z the Hartree-Fock wave function is fixed by requiring it to be in the lowest Landau level and by the symmetries of the classical solution (2) to be

$$|\Psi\rangle = \prod_{m=0}^{\infty}(u_m c^\dagger_{m\uparrow} + v_m c^\dagger_{m+Z\downarrow})\prod_{p=0}^{Z-1} c^\dagger_{p\downarrow}|0\rangle \quad \text{if} \quad Z>0,$$

$$|\Psi\rangle = \prod_{m=-Z}^{\infty}(u_m c^\dagger_{m\uparrow} + v_m c^\dagger_{m+Z\downarrow})|0\rangle \quad \text{if} \quad Z<0, \tag{5}$$

where $|u_m|^2+|v_m|^2=1$, $c_{m\sigma}|0\rangle=0$ and $c^\dagger_{m\sigma}$ creates electrons with spin σ in the angular momentum m state in the lowest Landau level. We choose u_m, v_m real and require $u_m\to1$ as $m\to\infty$. The coefficients u_m, v_m are determined by numerically iterating the Hartree-Fock equations until a self-consistent solution is found. Obviously, $|\Psi\rangle$ has electric charge $-Z|e|$ relative to the polarized groundstate $|\Psi_0\rangle=\Pi_{m=0}^{\infty}c^\dagger_{m\uparrow}|0\rangle$. Skyrmions with charge $Z<0$ are often referred to as anti-Skyrmions. In the actual numerical calculation we have a finite number of particles, $m\leq N$, and impose $u_N=1$ (or $u_N=u_{N-1}=1$; see below).

RESULTS

Our results are shown in Figs. 1–3. In Fig. 1 we give the energies to create one $Z=2$ Skyrmion and to create *two* $Z=1$ Skyrmions, both at fixed particle number and magnetic field, as functions of $\tilde{g}$. These energies are equal for the Skyrmion and the anti-Skyrmion (with the same $|Z|$). This means that we have shifted the Hartree-Fock result for the state (5) by a constant. For large $\tilde{g}$, the $Z=2$ Skyrmion has higher energy than two $Z=1$ Skyrmions. In this region the Skyrmions are small, much smaller than the system size, and the Hartree-Fock calculation is reliable.

The σ model is to be trusted only for small $\tilde{g}$ when the Skyrmions are large. It nevertheless gives energies that are close to, but slightly larger than, the Hartree-Fock energies for the range of $\tilde{g}$ shown. As $\tilde{g}$ decreases, the difference in energy between one $Z=2$ and two $Z=1$ Skyrmions decreases. We see that, except possibly for very small $\tilde{g}$, the $Z=2$ Skyrmion has larger energy. As noted earlier, in the limit $\tilde{g}\to0$ the Skyrmions become infinitely large and only the σ model term in Eq. (1) contributes to the energy. Thus both energies must eventually approach the pure σ model result $E(0)=8\pi\rho^s=\sqrt{\pi/8}e^2/\epsilon\ell$ as $\tilde{g}\to0$. To investigate the region of small $\tilde{g}$, we proceed in two different ways.

The numerical σ model method can be used for small $\tilde{g}$. The inset in Fig. 1 shows the difference in energy as a

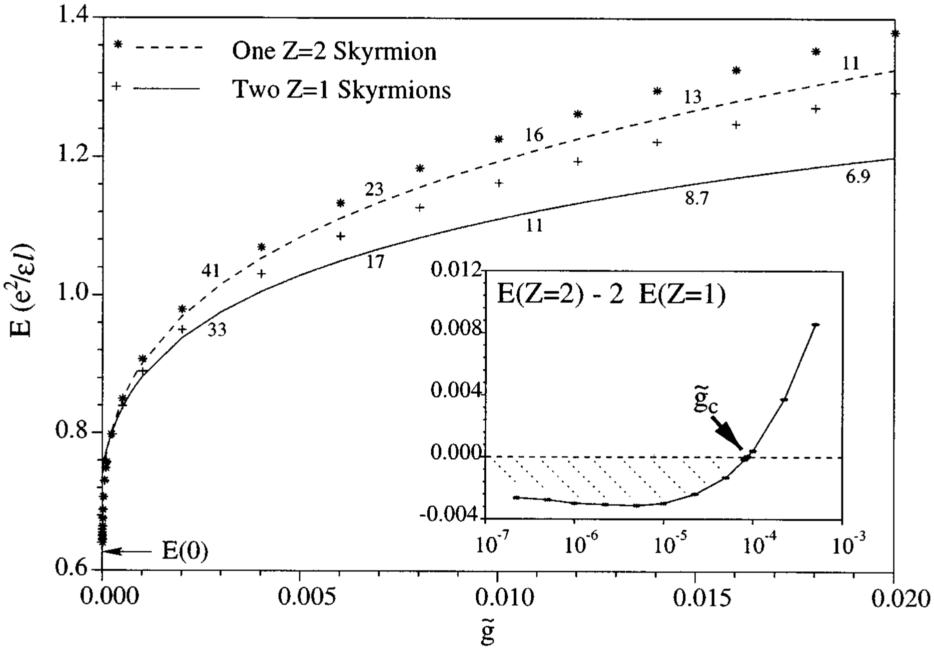

FIG. 1. Energies for one $Z=2$ Skyrmion and for *two* $Z=1$ Skyrmions. The lines are Hartree-Fock results for $N=1000$ and the points are σ model results. The spin of one $Z=2$ Skyrmion and two $Z=1$ Skyrmions is indicated along the lines. The inset shows that the energies cross at $\tilde{g}_c=8.4\times10^{-5}$ (σ model results only). $E(0)$ is the $\tilde{g}=0$ energy.

function of $\tilde{g}$ obtained using the σ model. This predicts a crossing at $\tilde{g}_c=(8.4\pm0.1)\times10^{-5}$. At this point the energy is $E_{\tilde{g}_c}=0.7506e^2/\epsilon\ell$ and there is a 10% change in the spin per charge (S/Z) of the lowest-energy excitation: $(S/Z)_{Z=1}=225$ and $(S/Z)_{Z=2}=245$. The size of the excitation also changes: The charge radii (per unit charge) are $r_{Z=1}=21\ell$ and $r_{Z=2}=31\ell$. We can also see that when $\tilde{g}$ decreases below $\tilde{g}_c$, the difference in energy first increases but then eventually starts to decrease as it must, since it should vanish at $\tilde{g}=0$.

Using the Hartree-Fock method for smaller $\tilde{g}$ than shown in Fig. 1, we find apparent crossing points, $\tilde{g}_c(N)$. However, the $\tilde{g}_c(N)$ are so small that the Skyrmions are large and there are large finite-size effects: $\tilde{g}_c$ depends on N and on whether one imposes as boundary condition that one or several of the spins are up at the edge of the system. It is difficult to increase N very much beyond $N=1000$ since the convergence is very slow for the small values of $\tilde{g}$ we are interested in. We conclude that within the Hartree-Fock method we are not able to *directly* probe a region of $\tilde{g}$ where an N-independent crossing might take place. However, we can reach smaller $\tilde{g}$ by finite-size scaling. In Fig. 2 we plot the crossing point $\tilde{g}_c(N)$ as a function of $1/N$ for three different sets of boundary conditions (as indicated in the

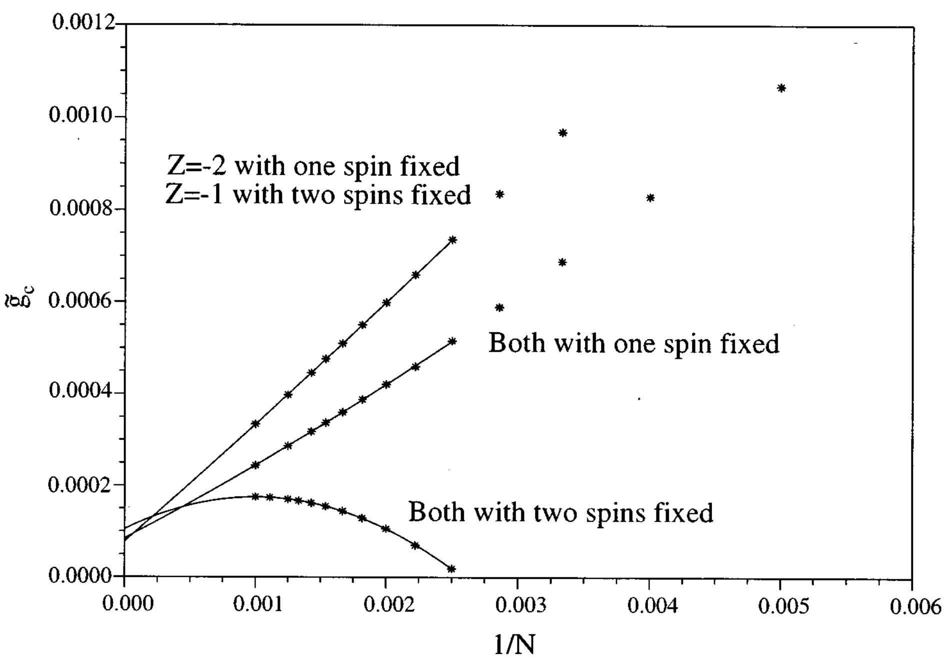

FIG. 2. Finite-size scaling of $\tilde{g}_c(N)$ using Hartree-Fock results for anti-Skyrmions with various boundary conditions (one or two spins up at the edge, as indicated in the figure). $\tilde{g}_c(N)$ is where the energy of one $Z=-2$ Skyrmion is equal to the energy of two $Z=-1$ Skyrmions in a Hartree-Fock calculation with N particles. Note that $\tilde{g}_c(N)$ depends strongly on N and on the boundary condition. Fitting to quadratic polynomials in $1/N$ gives $\tilde{g}_c\equiv\tilde{g}_c(\infty)=8.9\times10^{-5}$.

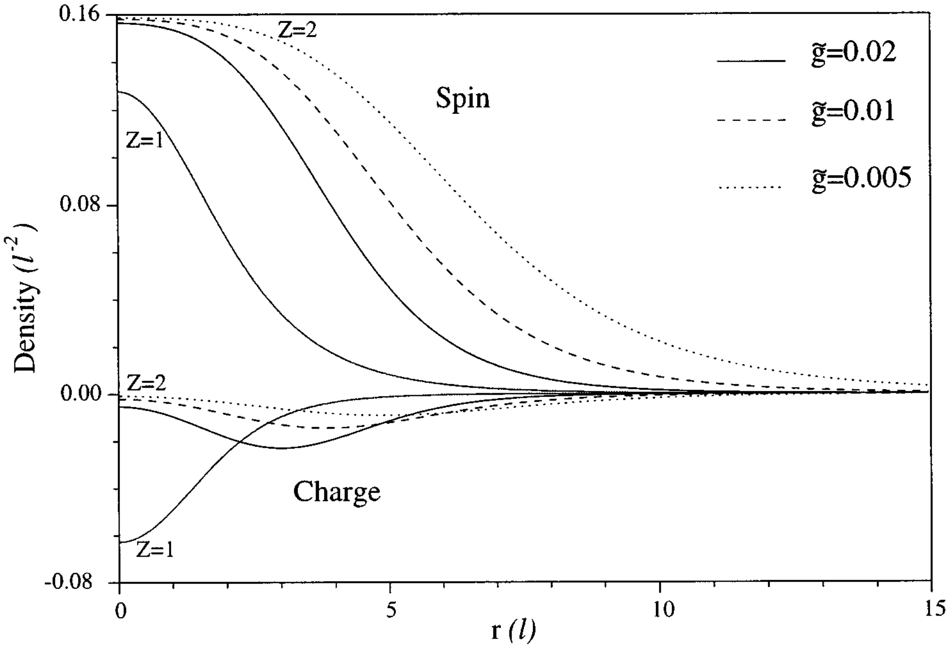

FIG. 3. Charge and spin densities at $\tilde{g}=0.02,0.01,0.005$ for one $Z=2$ Skyrmion (using the Hartree Fock method). Densities for one $Z=1$ Skyrmion at $\tilde{g}=0.02$ are included for comparison.

figure).[10] To each set we fit a quadratic polynomial in $1/N$ and read off $\tilde{g}_c \equiv \tilde{g}_c(\infty)=(8.9\pm 1.4)\times 10^{-5}$. The error is the standard deviation in the three extrapolated values. The errors in the fit for each curve are much smaller, reflecting the fact that $1/N^3$ terms are negligible compared to the quadratic terms in all our cases. Also, adding or subtracting a point at small N does not affect the results. That the three different sets, which give very different $\tilde{g}_c(N)$, predict virtually the same nonzero $\tilde{g}_c$ indicates strongly that there is a crossing. This value of $\tilde{g}_c$ agrees with the one obtained in the effective theory. Thus we conclude that the doubly charged Skyrmions have lower energy for $\tilde{g}<\tilde{g}_c=8.4\times 10^{-5}$.

It is straightforward to calculate not only the energy but also the total spin as well as spin density and charge density. The total spin, which is indicated along the line in Fig. 1, is much larger for the doubly charged Skyrmion than for the charge one Skyrmion. In Fig. 3 we give examples of densities for the doubly charged Skyrmion and compare them to *one* charge one Skyrmion. Note that the charge density has its maximum a finite distance away from the origin for the doubly charged Skyrmion (as the scale-invariant Skyrmion has).

The numerical σ model gives $\lim_{\tilde{g}\to 0}[E-\sqrt{\pi/8}e^2/\epsilon l]/\tilde{g}^{1/3}=2.86e^2/\epsilon l$ for $Z=2$ in agreement with Eq. (3). Thus there is no logarithm in the $Z=2$ energy and hence no reason to expect that $Z>2$ Skyrmions would ever have the lowest energy per charge. We have also checked that $Z=3$ Skyrmions have higher energy (per charge) than $Z=1,2$ Skyrmions for $\tilde{g}>10^{-7}$.

DISCUSSION AND EXPERIMENTAL IMPLICATIONS

Our analysis has established that two Skyrmions are unstable to pair-binding at very small Zeeman energies. At distances much larger than their size, the inter-Skyrmion potential must be repulsive as it will be dominated by their Coulomb interaction. It follows that the attraction that we have found sets in only when the Skyrmions overlap and their identities begin to merge. In computing the Skyrmion energies, we have ignored the effects of Landau-level mixing and of the softening of the Coulomb interaction at small distances by the finite extent of the subband wave functions transverse to the plane of the electron gas. It is likely that neither of these makes a huge difference to the energetics. In the σ model description, the dominant effect of Landau-level mixing is a renormalization of the spin stiffness, which does not affect the inter-Skyrmion energetics. The softening of the Coulomb interactions is significant at distances that are $O(\ell)$ and for much bigger Skyrmions (such as in the region of interest where they exceed 20ℓ) most of the self-interaction is unsoftened. (We also note that the long-wavelength arguments regarding the peculiarities of the $Z=1$ solution that underlie the pairing depend only on the form of the σ model action; they will apply *mutatis mutandis* to other ferromagnetic fillings as well.)

This is encouraging in that it suggests that it might be possible to see the binding of the Skyrmions as a sharp decrease in the spin polarization with decreasing $\tilde{g}$ in the vicinity of $\nu=1$, where the ground state will exhibit a dilute density of Skyrmions. Unfortunately, this is unlikely on two grounds. First, it would require an extremely fine tuning of $\tilde{g}$, which seems hard to accomplish. Second, there is the more fundamental problem that when the Skyrmions get to be very big, they become sensitive to the details of the disorder potential, which will tend to limit their size. Effectively, we expect the disorder to provide a lower limit to $\tilde{g}$ and, while we have not computed this for realistic disorder realizations, we are not greatly encouraged by the data in Ref. 5 which found a maximum Skyrmion size, even for $\tilde{g}$ reduced under pressure, considerably smaller than the critical one calculated in this paper.

Note added. While finishing this work we became aware of a recent paper by Nazarov and Khaetskii[11] that studies higher topological charge Skyrmions in the effective σ model in the spirit of our variational calculations. These authors also argue that the interaction between two charge one Skyrmions is attractive at intermediate and short distances and that the binding distance is zero, at which they form a charge two Skyrmion. However, their quoted results are independent of $\tilde{g}$, in contradiction with ours which hold only at small $\tilde{g}$, and they appear not to have noticed the existence or the extreme smallness of $\tilde{g}_c$.

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[8] For a review, see R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[9] Regrettably, the formula for the energy in Ref. 1 was in error by a factor of 1/2, which we have corrected here.

[10] The data in Fig. 2 are for anti-Skyrmions ($Z<0$). The reason for this is technical: While the Skyrmions and anti-Skyrmions have identical energies, it so happens that for very small $\tilde{g}$ the numerical errors in our calculations are much larger for the Skyrmions.

[11] Yu. V. Nazarov and A. V. Khaetskii (unpublished).