Soliton relaxation in magnets

V. G. Baryakhtar and B. A. Ivanov
Institute for Magnetizm AS, 252680, Kiev, Ukraine

A. L. Sukstanskii* and E. Yu. Melikhov
Physico-Technical Institute AS, 340114, Donetsk, Ukraine
(Received 7 October 1996; revised manuscript received 12 March 1997)

An equation set describing the evolution of the integrals of motion of solitons, induced by relaxation processes of both a relativistic and exchange nature is obtained within the framework of the phenomenological theory. Two-parameter one-dimensional ferromagnetic and antiferromagnetic solitons and two- and three-dimensional ferromagnetic precession solitons are analyzed. The corresponding integral curves are plotted, and the time dependences of the soliton parameters at various relaxation stages are discussed. [S0163-1829(97)06225-5]

I. INTRODUCTION

Relaxation processes are known to play a very important role in physics of magnetically ordered crystals. These processes are usually studied by means of two main approaches: microscopic and phenomenological. An advantage of the microscopic approach is that it enables one to find different dependences of relaxation characteristics on temperature and on parameters of the magnet under consideration. However, when applied to investigation of nonlinear waves, the microscopic approach appears to be rather intricate and, in fact, it can be used to analyze only the simplest kink-type solitons or domain walls (DW). A description of more complicated solitons (e.g., two-parameter solitons) and a generalization on non-one-dimensional excitations, etc., in the framework of the microscopic approach is a nontrivial problem because it requires one to know an exact spectrum and wave functions of magnons on the soliton background, while the latter are known for a moderate number of one-dimensional systems.

The phenomenological approach, suggested in the classical work by Landau and Lifshitz[1] well before the microscopic approach began to be developed, does not yield so comprehensive characteristics of relaxation processes. Nevertheless, it enables one to describe a whole picture of relaxation of a nonlinear excitation. In the framework of the phenomenological approach, an energy dissipation is taken into account by introducing relaxation terms into dynamic equations of motion (Landau-Lifshitz equations). In particular, the equation of motion for a magnetization vector **M** in one-sublattice ferromagnets (FM) has been proposed to be as follows:

$$\dot{\mathbf{M}}=-g[\mathbf{M},\mathbf{H}]+\mathbf{R}, \tag{1.1}$$

where **H** is the effective field, $\mathbf{H}=-\delta W/\delta\mathbf{M}$, W is the energy of the FM, g is the gyromagnetic ratio, the point means the time derivative.

The first term in the right-hand side of Eq. (1.1) describes the dynamics of the vector **M** whereas the second one describes a magnetization distribution approaching its equilibrium state. In Ref. 1 the dissipative term **R** has been chosen in the form

$$\mathbf{R}=\lambda gM\mathbf{H}_t, \quad \mathbf{H}_t=[\mathbf{M}[\mathbf{H},\mathbf{M}]]/M^2, \tag{1.2}$$

where $M=|\mathbf{M}|$, $\mathbf{H}_t$ has the meaning of the effective field component perpendicular to the magnetization vector **M**, λ is the sole dimensionless relaxation constant appearing in the theory. Taking into consideration that the vector $\mathbf{H}_t$ can be expressed by

$$\mathbf{H}_t=[\mathbf{M},\dot{\mathbf{M}}]/(gM^2), \tag{1.3}$$

the dissipative term can be rewritten in the Gilbert form[2]

$$\mathbf{R}=\frac{\lambda'}{M}[\mathbf{M},\dot{\mathbf{M}}], \quad \lambda'=\frac{\lambda}{1+\lambda^2}. \tag{1.4}$$

Both forms of **R** Eqs. (1.2) and (1.4) are equivalent to one another.

The equation of motion (1.1) with Landau-Lifshitz Eq. (1.2) or the Gilbert Eq. (1.4) dissipative term, as it is easy to see, conserves the length of the vector **M**, $|\mathbf{M}|=\text{const}$. So, as was outlined in Ref. 1, Eq. (1.1) is, in fact, the equation for the unit vector $\mathbf{m}=\mathbf{M}/M$. As a matter of fact, it is precisely the vector **m** that describes the magnetization distribution in the magnet. Besides, it was noted in Ref. 1 that such dissipative terms correspond to relaxation processes connected with relativistic interactions only. Really, calculating the energy dissipation rate $\dot{W}=-2Q$, Q is the so-called dissipative function, using Eqs. (1.2) or (1.4), one obtains

$$Q=-\frac{1}{2}\int d\mathbf{r}\mathbf{H}\dot{\mathbf{M}}=-\frac{1}{2}\int d\mathbf{r}\mathbf{H}\mathbf{R}=-\frac{\lambda}{2gM_0}\int d\mathbf{r}\dot{\mathbf{M}}^2.$$

Consequently, the energy dissipation takes place for a homogeneous precession of magnetization. Since only relativistic interactions result in relaxation of the homogeneous magnetization motion, the dissipative term **R** in the form of Eq. (1.2) or (1.4) has the relativistic origin.

Starting from Eq. (1.1) and **R** in the form of Eq. (1.2) or (1.4), it is rather easy to obtain such important relaxation characteristics of a magnet as a width of a ferromagnetic resonance (FMR) line, $\Delta\omega=\lambda\omega_0$, ($\omega_0$ is the FMR frequency), a dynamic slow-down coefficient of DW η,[1] a spin wave decrement,[3] and so on. However, detailed comparison of these results with experimental data and with microscopic calculations revealed some significant contradictions. Among them it should be primarily noted an erroneous dependence of the spin-wave decrement on the wave vector $k(\gamma(k)\sim k^2)$ whereas the microscopic calculation, made in Refs. 4 and 5 (see also Refs. 1 and 6), gives for short-wave magnons $(kx_0\gg 1)\,\gamma(k)\sim\omega^2(k)\sim k^4$. In FM's of the easy-plane type an absurd result is obtained: in the long-wave limit $(k\rightarrow 0)$, when $\omega(k)\sim|\mathbf{k}|\rightarrow 0$, a calculation by means of Eq. (1.2) or (1.4) gives $\gamma(k)\rightarrow\text{const}\neq 0$, i.e., $[\gamma(k)/\omega(k)]\rightarrow\infty$ at $k\rightarrow 0$ [the microscopic analysis leads to the hydrodynamic result $\gamma(k)\sim\omega^2(k)\sim k^2$ (Refs. 4 and 5)].

It should be noted that values of the relaxation constant λ obtained from the experimental data on FMR linewidth and on DW mobility in high-quality ferrite films can diverge considerably.[7] Furthermore, the microscopic calculation of the coefficient η (Refs. 8 and 9) showed that a DW slow-down is affected by processes of not only relativistic but exchange origin as well. Thus, the contradictions mentioned testifies that the phenomenological description of some relaxation processes in magnets by means of the dissipative term (1.2) or (1.4) is inadequate.

Significant progress in the development of the phenomenological approach has been achieved in Refs. 10–13. In these works a form of the dissipative term has been proposed, which takes into account relaxation processes of both relativistic and exchange origins. Besides, it was shown that a symmetry of a crystal and a hierarchy of different interactions affect the structure of dissipative terms and the hierarchy of relaxation constants.

To obtain the dissipative terms, in Refs. 10 and 11 Onsager equations have been used with components of the vector $\mathbf{M}$ as generalized coordinates, components of the effective field $\mathbf{H}$ being generalized forces. According to Ref. 10, the equation for the magnetization vector $\mathbf{M}$ can be written in the form

$$\dot{M}_i=\lambda_{ik}(\mathbf{M})H_k+\lambda_{ik,lm}(\mathbf{M})\,\frac{\partial^2 H_k}{\partial x_1\partial x_m}. \qquad (1.5)$$

The antisymmetric about the indices i and k parts of the tensors λ_{ik}, $\lambda_{ik,lm}$ define the dynamics of the vector $\mathbf{M}$ whereas the symmetric ones describe an energy dissipation.

A spatial dispersion of the system, described by the tensor $\lambda_{ik,lm}$ under small gradients of $\mathbf{M}$, should naturally be taken into consideration only in the exchange approximation. A higher symmetry of this approximation results in the conservation law for the total magnetic moment of the system:

$$\mathsf{M}(t)=\int d\mathbf{r}\mathbf{M}(\mathbf{r},t). \qquad (1.6)$$

In this case the corresponding term in Eq. (1.5) must have the form of a divergence. Besides, in the exchange approximation, the spin indices i, k in the tensor $\lambda_{ik,lm}$ must not "mix" with the coordinate indices l, m (the tensor $\lambda_{ik,lm}$ is symmetric about the latter). Taking also into account isotropy of the media about the spatial indices, the exchange relaxation term can be written in the form[10] $\lambda_{ik,lm}=-\lambda_e\delta_{ik}\delta_{lm}$ (the sign "minus" is chosen for convenience in such a way that $\lambda_e>0$).

The tensor λ_{ik} (more precisely, its symmetric part $\lambda_{ik}^{(s)}$) describes a contribution of different interactions of relativistic origin into dissipative processes. The form of $\lambda_{ik}^{(s)}$ is defined by the symmetry of the magnet, e.g., in a rhombic magnet the tensor $\lambda_{ik}^{(s)}$ is diagonal (in the main axes): $\lambda_{ik}^{(s)}=\text{diag}(\lambda_1,\lambda_2,\lambda_3)$.

Such a anisotropic structure of the relaxation term leads to that, not only a FMR frequency, but a FMR linewidth as well, turns out to be dependent of an orientation of the magnetization vector in the ground state of a FM (Ref. 14) (different orientations of the vector $\mathbf{M}$ can be obtained by applying an external magnetic field aligned with different principal axes of the magnet). This effect enables one to determine the values of relaxation constants experimentally by means of measuring FMR linewidth at different orientations of an external field (see details in Ref. 14).

A dynamic symmetry is of considerable importance in determining the structure of the tensor λ_{ik}. In particular, in the exchange approximation, the existence of the integral of motion M leads to the condition $\lambda_{ik}=0$. In the model of an uniaxial magnet (symmetry C_∞), one of the components of the vector M, namely, M_z (Z is the anisotropy axis) is an integral of motion. It follows that in an uniaxial magnet $\lambda_{ik}=\text{diag}(\lambda_1,\lambda_1,0)$ (the equality of the constants λ_x and λ_y follows from the equivalence of the axes X and Y).

The inclusion of an anisotropy in the basal plane changes both dynamic and dissipative terms in equations of motion. If the energy of the uniaxial anisotropy is much larger than that of interactions breaking the mentioned invariance, then similar hierarchy takes place for corresponding relaxation constants, the tensor λ_{ik} having the form [A theory with two different relaxation constants was proposed by Bloch[15] (see also Ref. 16) for description of magnetic relaxation in magnetically disordered systems. Relaxation times, $\tau_\parallel$ and $\tau_\perp$, introduced in Ref. 15, correspond to relaxation of longitudinal and transverse (with respect to an external magnetic field) components of magnetization vector $\mathbf{M}$. However, Bloch's theory does not take into consideration a symmetry of the crystal and a hierarchy of interactions. Only in some special cases it turns out to be possible to compare our relaxation constants and Bloch relaxation times. A detailed comparison of the present approach and Bloch's theory is carried out in Ref. 17.]

$$\lambda_{ik}=\text{diag}(\lambda_1,\lambda_1,\lambda_3),\quad \lambda_3\ll\lambda_1. \qquad (1.7)$$

From the aforesaid, the dissipative function of the FM can be represented as follows:[10,12]

$$Q=\frac{1}{2}\int d\mathbf{r}gM\{\lambda_{ik}H_iH_k+\lambda_e a^2(\nabla\mathbf{H})^2\}, \qquad (1.8)$$

where the factors gM and a^2 (a is the lattice constant) are introduced for convenience in order that the relaxation constants λ_{ik},λ_e be dimensionless. The equation of motion for the vector $\mathbf{M}$ with a relaxation term corresponding to such a dissipative function takes the form

$$\dot{\mathbf{M}} = -g[\mathbf{M},\mathbf{H}] + gM\{\lambda_{ik}H_i\mathbf{e}_k - \lambda_e a^2 \Delta\mathbf{H}\}. \qquad (1.9)$$

It is easy to verify that in the exchange approximation ($\lambda_{ik} = 0$) and in case of an uniaxial FM, the dissipative term (1.9) leads to the correct dependence of the spin-wave decrement on the wave vector in the long-wave approximation: this dependence is identical to that obtained by means of the microscopic calculations. Besides, two (or more) different relaxation constants presented in the theory make it possible to fit experimental data on FMR linewidth and on DW slowdown.[10]

Similar relaxation terms and a dissipative function, which takes into account a symmetry of the magnet and the exchange relaxation, for two-sublattice antiferromagnets has been obtained in Refs. 11 and 13. Such magnets are known to be convenient to describe in terms of vectors **M** and **L**, $\mathbf{M} = (\mathbf{M}_1 + \mathbf{M}_2)/2$, $\mathbf{L} = (\mathbf{M}_1 - \mathbf{M}_2)/2$, $\mathbf{M}_{1,2}$ are the magnetization vectors of the sublattices. Starting from the Onsager equations with components of the vectors **M** and **L** as generalized coordinates and the effective fields $\mathbf{H} = -\delta W/\delta\mathbf{M}$, $\mathbf{F} = -\delta W/\delta\mathbf{L}$ as generalized forces, the following dissipative function has been found:[11]

$$Q = \frac{1}{2}\int d\mathbf{r} g|\mathbf{L}|\{\lambda_{ik}H_iH_k + \lambda_e a^2(\nabla\mathbf{H})^2 + \lambda_0\mathbf{F}^2\}, \qquad (1.10)$$

where the tensor λ_{ik} is of a relativistic origin, whereas the relaxation constants λ_e and λ_0 are of an exchange origin. Relaxation terms in the equations of motion for the vectors **M** and **L** are defined by the relationships: $\mathbf{R}_m = \delta Q/\delta\mathbf{H}$, $\mathbf{R}_1 = \delta Q/\delta\mathbf{F}$ (see below).

It is important to note once more that the equation of motion (1.9) does not conserve the length of the magnetization vector, $|\mathbf{M}| \neq \text{const}$. Besides, the dissipative function (1.8) includes all the components of the effective field **H**, and not just its component $\mathbf{H}_t$ [see Eq. (1.2)]. These two circumstances complicate the analysis of the relaxation in the system substantially.

In the present work the main attention will be paid to the investigation of soliton relaxation in uniaxial magnets. Smallness of the relaxation constants involved enable one to develop the perturbation theory to describe the evolution of soliton's parameters. For solitons in completely integrable systems, there exist a specific form of the perturbation theory, based on the inverse scattering problem (e.g., see Refs. 18 and 19). We shall use a more simple version of the perturbation theory, based on the construction of evolution equations for integrals of motion of an undisturbed system. These equations describe a slow evolution of parameters of an initial excitation due to dissipation. The simplest variant of such an approach has been used in Ref. 20 when studying fluxons damping in Josephson's contacts in the framework of a sine-Gordon equation. An advantage of this approach is that it can be used even in the case when an initial (unperturbed) equation is not completely integrable, e.g., when analyzing two- or three-dimensional solitons.

II. RELAXATION IN FERROMAGNETS

To describe phenomenologically dissipation processes of nonlinear excitations in ferromagnets, we start from the equation of motion for the magnetization vector **M** Eq. (1.9). The relaxation term in a uniaxial FM has the form[12]

$$\mathbf{R} = gM\{\lambda_1\mathbf{H}_\perp - \lambda_e a^2\Delta\mathbf{H}\}, \qquad (2.1)$$

where $\mathbf{H}_\perp = \mathbf{H} - \mathbf{n}(\mathbf{n},\mathbf{H})$ is the component of the effective field perpendicular to the anisotropy axis (the axis Z, **n** is the unit vector along this axis); λ_e and λ_1 are exchange and relativistic relaxation constants, respectively ($\lambda_e, \lambda_1 \ll 1$). As was noted in the Introduction, if one takes into account small interactions breaking the uniaxiality of the magnet, then the Z component of the dissipative vector **R** becomes nonzero,

$$\mathbf{R} = gM\{\lambda_1\mathbf{H}_\perp + \lambda_3 H_z\mathbf{n} - \lambda_e a^2\Delta\mathbf{H}\}. \qquad (2.2)$$

As it will be shown below, the constant λ_3, even though it is small in comparison with $\lambda_1 (\lambda_3 \ll \lambda_1)$ is required when analyzing nonlinear wave relaxation.

Thus, the equation of motion (1.1) in uniaxial FM with a small deviation from uniaxiality has the form

$$\dot{\mathbf{M}} = -gM[\mathbf{M},\mathbf{H}] + gM\{\lambda_1\mathbf{H}_\perp + \lambda_3 H_z\mathbf{n} - \lambda_e a^2\Delta\mathbf{H}\} \qquad (2.3)$$

and the dissipative function Q is equal to

$$Q = \frac{1}{2}\int d\mathbf{r} gM\{\lambda_1\mathbf{H}_\perp^2 + \lambda_3 H_z^2 + \lambda_e a^2(\nabla\mathbf{H})^2\}. \qquad (2.4)$$

According to Eq. (2.3), the modulus of the vector **M** changes due to the relaxation terms. Multiplying Eq. (2.3) by **M**, one obtains[21]

$$\dot{M} = gM\{\lambda_1(\mathbf{m},\mathbf{H}_\perp) + \lambda_3 m_z H_z - \lambda_e a^2(\mathbf{m},\Delta\mathbf{H})\}. \qquad (2.5)$$

Combining Eqs. (2.3) and (2.5), it is easy to write the equation for the unit vector **m**:

$$\dot{\mathbf{m}} = -g[\mathbf{m},\mathbf{H}] + g\{\lambda_1[\mathbf{H}_\perp - \mathbf{m}(\mathbf{m},\mathbf{H}_\perp)] + \lambda_3[H_z\mathbf{n} - \mathbf{m}(m_zH_z)] - \lambda_e a^2[\Delta\mathbf{H} - \mathbf{m}(\mathbf{m},\Delta\mathbf{H})]\}. \qquad (2.6)$$

It should be noted that in the case $\lambda_i = \lambda_e \to 0$ $|\mathbf{M}| = \text{const}$, and the magnetization dynamics are entirely described by Eq. (2.6) for the unit vector **m**. As shown in Ref. 22, a nonconservation of $|\mathbf{M}|$ leads to an existence of the so-called dissipative linear spin mode.

To calculate an energy dissipation rate in the linear approximation with respect to the relaxation constants (we restrict ourselves to this approximation), the effective field **H** in the dissipative function Q should be calculated in the main (zero) approximation with respect to the these constants. In the nondissipative approximation, the perpendicular to **m** component of the effective field $\mathbf{H}_\perp$ can be readily found [see Eq. (1.3)]. But, as mentioned in the Introduction, one more peculiarity of Eqs. (2.3) and (2.6) is that these equations contain not only $\mathbf{H}_\perp$ but the collinear component of the effective field $\mathbf{H}_m$ as well, $\mathbf{H}_m = \mathbf{m}H_m$, $H_m = (\mathbf{mH})$. This component cannot be found from the nondissipative equation of motion and must be obtained independently.

So, the choice of the dissipative term **R** in the form (1.9) [or in the special case Eq. (2.2)] leads to that the equation for the normalized magnetization **m** becomes insufficient to describe a magnetization distribution. Naturally, one may analyze Eq. (2.3) for the magnetization vector **M** as such, but all soliton solutions are constructed on the basis of Eq. (2.6), and when analyzing their relaxation, it is also more convenient to start from the similar equation modified by its relaxation part properly.

To construct an equation for the collinear field $\mathbf{H}_m$, we use the explicit expression for the FM energy:

$$W=\int d\mathbf{r}\{f(M^2)+w(\mathbf{M})\}. \qquad (2.7)$$

Here the term $f(M^2)$ describes the isotropic exchange interaction which defines for the most part the length of the magnetization vector $M=|\mathbf{M}|$. The term $w(\mathbf{M})$ corresponds to the inhomogeneous exchange interaction, the anisotropy energy, the interaction with an external magnetic field, etc.:

$$w(\mathbf{M})=\frac{\alpha}{2}(\nabla\mathbf{M})^2+\frac{\beta}{2}\mathbf{M}_\perp^2-\mathbf{H}_e(\mathbf{M}-\mathbf{M}_0). \qquad (2.8)$$

Here $\mathbf{M}_0$ is the magnetization in the ground state, $\mathbf{H}_e$ is the external magnetic field.

It is easy to show that $H_m=-\delta W/\delta M$. From Eq. (2.8) one obtains

$$H_m=-2M\frac{df}{dM^2}-\mathbf{m}\frac{\delta w}{\delta\mathbf{M}}. \qquad (2.9)$$

Note, that if $w(\mathbf{M})$ is a homogeneous function of the second order with respect to the components of the vector **M**, then the second term in Eq. (2.9) reduces to $2M^{-1}w(\mathbf{M})$.

Under temperatures not very close to Curie temperature T_C, the function $f(M^2)$ has a sharp maximum at $M^2=M_0^2(T)$, $M_0(T)$ is the equilibrium value of magnetization [we chose the anisotropy energy so that $w(\mathbf{M})=0$ in the ground state]. In this case only the value of M close to M_0 may be considered to be actual, i.e., $\mu=(M-M_0)\ll M_0$, and one can write

$$\frac{df}{dM^2}=\frac{\mu}{\chi_\parallel M_0}, \quad \chi_\parallel^{-1}=4M_0^2\frac{d^2f(M_0^2)}{d(M_0^2)^2},$$

where the quantity $\chi_\parallel\ll 1$ has the meaning of the longitudinal susceptibility of the FM. If so doing,

$$H_m=-\frac{\mu}{\chi_\parallel}-\mathbf{m}\frac{\delta w}{\delta\mathbf{M}}. \qquad (2.10)$$

In the static case, even at an inhomogeneous distribution of magnetization (e.g., at the presence of a static domain wall or an inhomogeneous external field) $H_m=0$, and the length of magnetization vector depends on coordinates,

$$M=M_0\left[1-\chi_\parallel\left(\mathbf{m},\frac{\delta w}{\delta\mathbf{M}}\right)\right].$$

The latter result follows from the known fact that in the successive phenomenological theory, even at $T=\text{const}$, M is a function of a local intrinsic field. If a dynamic magnetization wave is present, then the quantity H_m, in general, does not equal 0. It is noteworthy that a contribution of the first term in Eq. (2.10) in H_m, although the value of μ is small, can be rather essential due to the presence of the large coefficient $\chi_\parallel^{-1}\gg 1$.

To calculate H_m and μ, it is necessary to use Eq. (2.5). Taking into account the relationship between H_m and μ, one obtains the equation for the collinear field H_m:

$$\frac{\chi_\parallel}{gM_0}\dot{H}_m-\lambda_e a^2\Delta H_m+\overline{\lambda(\mathbf{r},t)}H_m=-\frac{\chi_\parallel}{g}\left(\mathbf{M},\frac{\delta w}{\delta\mathbf{M}}\right)+\Lambda(\mathbf{r},t),$$

$$\overline{\lambda(\mathbf{r},t)}=\lambda_1\mathbf{m}_\perp^2+\lambda_3 m_z^2+\lambda_e a^2(\nabla\mathbf{m})^2,$$

$$\Lambda(\mathbf{r},t)=\frac{\lambda_1-\lambda_3}{g}m_z[\mathbf{m},\dot{\mathbf{m}}]_z+\frac{\lambda_e a^2}{g}(\mathbf{m},\Delta[\mathbf{m},\dot{\mathbf{m}}]). \qquad (2.11)$$

Equation (2.11) is a linear homogeneous equation of the diffusion-type with a right-hand side part. In combination with Eq. (2.6) for the unit vector **m** and with relationship (1.3) for the quantity $\mathbf{H}_t$, Eq. (2.11) completes the construction of the closed system of equations describing relaxation of different dynamic excitations in FM in the linear, with respect to relaxation constants, approximation. This system written in terms of **m** and H_m is more convenient for analysis than that written in the variables **m** and μ, since $\mu\to 0$ at $\chi_\parallel\to 0$ whereas the quantity H_m remains finite.

A general solution of Eq. (2.11) without the right-hand side describes the relaxation of H_m to the equilibrium value $H_m=0$, the characteristic relaxation time τ_0 is of the order $(\lambda_1 gM_0/\chi_\parallel)^{-1}$, and at $\chi_\parallel\to 0$ it is very small.[10] The same time τ_0 defines the rate of the homogeneous relaxation of M to its equilibrium value M_0. Based on the estimation $d^2f/d(M^2)^2\sim fM^{-4}$, $f\sim IM_0/\mu_0$, where I is the exchange integral value close to the Curie temperature $(I\sim T_C)$, μ_0 is the Bohr magneton, one obtains $\tau_e\sim\mu_0/(\lambda_1 gI)$. Consequently, the contribution of relativistic interactions (i.e., the const λ_1) to the relaxation of the magnetization length is amplified by the exchange interaction.[10] Previously this fact has been deduced on the basis of the microscopic analysis in Ref. 23. Such a coincidence of the character of magnetization length relaxation in the framework of the two approaches testifies that our generalized phenomenological equation (1.7) is adequate.

The inhomogeneous solution of Eq. (2.11) can be nonzero only in the presence of a dynamic magnetization wave. Note, that we are interested in the main approximation for H_m about relaxation constants and therefore a magnetization distribution in the nonlinear wave under consideration, calculated in the nondissipative approximation, must be substituted into the functions $\overline{\lambda(\mathbf{r},t)}$ and $\Lambda(\mathbf{r},t)$ in Eq. (2.11). Here it is also should be noted that a structure of this solution depends on the ratio between $\chi_\parallel$ and λ, and on the character of a magnetic excitation.

For a description of relaxation on the basis of the set of equations obtained, we use a simple version of the perturbation theory based on constructing of evolution equations for integrals of motion of the unperturbed system. This scheme is as follows: let the magnetization distribution in the nonlinear wave be determined by the set of parameters

$\alpha_1,\alpha_2,\ldots,\alpha_n$ which are constant in the nondissipative approximation. Taking into account relaxation terms, these parameters begin to depend on time. The corresponding evolution equations for α_j $(j=1,2,\ldots,n)$ can be obtained from the integrals of motion of the unperturbed system $I_1,I_2,\ldots,I_n$ (if the system under consideration has a solution with n parameters, then there exist at least n integrals of motion).

One of these integrals is the energy of the magnetic excitation, E. Its rate of change is determined by the dissipative function Q, $dE/dt=-2Q$. Let us find dE/dt, on the one hand, as a linear combination of the change rates of the nonlinear wave parameters, $d\alpha_j/dt$; on the other hand, one can calculate the value of Q as a function these parameters. Equating the corresponding values, we obtain the balance equation for the energy, which is one of the sought-for equations describing the evolution of the parameters α_j due to relaxation processes. Similar equations can be obtained by calculating the change rates of other integrals of motion. As a result, one obtains the system of n first-order differential equations for the parameters α_j.

The simplest variant of such an approach with one integral of motion (energy) has been used in Ref. 24 to analyze the dissipation of one-parameter nonlinear waves (domain walls, a parameter is a DW velocity) accounting relaxation and an applied force.

One important note should be made here. As mentioned above, the structure of the relaxation term in Eq. (2.2) is that it does not conserve the length of the magnetization vector $\mathbf{M}$ [see Eq. (2.5)]. Since the value M is also a function of a temperature, it means that an equation for the magnetization vector $\mathbf{M}$ with dissipative terms cannot, in general, be considered independently. There is a need to write a system of equations, which includes the Landau-Lifshitz equation as well as equations of thermoconductivity and of entropy balance. However, when studying the relaxation of a certain magnetic excitation in the temperature region well away from the Curie temperature (or Néel temperature for antiferromagnets), the relaxation is implicitly supposed to lead only to some changes of parameters characterizing the corresponding solution of the dynamic equations (slow relaxation). Both an initial excitation and a final state of the magnetic system are found from solutions of the dynamic equations in which the magnetization length is fixed. In doing so, we implicitly consider that the magnetic system is in a contact with a thermostat which immediately ‘‘compensates’’ any alternation of M by supplying or removing a quantity of heat to or from the magnetic system.

To take into consideration this implicit supposition when calculating the change rate of one or another integral of motion, it is sufficient to take into account only that relaxation which is not associated with changes of $|\mathbf{M}|$. With this aim in view, one should set $d\mathbf{M}/dt=M d\mathbf{m}/dt+d\mathbf{m}/dt$, and then take only the first addend. For example, taking into account Eq. (2.6), one obtains for dE/dt the following expression [in place of Eq. (2.4)]:

$$(\dot{E})_{M=\mathrm{const}}=-\int d\mathbf{r}M\dot{\mathbf{m}}\mathbf{H}=-\frac{M_0}{g}\int d\mathbf{r}\{\lambda_1[([\mathbf{m},\dot{\mathbf{m}}]_\perp)^2+gH_m(\mathbf{m}_\perp,[\mathbf{m},\dot{\mathbf{m}}]_\perp)]+\lambda_3[([\mathbf{m},\dot{\mathbf{m}}]_z)^2+gH_m(m_z[\mathbf{m},\dot{\mathbf{m}}]_z)]$$
$$+\lambda_e a^2[(\nabla[\mathbf{m},\dot{\mathbf{m}}])^2+gH_m(\mathbf{m},\Delta[\dot{\mathbf{m}},\mathbf{m}])]\}\equiv-2\overline{Q}. \tag{2.12}$$

The quantity $\overline{Q}$ hereafter is referred as a reduced dissipative function. In fact, the transition from Q to $\overline{Q}$ is due to a small value of the longitudinal susceptibility. The fact that $\chi_\parallel\neq 0$ is taken into consideration only at the intermediate stage of the collinear field H_m calculation, and in the final expressions, we must take the limit $\chi_\parallel\to 0$. So, according to Eq. (2.10), $\mu=\chi_\parallel(H_m+\mathbf{M}\delta W/\delta\mathbf{M})\to 0$. Note, that if so doing, $f(M^2)\sim\mu^2/\chi_\parallel\sim\mu\to 0$, and the corresponding term in the energy balance can be omitted.

Reduced expressions for rates of change of other integrals of motion can be found in a similar way. In particular, for an uniaxial FM, considered in the present section, it is convenient to use the integral of motion I_z equal to the total deviation of the Z component of magnetization from its equilibrium value. This value can be expressed by a number of magnons in a nonlinear wave N. For an easy-axis FM

$$N=\frac{1}{2\mu_0}\int d\mathbf{r}(M-M_z).$$

Calculating the change rate of this integral of motion with account of $M=\mathrm{const}$, one obtains

$$(\dot{N})_{M=\mathrm{const}}\equiv\overline{(\dot{N})}=\frac{d}{dt}\left\{\frac{M_0}{2\mu_0}\int d\mathbf{r}(1-m_z)\right\}$$
$$=\frac{M_0}{2\mu_0}\int d\mathbf{r}\{\lambda_1 m_z(\mathbf{m}_\perp,\mathbf{H}_\perp)+\lambda_3 m_z^2 H_z$$
$$-\lambda_e a^2[\Delta H_z-m_z(\mathbf{m},\Delta\mathbf{H})]\}. \tag{2.13}$$

It is sufficient to analyze two integrals of motion E and N for describing the evolution of a two-parameter soliton in a uniaxial FM. Using one more integral of motion—the momentum $\mathbf{P}$—does not lead to new equations due to the relationship $dE=\hbar\omega dN+\mathbf{V}d\mathbf{P}$ which is true for any FM admitting an existence of two-parameter solitons.[25] To analyze two-parameter nonlinear waves in two-axes magnets, wherein I_z is no longer an integral of motion, equations for dE/dt and $d\mathbf{P}/dt$ can be used.

A. Relaxation of two-parameter solitons

In this section we analyze relaxation of a two-parameter soliton in an easy-axis FM with the energy (2.8), taking into account a constant external magnetic field aligned with the easy axis Z. The corresponding dynamic solution of the equations of motion in an uniaxial FM has been investigated in Ref. 25 in detail. The phenomenological theory of soliton relaxation without an external magnetic field and with relativistic relaxation terms, which do not take into consideration the symmetry of a magnet, has been developed in Ref. 21.

It is easy to see that any solution of the dynamic equations in the presence of the external magnetic field H_e can be obtained from the corresponding solution calculated at $H_e=0$ by means of the simple substitution $\Omega\to\Omega-gH_e$, where Ω is the precession frequency about the anisotropy axis. The magnetization distribution in the two-parameter soliton in which we are interested, is defined by the formulas[25]

$$\varphi=\tilde{\Omega}t+\psi(x-Vt),\quad x_0\frac{d\psi}{dx}=-\frac{V}{V_m}\frac{1}{\cos^2(\theta/2)},$$

$$\tan^2\frac{\theta}{2}=\frac{\kappa^2}{A\cosh^2[\kappa/x_0(x-Vt)]+1/2(B-A)},\tag{2.14}$$

where $\tilde{\Omega}=\Omega-gH_e$, $V_m=2\omega_0x_0$; $\kappa=[1-\tilde{\Omega}/\omega_0-(V/V_m)^2]^{1/2}$; (x_0/κ) is the effective soliton width, $x_0(\alpha/\beta)^{1/2}$, $A^2=B^2+4\kappa^2(V/V_m)^2$, $B=\tilde{\Omega}/\omega_0+2(V/V_m)^2$. The angle variables θ and φ parametrize the unit magnetization vector $\mathbf{m}$,

$$m_x+im_y=\sin\theta e^{i\varphi},\quad m_z=\cos\theta.\tag{2.15}$$

The soliton structure is governed by two parameters: the velocity V and the precession frequency Ω. The localized soliton solution (2.14) exists in the region $\kappa^2>0$, or

$$\frac{\Omega-gH}{\omega_0}+\left(\frac{V}{V_m}\right)^2<1.\tag{2.16}$$

The values of the integrals of motion E and N, corresponding to the soliton solution (2.14), are equal (per the square a^2):

$$E=2E_0\left(\kappa+\frac{h}{2}I_0\right),\tag{2.17}$$

$$N=\frac{E_0}{\hbar\omega_0}I_0,\quad I_0=\tanh^{-1}\frac{2\kappa}{2-\omega+h},\tag{2.18}$$

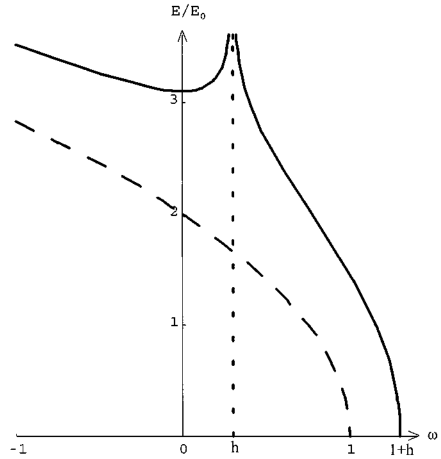

FIG. 1. The frequency dependence of the ferromagnetic soliton energy ($V=0$); solid line $-H\neq0$, dashed line $-H=0$.

where $E_0=2\beta M_0^2x_0$, $h=gH_e/\omega_0$, $\kappa=(1+h-\omega-u)^{1/2}$. Here we introduce the dimensionless variables $\omega=\Omega/\omega_0$, $u=(V/V_m)^2$ convenient for the further calculations.

Note, that the expression for the two-parameter soliton energy (2.17) cannot be obtained from the known result at $H_e=0$ (Ref. 25), by the simple substitution $\Omega\to\tilde{\Omega}$: the term in Eq. (2.17) proportional to hI_0 results in essentially different dependence of the soliton energy on its parameters than at $H_e=0$. In particular, for the soliton in rest ($u=0$), the two-parameter soliton energy $E(\omega)$ increases infinitely with $\omega\to h$, and at $\omega=0$ it has an additional minimum as compared with case $h=0$ (see Fig. 1).

The rate of the energy change, described under $\chi_\parallel\ll1$ by the reduced dissipative function Q (2.12), can be written in the form

$$\dot{E}=-\frac{1}{2}E_0q,\quad q=q_r+q_e,\tag{2.19}$$

where q_r and q_e are the contributions associated with the relativistic and exchange relaxation terms, respectively,

$$q_r=\lambda_1\langle\dot{\theta}^2+\sin^2\theta\cos^2\theta\dot{\varphi}^2-h_m\sin^2\theta\cos\theta\dot{\varphi}\rangle,\tag{2.20}$$

$$\begin{aligned}q_e=\lambda_e'\langle&\theta'^2+\dot{\theta}^2\varphi'^2+\theta'^2\dot{\varphi}^2+\dot{\varphi}'^2\sin^2\theta+\sin^2\theta\cos^2\theta\varphi'^2\dot{\varphi}^2+\sin2\theta(\dot{\theta}\varphi'\dot{\varphi}'+\theta'\dot{\varphi}\dot{\varphi}'-\dot{\theta}'\varphi'\dot{\varphi})+2\cos2\theta\theta'\dot{\theta}\varphi'\dot{\varphi}\\&-h_m[\dot{\varphi}\cos\theta(2\theta'^2+\sin^2\theta\varphi'^2)+\sin\theta(\theta''\dot{\varphi}-\dot{\theta}\varphi''+2\theta'\dot{\varphi}'-2\dot{\theta}'\varphi')]\rangle,\end{aligned}\tag{2.21}$$

where $\lambda_e'=\lambda_e(a/x_0)^2$, $h_m=H_m/(\beta M_0)$; prime and angular brackets mean differentiation and integration with respect to the dimensionless space variable $\xi=x/x_0$, respectively, and a point means differentiation with respect to dimensionless time $\tau=\omega_0t$. The terms proportional to λ_3 are omitted in Eq. (2.19) because at $\lambda_3\ll\lambda_1$ these terms are shown to have no crucial significance in the problem of soliton relaxation.

For the change rate of the second integral of motion N, one obtains from Eq. (2.13)

$$\overline{\dot{N}}=-\frac{E_0}{2\hbar\omega_0}\eta,\quad\eta=\eta_r+\eta_c,\tag{2.22}$$

$$\eta_r=\lambda_1\langle\sin^2\theta\cos\theta(\dot{\varphi}\cos\theta-h_m)\rangle, \tag{2.23}$$

$$\eta_e=\lambda_e'\langle\theta'^2\dot{\varphi}+\cos 2\theta\dot{\theta}\varphi'+\sin^2\theta\cos^2\theta\varphi'^2\dot{\varphi}+\sin\theta\cos\theta(\theta'\dot{\varphi}'-\dot{\theta}'\varphi')-h_m[2\cos\theta\theta'^2+\sin\theta\theta''+\sin^2\theta\cos\theta\varphi'^2]\rangle. \tag{2.24}$$

On the other hand, let us calculate dE/dt and dN/dt as linear combinations of the derivatives du/dt and $d\omega/dt$ by means of the explicit expressions (2.17) and (2.18). Equating the expressions found from Eqs. (2.19) and (2.22), respectively, the set of differential equations for the soliton parameters ω and u can be obtained:

$$\dot{\omega}=g_\omega(\omega,u)\equiv\frac{1}{4\kappa}(qb_2-\eta b_1),$$
$$\dot{u}=g_u(\omega,u)\equiv\frac{1}{4\kappa}(\eta a_1-qa_2), \tag{2.25}$$

where

$$a_1=\omega(\omega-h)+2u(h+2),\quad a_2=\omega-h+2u,$$

$$b_1=\omega(\omega-3h)+2(h^2+h-2u),\quad b_2=2+h-\omega.$$

According to Eqs. (2.19) and (2.22), the functions g_ω and g_u can be also represented as a sum of two terms proportional to the relativistic and exchange relaxation constants, respectively,

$$g_\omega=g_\omega^{(r)}+g_\omega^{(e)},\quad g_u=g_u^{(r)}+g_u^{(e)}.$$

To calculate the collinear field $h_m=H_m/(\beta M_0)$, let us come back to Eq. (2.11). Coordinate and time dependences of the coefficient $\lambda(\mathbf{r},t)$ and of the right-hand side of this equation are determined by the soliton solution (2.15). Such a cumbersome structure of this solution makes it impossible to solve Eq. (2.11) in the general case. That is why we restrict ourselves to consideration of two limiting cases: $\chi_\parallel\ll\lambda$ and $\chi_\parallel\gg\lambda$ (λ is the characteristic value of the relaxation constants).

Functional relationship between the collinear field H_m and the magnetization distribution $\mathbf{m}(x,t)$ in these two cases are shown to be quite different. We are interested, in fact, in the limiting value of H_m at $\chi_\parallel\to0$, $\lambda\to0$. Consequently, the problem of the calculation of H_m reveals a nonanalyticity: the value of H_m at small $\chi_\parallel$ and λ depends on the sequence of the limiting transitions $\chi_\parallel\to0$ and $\lambda\to0$.

Let $\chi_\parallel\ll\lambda$. In this case one can set $\chi_\parallel=0$ in Eq. (2.11) from the very beginning. Besides, let us also assume that the soliton is sufficiently narrow, i.e., its width $\Delta=x_0/\kappa\ll l_d$, $l_d=(\lambda_e a^2/\lambda_3)^{1/2}$ is a some characteristic diffusion length. In this limiting case, the magnetization distribution in the soliton can be considered as a δ function, $\sin^2\theta(\xi)=2\Delta\delta(\xi)$.

If doing so, Eq. (2.11) can be readily solved:

$$H_m(\xi)=H_m(0)\exp(-|\xi|/I_d), \tag{2.26}$$

$$H_m(0)=\frac{\Lambda(0)\beta M_0}{\overline{\lambda}(0)+2(\lambda_3\lambda_e a^2)^{1/2}\Delta^{-1}}. \tag{2.27}$$

Here $\Lambda(0)$ and $\overline{\lambda}(\mathbf{r},t)$ are defined by the value of the vector $\mathbf{m}$ in the soliton center:

$$\Lambda(0)=-4\lambda_1 s_0^2[2\omega c_0^4+(4u-\omega)c_0^2-2u]-4\lambda_e' s_0^2[6\omega c_0^4 -\omega c_0^2(3+5\omega-5h)+2(u+uh-2u\omega+\omega^2-\omega h)],$$

$$\overline{\lambda}(0)=4[\lambda_1 c_0^2+\lambda_e'(c_0^2-\omega+h)]s_0^2,$$

$$c_0^2\equiv\cos^2\theta(0)=\frac{1}{2}\{[(\omega-h)^2+4u]^{1/2}+\omega-h\},$$

$$s_0\equiv\sin\theta(0).$$

Note, that the quantity h_m in the expressions for q and η Eqs. (2.20)–(2.24) is multiplied by the function which differs essentially from 0 only in the region of the soliton localization, $\Delta\ll l_d$. Therefore, when integrating over ξ in the expressions for q and η, the exponential factor in $H_m(\xi)$ Eq. (2.26) can be set equal to 1, and the quantity λ_3 drops out of the final results [at $\lambda_3\ll\lambda_1$, the second term in the denominator of (2.27) is small as compared with the first one].

It should be noted that formulas (2.26) and (2.27) are obtained in the main approximation with respect to the relaxation constants. If one of them is small as compared with another ($\lambda_e'\ll\lambda_1$ or $\lambda_e'\gg\lambda_1$), then H_m does not depend on the relaxation constants at all.

In another limiting case, when the diffusion length is much smaller than the soliton width ($\Delta\gg l_d$), the collinear field H_m differs from 0 only in the soliton localization region

$$H_m(\xi)\approx\frac{\Lambda(\xi)}{\overline{\lambda}(\xi)}. \tag{2.28}$$

At $\lambda_3\ll\lambda_e'$, λ_1 the condition $\Delta\gg l_d$ for localized excitations is realized only in exceptional cases, and we shall not consider it further.

Let now $\chi_\parallel\gg\lambda$. It is easy to see that in this case a characteristic value of the soliton velocity V_* appears in the problem, and the result nonanalytically depends on the ratio (V_*/V). Under small velocities, $V\ll V_*$, the terms in Eq. (2.11) proportional to $\chi_\parallel$ can be neglected and the equation reduces to the case $\chi_\parallel\ll\lambda$ considered above.

In the more interesting case, $V\gg V_*$, one can omit all the terms proportional to the relaxation constants, and the collinear field H_m turns out to be the equal to

$$H_m=-\mathbf{m}\frac{\delta W}{\delta\mathbf{M}}=-\beta M_0\{\theta'^2+\sin^2\theta(1+\varphi'^2) +h(1-\cos\theta)\}. \tag{2.29}$$

It should be noted that in the static case, $\omega = u = 0$, the collinear field, defined by Eq. (2.29), is not equal to 0. It means that Eq. (2.29) under small values of the soliton velocity is not adequate: even though the characteristic velocity V_* is very small, there is a velocity interval in which the collinear field H_m is determined by Eqs. (2.26) and (2.27), and $H_m \to 0$ at $u \to 0$, $\omega \to 0$.

Thus, in the case $\chi_\parallel \ll \lambda$ and in the case $\chi_\parallel \gg \lambda$, $V < V_*$ the collinear field H_m can be considered constant in the soliton localization region, $H_m \approx H_m(0)$, where $H_m(0)$ is defined by Eq. (2.27). If $\chi_\parallel \gg \lambda$ and $V > V_*$, then H_m depends on the magnetization distribution in the soliton [see Eq. (2.29)].

Substituting one or another expression for H_m in the formulas (2.20)–(2.23) and using the soliton magnetization distribution (2.14), we obtain, after simple but rather tedious calculations, the desired evolution equations for soliton parameters u and ω.

The functions $g_{u,\omega}^{(r),(e)}$ are very cumbersome and therefore we shall write down only one of them, namely, $g_u^{(r)}(\omega,u)$ in the case $\chi_\parallel \ll \lambda$ (or $\chi_\parallel \gg \lambda$, $V < V_*$) with the collinear field H_m being defined by Eqs. (2.26) and (2.27):

$$g_u^{(r)}(\omega,u) = 2\lambda_1 u \left\{ -\frac{5}{2}\omega_1^4 + \frac{10}{3}\omega_1^3 - \omega_1^2 - \frac{4}{3}\omega_1 - 12u + \frac{52}{3}\omega_1 u - \frac{44}{3}\omega_1^2 u - 16u^2 + h\left(-\frac{5}{2}\omega_1^3 - \frac{10}{3}\omega_1^2 - \omega_1 + \frac{2}{3} + \frac{28}{3}u - \frac{26}{3}\omega_1 u \right) - h_m\left(-\frac{3}{2}\omega_1^2 + \omega_1 - 4u \right) + \left(\frac{I_0}{\kappa}\right)\left[-\frac{5}{4}\omega_1^5 + \frac{5}{2}\omega_1^4 - \frac{3}{2}\omega_1^3 + \omega_1^2 + 4u - 6u\omega_1 + 16u\omega_1^2 - 9u\omega_1^3 + 24u^2 - 16u^2\omega_1 + h\left(-\frac{5}{2}\omega_1^4 + \frac{5}{2}\omega_1^3 - \frac{3}{2}\omega_1^2 - 6u + 10u\omega_1 - 6u\omega_1^2 - 4u^2 \right) - h_m\left(-\frac{3}{4}\omega_1^3 + \omega_1^2 + 4u - 3\omega_1 u \right) \right] \right\}, \tag{2.30}$$

where $\omega_1 = \omega - h$.

Note, that in the case under consideration the separation of contributions of relativistic and exchange terms in the soliton relaxation is a matter of convention because the quantity h_m, defined by Eq. (2.26), depends on both relaxation constants.

The integral curves of equations (2.25), obtained by the numerical integration for different values of the relaxation constants λ_1 and λ_e' are shown in Figs. 2(a) and 2(b). It is interesting to note that at any values of the soliton parameters, with the exception of a narrow region of small soliton velocities, the derivative $d\omega/dt > 0$. The region of parameters in which $d\omega/dt < 0$ is represented in Fig. 2(c).

Analytical solutions of the evolution equations can be obtained in some limiting cases.

Small-amplitude solitons. Such solitons exist in the region of the parameters (u,ω) in the vicinity of the straight line $u + \omega = 1 + h$, wherein $\kappa \ll 1$ and $\theta_{\max} \sim \kappa \ll 1$.

In case $\chi_\parallel \ll \lambda$ (or $\chi_\parallel \gg \lambda$, $V < V_*$) one can set $s_0^2 \approx \chi^2(2-\omega_1)^{-1} \ll 1$, $c_0^2 \approx 1$ in the expressions for h_m Eqs. (2.26) and (2.27) and immediately obtains $h_m = -(1+h+u)$. Substituting this expression into the functions $g_{u,\omega}^{(r),(e)}$, the evolution equations in the asymptotic case under consideration can be written in the form

$$\dot{u} = \frac{8}{3} u\kappa^2 \left\{ \lambda_1 \left(3 + \frac{4h}{1+u} \right) - \lambda_e' \left(2 - u + \frac{2h\omega_1}{1+u} \right) \right\},$$

$$\dot{\omega} = \frac{8}{3} \kappa^2 \left\{ \lambda_1 \left(3 + \frac{h(2+\omega_1)}{1+u} \right) + \lambda_e' u \left(5 + 2u + \frac{h(5+u)}{1+u} \right) \right\}. \tag{2.31}$$

At small velocity ($u \ll 1$) the equations of the small-amplitude soliton relaxation can be substantially simplified,

$$\dot{u} = \frac{8}{3} u\kappa^2 \{ \lambda_1(3+4h) - 2\lambda_e'(1+h) \},$$

$$\dot{\omega} = 8\kappa^2(1+h)\left(\lambda_1 + \frac{5}{3}\lambda_e' u \right), \tag{2.32}$$

and can be solved analytically. The solution of Eqs. (2.38) with the initial conditions $u = u(0)$, $\omega = \omega(0)$ at $\tau = 0$ has the form

$$u(\tau) = u(0)\exp\{ d[1 - \exp(-8\lambda_1 \tau(1+h))] \},$$

$$\kappa(\tau) = \kappa(0)\exp\{ -4\lambda_1 \tau(1+h) \},$$

$$d = \frac{8}{3}\kappa^2(0)[\lambda_1(3+4h) - 2\lambda_e'(1+h)]. \tag{2.33}$$

So, in the region $\kappa \ll 1$, $u \ll 1$, the soliton relaxation is largely governed by relativistic interactions.

At $\tau \to \infty$ the parameter $\kappa \to 0$ and $u \to u_1 = u(0)e^d$, i.e., the integral curve of the evolution equations $\omega = \omega(u)$ terminates at the point lying on the straight line $u + \omega = 1 + h$ where the soliton amplitude is equal to 0, the effective width (x_0/κ) approaches infinity, and the soliton degrades into the homogeneous ground state, the final value of the soliton velocity remaining finite. Note that the value u_1 can be greater or smaller than the initial value $u(0)$, depending on the sign of the parameter d, i.e., on the ratio between relativistic and exchange relaxation constants. If

$$\lambda_1 > 2\lambda_e' \frac{1+h}{3+4h},$$

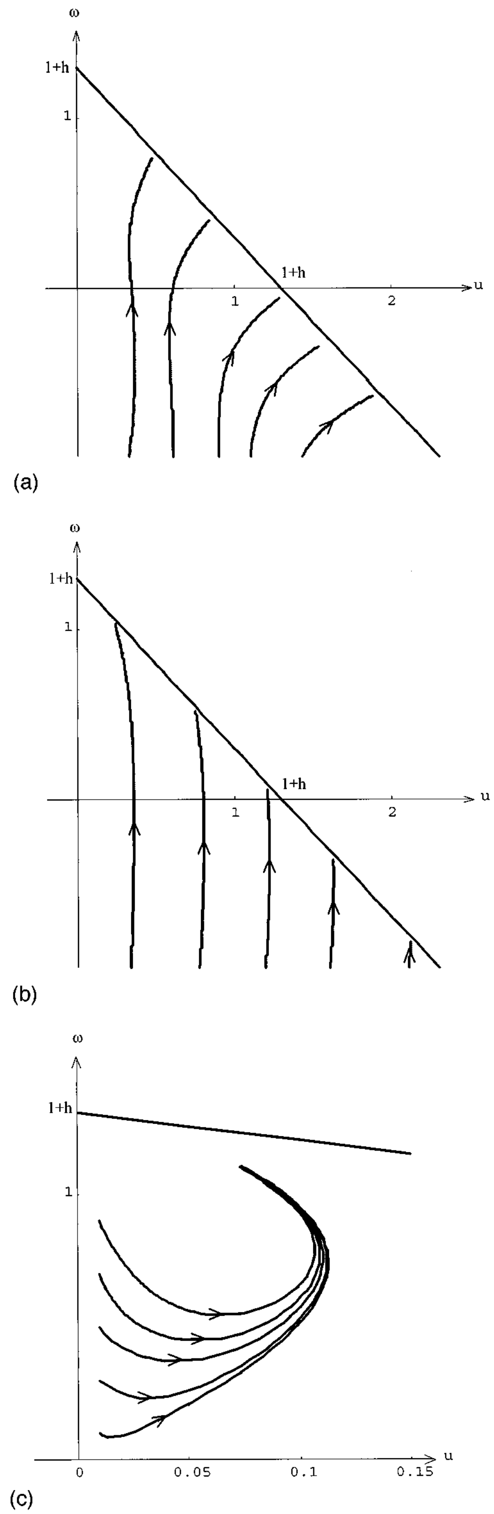

FIG. 2. The ferromagnetic soliton parameters evolution; (a) relativistic relaxation ($\lambda_1 \neq 0$, $\lambda_e = 0$); (b) exchange relaxation ($\lambda_e \neq 0$, $\lambda_1 = 0$); (c) exchange relaxation within the region whereby $d\omega/dt < 0$.

then $d > 0$ and the soliton accelerates in the course the relaxation, $u_1 > u(0)$, whereas under the opposite inequality taking place, $d < 0$, $u_1 < u(0)$, i.e., the soliton slows down.

Under $\kappa \ll 1$ $E \sim \kappa(1+h)$ [see Eq. (2.17)], and the exponential dependence $\kappa(t)$ enables one to introduce an effective soliton lifetime τ_s, $\tau_s = [4\lambda_1(1+h)]^{-1}$. Note that under $u \ll 1$ this time does not depend on the initial soliton parameters but decreases as the external magnetic field increases.

Under large values of the velocity ($u \gg 1$) the evolution equations are as follows:

$$\dot{u} = \frac{8}{3} u \kappa^2 (3\lambda_1 + u\lambda_e'),$$

$$\dot{\omega} = \frac{8}{3} \kappa^2 [\lambda_1(3-h) - 2\lambda_e' u \omega_1]. \tag{2.34}$$

The solution of the system (2.34) at $\lambda \sim \lambda_e'$ has the form

$$u(\tau) = u(0) + \frac{1}{3} \kappa^2(0)[1 - \exp(-8\lambda_e' u^2(0)\tau)],$$

$$\kappa(\tau) = \kappa(0)\exp[-4\lambda_e' u^2(0)\tau]. \tag{2.35}$$

Consequently, the relaxation of the small-amplitude soliton under $u \gg 1$ is defined mainly by exchange interactions, the lifetime $\tau_s = [4\lambda_e' u^2(0)]^{-1}$ being inversely proportional to the fourth power of the initial soliton velocity, $\tau_s \sim v^{-4}(0)$. The final value of the soliton velocity $u_1 = u(0) + \kappa^2(0)/3 > u(0)$, i.e., the soliton accelerates in the course of relaxation.

An analysis of the system of the evolution equations for the small-amplitude soliton (2.31) shows that an exponential dependence of the soliton parameters on time takes place, regardless of the initial values $u(0)$ and $\omega(0)$. It is also noteworthy that the integral of motion N Eq. (2.18) in a small-amplitude soliton is proportional to κ, and therefore the number of magnons in soliton exponentially tends to zero in the course of relaxation.

In case $\chi_\parallel \gg \lambda$, $V > V_*$, when the collinear field h_m is determined by Eq. (2.29), the evolution equations for the small-amplitude soliton are somewhat different from Eq. (2.31). However, in this case all the characteristic properties of these equations are just the same as those of the system (2.31), and therefore we shall not discuss them here.

Precession soliton. If the initial soliton velocity is equal to 0 (but $\omega \neq 0$) then the soliton remains immobile at all succeeding moments during its relaxation (such an excitation can be referred to as a precession soliton). In this case the integral curve is some interval of the ordinate axis ω. Naturally, that under $V = 0$ the collinear field is determined by the formulas (2.26) and (2.27) and the analytical expression for h_m depends on the sign of $\omega_1 = \omega - h$:

$$h_m = \begin{cases} 2\omega, & \omega < h, \\ \omega\left\{(1-2\omega_1) + \dfrac{\lambda_e'}{\lambda_1}(1-\omega_1)\right\}, & \omega < h. \end{cases} \tag{2.36}$$

Such a difference is due to the fact that at $\omega < h$ the soliton amplitude $\theta_0 = \pi$, whereas at $\omega > h$ $\cos^2(\theta_0/2) = \omega - h < 1$, i.e., the soliton amplitude becomes a function of the frequency.[25]

The character of the precession soliton relaxation essentially depends on the presence of an external magnetic field.

Let us consider first the case $H=0$. In this case and under $|\omega|\ll 1$ the evolution equation takes the form:

$$\dot{\omega}=2\overline{\lambda}\omega,\quad \omega(\tau)=\frac{\omega(0)}{1-2\omega(0)\overline{\lambda}\tau},\tag{2.37}$$

where $\overline{\lambda}=\lambda_1/3+\lambda_e'$. We see that at $\omega(0)>0$ the precession frequency ω rapidly increases and in a finite time reaches the value $\omega\sim 1$. If $\omega(0)>0$ then at $\tau\to\infty$ the frequency asymptotically approaches 0. Hence, the precession soliton with $\omega>0$ becomes a small-amplitude one in a finite time and then degrades exponentially (see above), whereas under $\omega<0$ the soliton transforms into a special solution of the equations of motion with $\omega=V=0$ which describes two domain walls separated infinitely.

Let now $H\neq 0$. In this case there are two characteristic frequency values in the problem: $\omega=0$ and $\omega=h$. If $\omega\to h+0$ $(\omega_1\to+0)$ then, in accordance with Eq. (2.36), $h_m=h(1+\lambda_e'/\lambda_1)$ and we obtain for ω:

$$\dot{\omega}=2\overline{\lambda}h(1+\lambda_e'/\lambda_1)(\omega-h),$$
$$\omega(\tau)=h+[\omega(0)-h]\exp[2\overline{\lambda}h\tau(1+\lambda_e'/\lambda_1)],\tag{2.38}$$

i.e., the soliton frequency increases exponentially rather than by the power law as in case $h=0$.

If $\omega\to h-0$ $(\omega_1\to-0)$ then $h_m=2\omega$, and the soliton frequency decreases exponentially,

$$\dot{\omega}=2\overline{\lambda}h(\omega-h),\quad \omega(\tau)=h+[\omega(0)-h]\exp(2\overline{\lambda}h\tau).\tag{2.39}$$

In the case $|\omega|\ll h\sim 1$, one obtains for the soliton frequency

$$\dot{\omega}=-\Gamma(h)\omega,\quad \omega(\tau)=\omega(0)e^{-\Gamma(h)\tau},\tag{2.40}$$

where the inverse soliton lifetime $\Gamma=\Gamma(h)$ is determined by a rather cumbersome expression not shown here. At small field, this lifetime is proportional to h, $\Gamma(h)=2h(\lambda_1/3+\lambda_e')=2\overline{\lambda}h$.

So, the entire picture of the precession soliton relaxation under an external magnetic field is as follows: under $\omega<0$ or $0<\omega<h$ the precession frequency tends to 0 with a characteristic lifetime $\Gamma(h)$. Under $\omega>h$ the soliton frequency increases, it becomes a small-amplitude one and then degrades according to Eq. (2.33). Such a character of the soliton relaxation is in perfect agreement with the fact that the soliton energy decreases in the course of relaxation (see Fig. 1): either $E(\omega)\to E_{\min}(\omega=0)$ at $\omega<0$ and at $0<\omega<h$, or $E(\omega)\to E(\omega=1+h)=0$ at $\omega>h$.

Soliton with a small velocity. If the initial value of the soliton velocity is small $(u\ll 1)$, then in the small-amplitude region, (at $\omega\sim 1+h$) this velocity can increase as well as decrease as dictated by the sign of the parameter d [see Eq. (2.33)]. In the case $|\omega|\gg 1(\omega<0)$ the evolution equation for u can be written in the form

$$\dot{u}=\frac{16}{3}\lambda_e'u\omega^2-\frac{10}{3}\lambda_1u\omega^3>0.\tag{2.41}$$

Consequently, the soliton with a small velocity accelerates due to both relativistic and exchange relaxation.

B. Non-one-dimensional solitons

Let us now consider a relaxation of non-one-dimensional solitons in a uniaxial FM. Dynamical properties of such excitations have been studied in detail in Ref. 25.

We restrict ourselves to an analysis of precession solitons $(\mathbf{V}=0)$ in the absence of an external magnetic field. For such solitons of any dimensionality

$$\dot{\omega}=\frac{d\omega}{dN}\dot{N}=-\frac{d\omega(N)}{dN}\int d\mathbf{r}\dot{\varphi}\{\lambda_1\sin^2\theta\cos^2\theta+\lambda_e a^2[(\nabla\theta)^2+\sin^2\theta\cos^2\theta(\nabla\varphi)^2]\}\tag{2.42}$$

(we assumed that at $\mathbf{V}=0$, $d\theta/dt=0$ and $d(\nabla\varphi)/dt=0$ and set $\mathbf{H}_m=0$).

It has been shown in Ref. 25 that two- and three-dimensional solitons exist only at positive precession frequencies ω in contrast to the one-dimensional case in which solitons exist both at positive and negative signs of ω. If $\omega>0$, then the integrals in Eq. (2.39) are positive and the sign of the derivative $d\varphi/dt$ is opposite to that of $d\omega(N)/dN$.

Two-dimensional (2D) solitons. The 2D soliton is characterized by

$$\theta=\theta(\rho),\quad \varphi=\omega t+n\chi,\quad \theta(\rho)\to 0\quad \text{if}\ \rho\to\infty,\tag{2.43}$$

where, ρ, χ are the polar coordinates of the 2D magnet, the integer n is the topological charge of the soliton. For the 2D soliton at all N, $d\omega/dN<0$,[25] therefore in the course of relaxation the soliton frequency increases. At $\omega\ll\omega_0$ for any n one can assume that[25]

$$\cos\theta=\tanh\frac{\rho-R_0}{l_0},\quad \omega(N)=\omega_0\left(\frac{N_2}{N}\right)^{1/2},\tag{2.44}$$

where $R_0=l_0\omega_0/\omega\gg l_0$, $N_2=2\pi s(l_0/a)^2\gg 1$; s is the atom spin, a is the lattice constant. Evaluating the integrals in Eq. (2.42) on the basis of Eq. (2.44), one obtains

$$\dot{\omega}=\frac{\overline{\lambda}}{\omega_0}\omega^3,\quad \omega(t)=\frac{\omega(0)}{[1-2\overline{\lambda}\omega^2(0)t/\omega_0]^{1/2}},\tag{2.45}$$

where $\lambda'=\lambda_1\approx\lambda_e'$, i.e., similar to the one-dimensional (1D) case the soliton frequency changes from its initial value $\omega(0)$ to $\omega=\omega_0$ in a period $t_0\sim\omega_0/[\overline{\lambda}\omega^2(0)]$. Further evolution of the 2D soliton is entirely different from that of the 1D soliton [see Eq. (2.37)].

The soliton without topological charge $(n=0)$ also exists at $\omega<\omega_0$ similar to the 1D case; at $\omega\to\omega_0$

$$\theta(\rho)=\kappa\psi\left(\frac{\kappa\rho}{l_0}\right),\quad \kappa=\left[\frac{\omega_0-\omega}{\omega_0}\right]^{1/2},\tag{2.46}$$

but at $\omega\to\omega_0$ the value of N tends to the finite limit[25]

$$N=\widetilde{N}_2+A_0N_2\frac{\omega-\omega_0}{\omega},\quad \widetilde{N}_2=1.8N_2,\quad A_0\sim 1.$$

An approximate estimation of the integral in Eq. (2.42) on the basis of Eq. (2.46) shows that at $\omega\to\omega_0$, $d\omega/dt\sim\lambda_1$ and the degeneration point $(\omega=\omega_0)$ is achieved in a finite time $t_1=(\lambda_1\omega_0)^{-1}$, i.e., much quicker than in the 1D case.

In the degeneration point ($\omega=\omega_0$) the number of magnons in the soliton remains finite ($N=\widetilde{N}_2\gg 1$) and the soliton width approaches infinity. Thus, at the final stage soliton relaxation can be described as a transition of the soliton to $\widetilde{N}_2$ magnons of the continuous spectrum.

For the topological soliton the frequency cannot exceed $\omega_0/|n|$; at $N\ll N_2$ we have[25]

$$\tan\frac{\theta}{2}\cong\left(\frac{R}{\rho}\right)^{|n|},\quad \frac{N}{N_2}\cong A_n\left(\frac{1}{|n|}-\frac{\omega}{\omega_0}\right),\quad A_n>0. \tag{2.47}$$

The final stage of relaxation of this soliton is different. The main contribution to the integral in Eq. (2.42) comes from the terms with gradients. Calculating the integral on the basis of Eq. (2.47), one obtains

$$\frac{dN}{dt}=-\frac{16\pi s}{3}\lambda_e'\omega_0, \tag{2.48}$$

and the number of magnons in the soliton approaches the terminal value $N=0$ in a finite time interval $t_e\sim(\omega_0\lambda_e' s)^{-1}$, and $\omega\to\omega_0/|n|$. Consequently, the 2D soliton finally ends up as the singular vortex state with $R\to 0$ ($N\to 0$). The further evolution should go with the change of the vortex line topological charge. The minimum threshold energy value for this process seems to be associated with the break of a soliton line forming two vortex lines with free ends. The process of this type is described for topological vortices in superfluid helium-3.[26]

Three-dimensional (3D) solitons. In the case of dynamic central-symmetric solitons $\theta=\theta(r)$, $r=|\mathbf{r}|$, the form of the function $\theta(r)$ is of type (2.44) and (2.46) at $\omega\ll\omega_0$ and at $(\omega_0-\omega)\ll\omega_0$, respectively. Thus[25]

$$R\cong 2 1_0\frac{\omega_0}{\omega},\quad N\cong 16N_3\frac{\omega_0^3}{3\omega^3},\quad \omega\ll\omega_0,$$

$$N\cong A_3\frac{N_3}{(\omega_0-\omega)^{1/2}},\quad \omega\to\omega_0, \tag{2.49}$$

where $N_3=4\pi s(1_0/a)^3$, $A_3\sim 1$.

It is seen that the derivative $dN/d\omega$ is negative at small precession frequencies ω and is positive at $\omega\to\omega_0$, changing its sign at $\omega=\omega_*\approx 0.915\omega_0$.[25]

According to Eq. (2.49), the soliton frequency decreases at $\omega>\omega_*$ and grows at $\omega<\omega_*$ both due to relativistic and exchange relaxation. This behavior is completely different from that of the 1D case, namely, in the course of relaxation the frequency of the 3D soliton, at any initial value of ω, approaches ω_*.

As shown in Ref. 27 on the basis of the Lyapunov theory, the 3D solitons with $\omega>\omega_*$ are unstable and it is not reasonable to consider them within the slow evolution framework. For the low-frequency solitons ($\omega<\omega_*$), according to Eqs. (2.42), (2.44), and (2.49),

$$\dot{\omega}=\begin{cases}\overline{\lambda}\omega^3/\omega_0, & \omega\ll\omega_0\\ \widetilde{\lambda}\omega_0^3/(\omega_*-\omega), & \omega_*-\omega\ll\omega_0,\end{cases}$$

where $\widetilde{\lambda}\sim\overline{\lambda}=\lambda_1+\lambda_e'$. Integrating this equation, one obtains

$$\omega(t)=\begin{cases}\omega(0)[1-\overline{\lambda}\omega^2(0)t/\omega_0]^{-1/2}, & \omega\ll\omega_0\\ \omega_*-\{[\omega_*-\omega(0)]^2-2\widetilde{\lambda}\omega_0^3 t\}^{1/2}, & \omega\ll\omega_*.\end{cases} \tag{2.50}$$

These formulas describe the following picture of the three-dimensional soliton relaxation. In the course of evolution the soliton frequency increases from any initial value to ω_* in a finite period of time [of the order $\omega_0/(\overline{\lambda}\omega^2(0))$], then it remains constant. The soliton with $\omega=\omega_*$ does not evolve at the expense of the slow relaxation processes described by Eq. (2.42). This fact may seem to be paradoxical but actually it can be well explained by the specific character of the $E(\omega)$ or $N(\omega)$ dependences of the 3D solitons: the energy E as a function of ω grows infinitely both at $\omega\to\omega_0$ [$E\sim(\omega_0-\omega)^{-1/2}$] and at $\omega\to 0$ ($E\sim\omega^{-2}$), i.e., the function $E=E(\omega)$ has a minimum at $\omega=\omega_*$. Thus, the soliton with $\omega=\omega_*$ is the natural final result of slow relaxation processes.

In order to understand the transition of the soliton with $\omega=\omega_*$ to the ground state, one has to go beyond the slow relaxation approximation. The characteristic feature of the 3D soliton with $\omega\sim\omega_*$ is the fact that its energy is higher than the energy of the same number of magnons of the continuous spectrum,[25] $E(N_*)=1.034\,\hbar\omega_0 N_*$, $N_*=N(\omega_*)=9.08N_3$, and the transition of the soliton to N_* free magnons with energies $h\omega_0 N_*$ is energetically favorable. However, the condition of radiation of a small number of magnons $n\ll N$, which has the form $E(N)>E(N-n)+\hbar\omega_0 n$, is not fulfilled at $\omega=\omega_*<\omega_0$ because $dE/dN=\hbar\omega$. Nevertheless, the decay of the soliton into free magnons is possible by way of N-particle process ($N\sim N_*$), although the probability of such a process is exponentially small.

III. RELAXATION IN ANTIFERROMAGNETS

Our analysis of relaxation in antiferromagnets (AFM's) will be done in the framework of the two-sublattice model of AFM's starting from the equations of motion for the vectors **M** and **L**:

$$\dot{\mathbf{M}}=-\frac{2}{g}\{[\mathbf{M},\mathbf{H}]+[\mathbf{L},\mathbf{F}]\}+\mathbf{R}_M,$$
$$\dot{\mathbf{L}}=-\frac{2}{g}\{[\mathbf{M},\mathbf{F}]+[\mathbf{L},\mathbf{H}]\}+\mathbf{R}_L, \tag{3.1}$$

where $\mathbf{H}=-\delta W/\delta\mathbf{M}$, $\mathbf{F}=-\delta W/\delta\mathbf{L}$ are the effective fields, $\mathbf{R}_M,\mathbf{R}_L$ are the dissipation terms defined by the dissipative function ($\mathbf{R}_M=\delta Q/\delta\mathbf{H}$, $\mathbf{R}_L=\delta Q/\delta\mathbf{F}$). The structure of the latter, taking into account both exchange and relativistic relaxation processes, has been found in Refs. 11 and 13, and can be written in the form (1.8).

As in FM's, a symmetry of the relativistic relaxation constants λ_{ik} is governed by symmetry and hierarchy of relativistic interactions. In particular, in uniaxial AFM's the tensor λ_{ik} has the form (1.7) and the dissipative terms $\mathbf{R}_M$ and $\mathbf{R}_L$ can be written as follows:

$$\mathbf{R}_M=g|\mathbf{L}|(\lambda_1\mathbf{H}_\perp+\lambda_3 H_z\mathbf{n}-\lambda_e a^2\Delta\mathbf{H}),\quad \lambda_3\ll\lambda_1$$
$$\mathbf{R}_L=g|\mathbf{L}|\lambda_0\mathbf{F}. \tag{3.2}$$

As in the previous sections, we shall analyze relaxation of nonlinear waves in the linear approximation with respect to relaxation constants, using the method of construction of evolution equations for nonlinear wave parameters, described above. Calculating the rate of change of integrals of motion I_i on the basis of Eq. (3.1), only the relaxations which are not connected with the change of the sublattices magnetization vectors $M_\alpha = |\mathbf{M}_\alpha|$, $\alpha = 1,2$, should be taken into consideration. In particular, one must use the reduced dissipative function $\bar{Q}$ instead of the full dissipative function Q (1.8).

To separate in Q the terms corresponding to the change of $|\mathbf{M}_\alpha|$, let us rewrite the equations of motion in terms of the vectors $\mathbf{M}_\alpha$,

$$\dot{\mathbf{M}} = -g[\mathbf{M}_\alpha, \mathbf{H}_\alpha] + \mathbf{R}_\alpha, \tag{3.3}$$

$$\mathbf{R}_1 = \mathbf{R}_M + \mathbf{R}_L, \quad \mathbf{R}_2 = \mathbf{R}_M - \mathbf{R}_L.$$

Hence

$$\dot{M}_1 = \mathbf{m}_1(\mathbf{R}_M + \mathbf{R}_L), \quad \dot{M}_2 = \mathbf{m}_2(\mathbf{R}_M - \mathbf{R}_L), \quad \mathbf{m}_\alpha = \frac{\mathbf{M}_\alpha}{M_\alpha}. \tag{3.4}$$

Taking into account Eq. (3.4), one readily obtains the reduced dissipative function $\bar{Q}$ in the form

$$\dot{\bar{Q}} = -\frac{1}{2}\left(\frac{dE}{dt}\right)_{M_\alpha = \text{const}} = \frac{1}{2}\int d\mathbf{r}[M_1\dot{\mathbf{m}}_1\mathbf{H}_1 + M_2\dot{\mathbf{m}}_2\mathbf{H}_2] = \frac{1}{2}\int d\mathbf{r}\Big\{\mathbf{H}\mathbf{R}_M + \mathbf{F}\mathbf{R}_L - \frac{1}{M_0^2}[(H_M + F_L)(\mathbf{M}\mathbf{R}_M + \mathbf{L}\mathbf{R}_L) + (H_L + F_M)(\mathbf{L}\mathbf{R}_M + \mathbf{M}\mathbf{R}_L)]\Big\}, \tag{3.5}$$

where $H_L = (\mathbf{HL})$, $F_L = (\mathbf{FL})$, $H_M = (\mathbf{HM})$, $F_M = (\mathbf{FM})$ [two first addends in Eq. (3.5) correspond to the full dissipative function Q].

When calculating $\bar{Q}$ in the linear approximation with respect to relaxation constants, the effective fields $\mathbf{H}$ and $\mathbf{F}$ in Eqs. (3.2), (3.3) should be substituted in the main (zero) approximation [we already used this circumstance when calculating Eq. (3.5) by setting $M_1 = M_2 = M_0$]. In the nondissipative approximation, the vectors $\mathbf{H}$ and $\mathbf{F}$ can be expressed in terms of two scalar values H_L and F_L which may be referred to as collinear fields (in analogy with H_m in FM's),

$$\mathbf{H} = \frac{1}{L^2}\left\{\frac{2}{g}[\mathbf{L}, \dot{\mathbf{L}}] + \mathbf{L}H_L + \mathbf{M}F_L\right\},$$
$$\mathbf{F} = \frac{1}{L^2}\left\{\frac{2}{g}[\mathbf{L}, \dot{\mathbf{M}}] + \mathbf{M}H_L + \mathbf{L}F_L\right\}. \tag{3.6}$$

The collinear fields H_L and F_L can be obtained from the explicit expression for the AFM energy:

$$W = \int d\mathbf{r}[f(M_1^2) + f(M_2^2) + w_0(\mathbf{M}, \mathbf{L})], \tag{3.7}$$

$$w_0(\mathbf{M}, \mathbf{L}) = \frac{\alpha}{2}(\nabla\mathbf{L})^2 + \frac{\delta}{2}\mathbf{M}^2 + w_a - 2\mathbf{M}\mathbf{H}_e, \tag{3.8}$$

where the function $f(M_\alpha^2)$ defines the density of the intersublattice exchange interaction which forms the length of the sublattice magnetization vectors, w_a is the anisotropy energy, $\mathbf{H}_e$ is an external magnetic field.

The equations of motion (3.1) in the nondissipative approximation are known to have two integrals of motion,

$$p \equiv \mathbf{ML} = 0, \quad s \equiv \mathbf{M}^2 + \mathbf{L}^2 - M_0^2 = 0. \tag{3.9}$$

When the dissipative terms in Eq. (3.1) are taken into account, the quantities p and s are no longer constant:

$$\dot{p} = \mathbf{R}_M\mathbf{L} + \mathbf{R}_L\mathbf{M}, \quad \dot{s} = 2(\mathbf{R}_M\mathbf{M} + \mathbf{R}_L\mathbf{L}). \tag{3.10}$$

The function $f(x)$ in Eq. (3.7) under temperature far below Néel temperature has a deep minimum at $x = M_0^2(T)$, $M_0(T)$ is the equilibrium value of the sublattice magnetization length. Therefore, approximating the function $f(x)$ by the expression $f(x) = (x - M_0^2)^2/(4\chi_\parallel M_0^2)$, $\chi_\parallel \ll 1$, one obtains from the energy (3.7):

$$H_L = \frac{4}{\chi_\parallel}p + H_0, \quad H_0 = -\left(\frac{\delta w_0}{\delta\mathbf{M}}, \mathbf{L}\right),$$
$$F_L = -\frac{2}{\chi_\parallel}s + F_0, \quad F_0 = -\left(\frac{\delta w_0}{\delta\mathbf{L}}, \mathbf{L}\right). \tag{3.11}$$

As mentioned in the Sec. II, the longitudinal susceptibility $\chi_\parallel$ should be set equal to 0 in the final results. However, contributions of two first terms in Eq. (3.11) are finite because $p \sim \chi_\parallel$, $s \sim \chi_\parallel$. In the static case, $H_L = F_L = 0$ but the quantities p and s are not equal to zero:

$$p = -\frac{\chi_\parallel}{4}H_0, \quad s = \frac{\chi_\parallel}{2}F_0.$$

In the presence of a dynamic magnetization wave, the fields H_L and F_L are not, in general, equal to 0. The equations for these quantities can be obtained by differentiating Eq. (3.11) with respect to time and using Eqs. (3.2) and (3.10):

$$\frac{\chi_\parallel L}{4g}\dot{H}_L-\lambda_e a^2\mathbf{L}^2\Delta H_\perp-2\lambda_e a^2[(\mathbf{L},\nabla\mathbf{L})\nabla H_L+(\mathbf{L},\nabla\mathbf{M})\nabla F_L]+H_L[\lambda_1\mathbf{L}_\perp^2+\lambda_3 L_z^2+\lambda_0\mathbf{M}^2-\lambda_e a^2(\mathbf{L},\Delta\mathbf{L})]$$
$$+F_L[(\lambda_1-\lambda_3)\mathbf{L}_\perp\mathbf{M}_\perp-\lambda_e a^2(\mathbf{L},\Delta\mathbf{M})]=\frac{\chi_\parallel L}{4g}\dot{H}_0+\frac{2}{g}\{(\lambda_1-\lambda_3)L_z[\mathbf{L},\dot{\mathbf{L}}]_z+\lambda_0(\mathbf{M},[\mathbf{L},\dot{\mathbf{M}}])+\lambda_e a^2(\mathbf{L},\Delta[\mathbf{L},\dot{\mathbf{L}}])\},\tag{3.12}$$

$$\frac{\chi_\parallel L}{4g}\dot{F}_L-\lambda_e a^2\mathbf{M}^2\Delta F_\perp-2\lambda_e a^2[(\mathbf{M},\nabla\mathbf{M})\nabla F_L+(\mathbf{M},\nabla\mathbf{L})\nabla H_L]F_L[\lambda_1\mathbf{M}_\perp^2+\lambda_3 M_z^2+\lambda_0\mathbf{L}^2-\lambda_e a^2(\mathbf{M},\Delta\mathbf{M})]$$
$$+H_L[(\lambda_1-\lambda_3)\mathbf{M}_\perp\mathbf{L}_\perp-\lambda_e a^2(\mathbf{M},\Delta\mathbf{L})]=\frac{\chi_\parallel L}{4g}\dot{F}_0+\frac{2}{g}\{(\lambda_1-\lambda_3)\mathbf{M}_\perp[\dot{\mathbf{L}},\mathbf{L}]_\perp+\lambda_e a^2(\mathbf{M},\Delta[\mathbf{L},\dot{\mathbf{L}}])\}.\tag{3.13}$$

The solution of Eqs. (3.12) and (3.13) without the right-hand side describes relaxation of H_L and F_L to their equilibrium values. An inhomogeneous solution of this system of equations can be nonzero only in the presence of a dynamic magnetization wave.

It should be noted once more that the system (3.12) and (3.13) is obtained in the linear approximation with respect to the small parameters λ and $\chi_\parallel$ [as is Eq. (2.11) for the collinear field H_m in FM's]. Hence, all the coefficients and the right-hand side part of these equations are defined by the magnetization distribution in the excitation under consideration, calculating in the nondissipative approximation.

Further we analyze relaxation in an easy-axis AFM with the anisotropy energy w_a

$$w_a=\frac{1}{2}\beta\mathbf{L}_\perp^2-\frac{1}{4}b(\mathbf{L}_\perp^2)^2,\quad \beta>0,\quad b>0.\tag{3.14}$$

As an example, we shall discuss relaxation of the one-dimensional two-parameter soliton. The corresponding solution of the dynamic equations has been obtained in Ref. 28. At $b=0$ this solution has the form:

$$\varphi=kx-\Omega t,\quad \cos\theta=-\tanh[\tilde{\kappa}(x-Vt)],$$
$$k=\frac{(\Omega+\Omega_e)V}{c^2},\quad \tilde{\kappa}^2=\frac{\beta}{\alpha(1-V^2/c^2)}-\frac{(\Omega+\Omega_e)^2}{c^2},\tag{3.15}$$

where $\Omega_e=gH_e$, an external magnetic field is considered to be aligned with the anisotropy axis. The angle variables θ and φ parametrize the unit vector $\mathbf{l}=\mathbf{L}/|\mathbf{L}|$,

$$l_x+il_y=\sin\theta e^{i\varphi},\quad l_z=\cos\theta.\tag{3.16}$$

This solution describes the magnetization distribution in the two-parameter topological soliton of the kink-type $[\theta(-\infty)=0,\ \theta(+\infty)=\pi]$. The soliton solution (3.15) exists in the region $\tilde{\kappa}^2>0$, i.e., its parameters (the velocity V and the precession frequency Ω) satisfy the inequality

$$(\Omega+\Omega_e)^2<\frac{\omega_0^2}{1-V^2/c^2}.$$

The value of the magnetic field is limited by the condition of the static soliton existence, $H_e<M_0(\beta\delta)^{1/2}$, or $H_e<M_0(\beta\delta)^{1/2}$, i.e., by the spin-flop field.

To construct the evolution equations for the soliton parameters V and Ω, we use, as in FM's, the soliton energy (E) and the total Z projection of magnetization,

$$N=\frac{1}{\hbar g}\int d\mathbf{r}M_z\tag{3.17}$$

(in the nondissipative approximation, the latter conserves due to the uniaxiality of AFM's).

The values of the integral of motion E and N, corresponding to solution (3.15), are equal to

$$E=\frac{2E_0}{\kappa}\left\{\frac{1}{1-u}-\omega_e(\omega+\omega_e)\right\},$$
$$N=\frac{2n_0}{\kappa}(\omega+\omega_e),\tag{3.18}$$

where

$$E_0=\beta M_0^2 x_0,\quad n_0=E_0/(2\hbar\omega_0),$$
$$\kappa^2=1/(1-u)-(\omega+\omega_e)^2;\quad \omega_e=\Omega_e/\omega_0,\quad \omega=\Omega/\omega_0,$$
$$u=(V/c)^2$$

are the dimensionless parameters convenient for further calculations. The frequency dependence of the AFM soliton energy at $V=0$ is represented in Fig. 3.

In the main approximation with respect to the small parameter (β/δ), all terms in the reduced dissipative function (3.5), proportional to $\mathbf{M}$, can be omitted and, taking into account Eqs. (3.2) and (3.6), the function $\overline{Q}$ can be written in the form

$$\overline{Q}\cong\frac{1}{2gM_0^2}\int d\mathbf{r}\mathbf{R}_M[\mathbf{L},\dot{\mathbf{L}}]=\frac{gM_0\delta}{2}E_0 q.\tag{3.19}$$

The function q is defined by the sum of the relativistic and exchange terms, $q=q_r+q_e$,

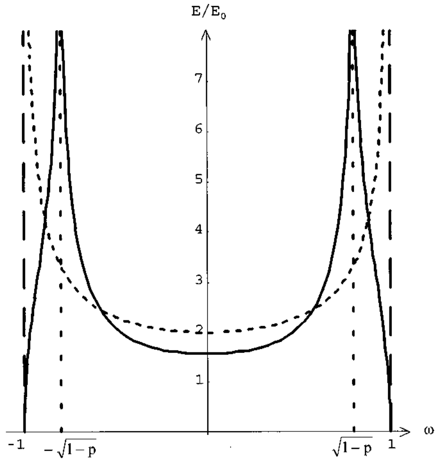

FIG. 3. The frequency dependence of the antiferromagnetic soliton energy ($V=0$); solid line $-H\neq 0$, dashed line $-H=0$.

$$q_r=\lambda_1\langle([\mathbf{l},\dot{\mathbf{l}}]_\perp)^2-h_L l_z[\mathbf{l},\dot{\mathbf{l}}]_z\rangle,$$

$$q_e=\lambda_e'=\langle([\mathbf{l},\dot{\mathbf{l}}]')^2+h_L(\dot{\mathbf{l}}'[\mathbf{l},\mathbf{l}']-\mathbf{l}''[\mathbf{l},\dot{\mathbf{l}}])\rangle, \quad (3.20)$$

where $\lambda_e'=\lambda_e(a/x_0)^2$, $h_L=gH_L/(2M_0\omega_0)$.

The reduced rate of change of the second integral of motion $\overline{\dot{N}}$ is obtained in a similar manner:

$$\overline{\dot{N}}=-\frac{1}{\hbar g M_0^2}\int d\mathbf{r}L_z(\mathbf{R}_M,\mathbf{L})=gM_0\delta n_0\eta, \quad \eta=\eta_r+\eta_e, \quad (3.21)$$

$$\eta_r=\lambda_1\langle l_z^2[\mathbf{l},\dot{\mathbf{l}}]_z-h_L l_z \mathbf{l}_\perp^2\rangle,$$

$$\eta_e=\lambda_e'\langle l_z(\mathbf{l},[\mathbf{l},\dot{\mathbf{l}}]'')+h_L(l_z''-l_z\mathbf{l}'^2)\rangle. \quad (3.22)$$

According to the general scheme, one obtains from E and N Eq. (3.18) the following evolution equations for parameters ω and u:

$$\dot{u}=-\delta\kappa(1-u)^2(q+\omega\eta), \quad (3.23)$$

$$\dot{\omega}=-\frac{\delta\kappa}{2}\{(\omega+\omega_e)(1-u)q-[1-(1-u)(\omega+\omega_e)(2\omega+\omega_e)]\eta\}. \quad (3.24)$$

Equations (3.12) and (3.13), defining the quantities H_L and F_L in the main approximation with respect to the small parameter (β/δ), are divorced from one another, and for the dimensionless quantity h_L we have an equation

$$\tilde{\chi}\dot{h}_L-\lambda_e'h_\perp''+h_L\{\lambda_3+(\lambda_1-\lambda_3)\mathbf{l}_\perp^2+\lambda_e'\mathbf{l}'^2\}=(\lambda_1-\lambda_3)l_z[\mathbf{l},\dot{\mathbf{l}}]_z+\lambda_e'(\mathbf{l},[\mathbf{l},\dot{\mathbf{l}}]'')+\tilde{\chi}\omega_e\dot{l}_z, \quad (3.25)$$

where $\tilde{\chi}=\chi_\parallel\omega_0/(4gM_0)$.

With accuracy at designations, Eq. (3.31) coincides with Eq. (2.11) for the collinear component of the effective field in a uniaxial FM but the last term in the right-hand side of Eq. (3.25) somewhat differs. We will not dwell on the analysis of this equation in detail. The results of this analysis in the case under consideration are as follows: if the soliton velocity V exceeds the characteristic value $V_*\sim\lambda\omega_e x_0/\chi_\parallel$, then the quantity h_L is defined by the simple formula

$$h_L=\omega_e\cos\theta. \quad (3.26)$$

It is obvious that such a situation can be realized only in the limiting case $\tilde{\chi}\gg\lambda$ and in the presence of an external magnetic field. If $\tilde{\chi}\ll\lambda$, then the quantity h_L in Eqs. (3.20) and (3.22) should be set equal to 0, regardless of an external field and a soliton velocity.

The evolution equations for the soliton parameters have the simplest form under $H_e=0$. Substituting Eq. (3.18) into Eqs. (3.20)–(3.22) and setting $h_L=0$, one obtains

$$\dot{u}=f_r(u,\omega)+f_e(u,\omega), \quad \dot{\omega}=g_r(u,\omega)+g_e(u,\omega),$$

$$f_r=-2\lambda_1\delta u\kappa^2(1-u)^2, \quad g_r=-\lambda_1\delta\kappa^2\omega(1-u)\left(\frac{1}{3}+u\right),$$

$$f_e=-2\lambda_e'\delta u\kappa^2(1-u)\left[\frac{1}{3}+\omega^2(1-u)^2\right],$$

$$g_e=-2\lambda_e'\delta\kappa^2\omega\left[1-\frac{u}{3}-\omega^2(1+u)(1-u)^2\right]. \quad (3.27)$$

The analysis of these equations shows that an influence of relativistic and exchange dissipative terms on soliton relaxation differs essentially. In the whole region of the soliton parameters (3.26), the relativistic relaxation leads to monotonic decreasing of both the soliton velocity ($f_r<0$) and the absolute value of the precession frequency [sgn(g_r)= −sgn(ω)]. The character of relativistic relaxation is readily illustrated by the integral curves of equations $\dot{u}=f_r$, $\dot{\omega}=g_r$

$$\omega=\omega(0)\left[\frac{u}{u(0)}\right]^{1/6}\left[\frac{1-u(0)}{1-u}\right]^{1/3},$$

where $u(0)$ and $\omega(0)$ are the initial values of the soliton parameters. These curves are schematically represented in Fig. 4(a).

As to the contribution of exchange relaxation, the functions f_e and g_e can change their signs, and in the course of exchange relaxation the soliton parameters u and ω change nonmonotonically. The corresponding integral curves of the equations $\dot{u}=f_e$, $\dot{\omega}=g_e$, obtained by the numerical integration, are shown in Fig. 4(b). The integral curves in the case $\lambda_1=\lambda_e'$ are shown in Fig. 4(c).

In the case $u(0)=0$, the functions $f_r=f_e=0$; it means that if the initial soliton velocity is equal to zero then the soliton remains immobile at all succeeding moments (as it does in FM's). Similarly, if initially the precession frequency is equal to 0, then $g_r=g_e=0$ and $\omega=0$ in the course of relaxation.

If an external magnetic field is large enough and $V_*<c$, then the picture of soliton relaxation becomes more complicated. In the case $V>V_*$, when h_L is determined by the formula (3.31), the functions $g(u,\omega)$ and $f(u,\omega)$ have the form

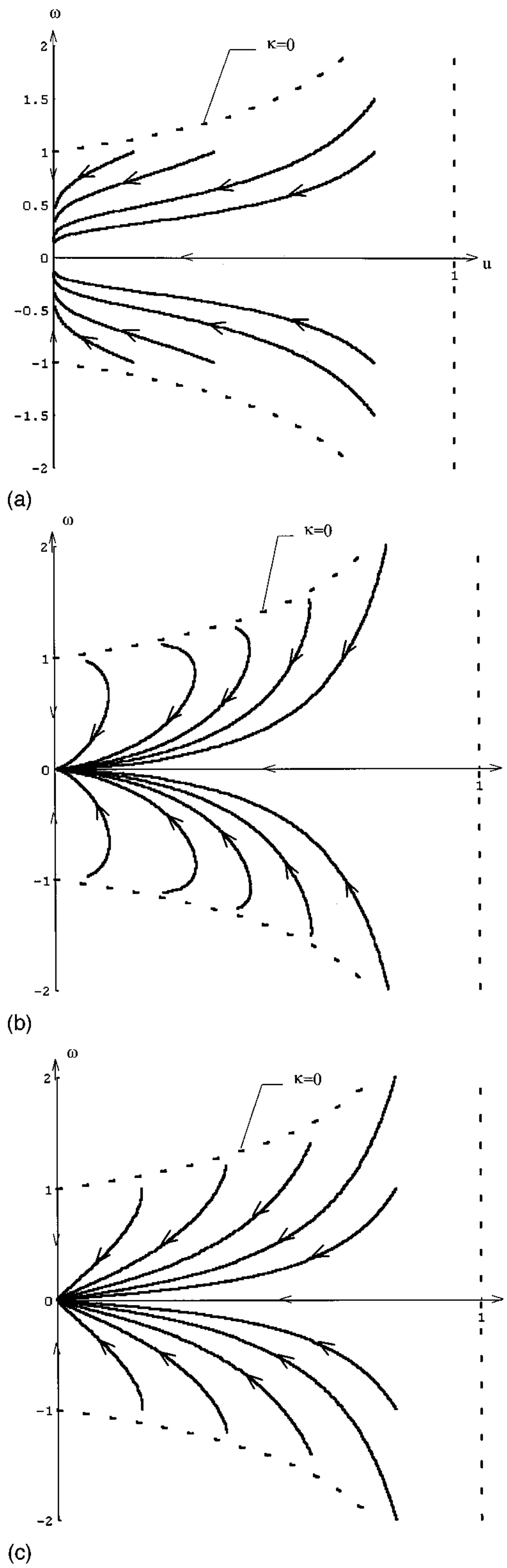

FIG. 4. The antiferromagnetic soliton parameters evolution at $b=0$; (a) relativistic relaxation ($\lambda_1 \neq 0$, $\lambda_e=0$); (b) exchange relaxation ($\lambda_e \neq 0$, $\lambda_1=0$); (c) relaxation at $\lambda_e=\lambda_1$.

$$f_r=-2\lambda_1 \delta u \kappa^2(1-u)^2,$$

$$g_r=-\lambda_1 \delta \kappa^2(\omega+\omega_e)(1-u)\left(\frac{1}{3}+u\right),$$

$$f_e=-2\lambda_e' \delta u \kappa^2(1-u)\left[\frac{1}{3}+(\omega+\omega_e)^2(1-u)^2\right],$$

$$g_e=-2\lambda_e' \delta \kappa^2(\omega+\omega_e)\left[1-\frac{u}{3}-(\omega+\omega_e)^2(1+u)(1-u)^2\right]. \tag{3.28}$$

It is easy to see that the functions in Eqs. (3.28) can be obtained from Eqs. (3.27) by the simple substitution $\omega \to \omega + \omega_e$. The region of the soliton existence can also be obtained from that calculated at $\omega_e=0$ by the frequency axis shift on the value $(-\omega_e)$, since the integral curves of Eqs. (3.28) are just the same as those of Eqs. (3.27) but shifted on the value $(-\omega_e)$. In this case the limiting state of the soliton turns out to be that with precession: $u \to 0$, $\omega \to -\omega_e$. However, it should be recollected that formula (3.28) under small values of the soliton velocity is inadequate because under $V<V_*$ one must take $h_L=0$. If so doing, we have the following evolution equations rather than Eq. (3.28):

$$f_r=-2\lambda_1 \delta u \kappa^2(1-u)^2,$$

$$g_r=-\lambda_1 \delta \kappa^2(1-u)\left[u(\omega+\omega_e)+\frac{1}{3}\,\omega\right],$$

$$f_e=-2\lambda_e' \delta u \kappa^2(1-u)\left[\frac{1}{3}-\omega(\omega+\omega_e)^2(1-u)^2 - \omega_e(\omega+\omega_e)\left(\frac{1}{3}-u\right)\right],$$

$$g_e=-2\lambda_e' \delta \kappa^2(1-u)\left[\kappa^2\left(\omega-\frac{u}{3}\,(\omega+\omega_e)\right) + u(\omega+\omega_e)^2\left(u(\omega+\omega_e)-\frac{\omega}{3}\right)\right]. \tag{3.29}$$

It can be readily verified that these equations result in the final state of the soliton relaxation with $u=\omega=0$ as it must.

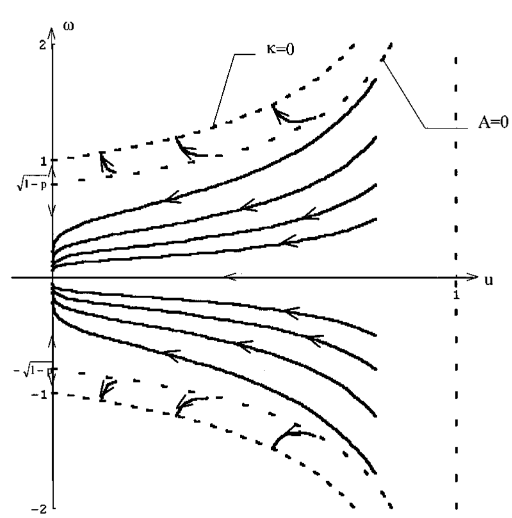

FIG. 5. The antiferromagnetic soliton parameters evolution at $b \neq 0$, $\lambda_e=0$.

If the characteristic velocity $V_* < c$, then the soliton relaxation is described by Eqs. (3.28) at $V > V_*$ and by Eqs. (3.29) at $V < V_*$. Naturally, in the case $V_* < c$ there exists a certain intermediate region between one regime to another. An analysis of this intermediate region requires calculating h_L from Eq. (3.25) in the general case, which is beyond the scope of the present paper.

In the more general model of AFM's, which takes into account the anisotropy energy of the fourth order, i.e., $b \neq 0$ [see Eq. (3.14)], the solitonlike solution of the dynamic equations can be represented in the form (at $H_e = 0$):[28]

$$\tan\theta = \begin{cases} \dfrac{\kappa}{A^{1/2}} \dfrac{1}{\sinh\left(\dfrac{\kappa}{x_0}(x - Vt)\right)}, & A > 0 \\ \dfrac{\kappa}{|A|^{1/2}} \dfrac{1}{\cosh\left(\dfrac{\kappa}{x_0}(x - Vt)\right)}, & A_{\min} < A < 0, \end{cases} \tag{3.30}$$

where $A = (1-p)/(1-u) - \omega^2$, $p = b/(2\beta)$; the dimensionless parameters ω, u, and κ are defined above. The restriction $A > A_{\min} = -p/(1-u) < 0$ is connected with the existence condition of the solution (3.30) ($\kappa^2 > 0$).

At $A > 0$, solution (3.30), as previously, describes the soliton with a topological charge [kink, or domain wall, $\theta(-\infty) = 0$, $\theta(+\infty) = \pi$]; in the case $A < 0$ then $\theta(\pm\infty) = 0$ and solution (3.30) describes a dynamic soliton without a topological charge. The soliton energy and the integral of motion N, corresponding to Eq. (3.30), are equal to

$$E = E_0 \left\{ \frac{D_k}{2} \left(\frac{1-p}{1-u} + \omega^2 \right) + \kappa \right\}, \quad N = n_0 \omega D_k, \tag{3.31}$$

where $k = 1,2$; at $A > 0$ $k = 1$ and at $A < 0$ $k = 2$,

$$D_k = 2 \left(\frac{1-u}{p} \right)^{1/2} \begin{cases} \sinh^{-1}\left[\dfrac{p}{(1-u)\kappa^2 - p} \right]^{1/2}, & k = 1 \\ \cosh^{-1}\left[\dfrac{p}{p - (1-u)\kappa^2} \right]^{1/2}, & k = 2. \end{cases}$$

Note that the existence regions of two types of solitons (with and without a topological charge) are separated by the infinitely high-energy barrier, $E(A \to \pm 0) \to \infty$. Therefore in the course of relaxation, a transformation of a dynamic soliton in the topological one and vice versa is impossible. The analysis of the soliton relaxation at $b \neq 0$ is performed in the same manner as it done above for the case $b = 0$, and therefore we shall present only the final results.

The evolution equations for the soliton parameters u and ω, which are connected with relativistic relaxation, have the following form:

$$\dot{u} = -2\lambda_1 \delta u (1-u) \frac{[(1-p)/(1-u) - \omega^2] D_k + 2\kappa}{[(1-p)/(1-u) + \omega^2] D_k + 2\kappa},$$

$$\dot{\omega} = -\lambda_1 \delta \omega \left\{ u \left[\left(\frac{1-p}{1-u} - \omega^2 \right) D_k + 2\kappa \right] \left[\left(\frac{1-p}{1-u} + \omega^2 \right) D_k + 2\kappa \right]^{-1} - \frac{1 - p - \omega^2 + u\omega^2}{2p} \left[\left(\frac{1-p}{1-u} - \omega^2 \right) D_k - 2\kappa \right] \right.$$
$$\left. \times \left[\left(\frac{1-p}{1-u} - \omega^2 \right) D_k + \frac{2\omega^2}{\kappa} \right]^{-1} \right\}. \tag{3.32}$$

The exchange terms have similar but more cumbersome form and we do not write them down here.

The integral curves of Eqs. (3.32) are shown in Fig. 5. We see that topological solitons approach (as at $b = 0$) the equilibrium state with a domain wall at rest whereas dynamical solitons degenerate into the homogeneous ground state of AFM's.

IV. CONCLUSION

The analysis performed demonstrates a rather complicated character of soliton parameters evolution in the course of relaxation. It should be stressed once more that an adequate description of a nonlinear excitation in magnets is impossible without taking into account the exchange relaxation processes. They affect not only quantitative estimations, as took place for domain walls,[10] but can result in a qualitatively different picture of the soliton evolution.

ACKNOWLEDGMENT

This work was supported, in part, by the International Soros Science Education Program (ISSEP) through Grant No. 452.

*Electronic address: sukstan@purr.dipt.donetsk.ua

[1] L. D. Landau and E. M. Lifshitz, Phys. Z. Sowjetunion **8**, 153 (1935).
[2] T. L. Gilbert, Phys. Rev. **100**, 1243 (1955).
[3] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **15**, 1 (1945).
[4] F. Dyson, Phys. Rev. **102**, 1217 (1956).
[5] V. N. Koshcheev and M. A. Krivoglaz, Fiz. Tverd. Tela (Leningrad) **3**, 1541 (1961).
[6] V. G. Baryakhtar, V. N. Krivoruchko, and D. A. Yablonskii, *Green Functions in the Theory of Magnetism* (Nauk-dumka, Kiev, 1984) (in Russian).
[7] A. P. Malozemoff and J. C. Slonchewski, *Magnetic Domain*

Walls in Bubbles Materials (Academic, New York, 1979).
[8] A. S. Abyzov and B. A. Ivanov, Zh. Eksp. Teor. Fiz. **76**, 1700 (1979).
[9] B. A. Ivanov, Yu. N. Mitsay, and N. V. Shakhova, Zh. Eksp. Teor. Fiz. **87**, 289 (1984).
[10] V. G. Baryakhtar, Zh. Eksp. Teor. Fiz. **87**, 1501 (1984).
[11] V. G. Baryakhtar, Fiz. Nizk. Temp. **11**, 1198 (1985).
[12] V. G. Baryakhtar, Fiz. Tverd. Tela (Leningrad) **29**, 1317 (1987).
[13] V. G. Baryakhtar, Zh. Eksp. Teor. Fiz. **94**, 196 (1988).
[14] V. G. Baryakhtar, *Integrability and Kinetic Equations for Solitons* (Nauk-Dumka, Kiev, 1990), pp. 6–43 (in Russian).
[15] F. Bloch, Phys. Rev. **70**, 460 (1946).
[16] R. K. Wangness, Phys. Rev. **98**, 927 (1955).
[17] V. G. Baryakhtar (unpublished).
[18] D. J. Kaup and A. C. Newell, Proc. R. Soc. London, Ser. A **361**, 413 (1978).
[19] V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 538 (1977).
[20] D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).
[21] V. G. Baryakhtar, B. A. Ivanov, and A. L. Sukstanskii, Zh. Eksp. Teor. Fiz. **91**, 1454 (1986).
[22] I. E. Dzyaloshinskii and B. G. Kukharenko, Zh. Eksp. Teor. Fiz. **70**, 2360 (1976).
[23] A. I. Akhiezer, V. G. Baryakhtar, and S. V. Peletminskii, *Spin Waves* (Nauka, Moscow, 1967).
[24] V. G. Baryakhtar, B. A. Ivanov, T. K. Soboleva, and A. L. Sukstanskii, Zh. Eksp. Teor. Fiz. **68**, 1509 (1980).
[25] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Magnetization Waves. Dynamic and Topological Solitons* (Nauk-Dumka, Kiev, 1983).
[26] G. E. Mineev, Zh. Eksp. Teor. Fiz. **72**, 2256 (1977).
[27] B. A. Ivanov and A. L. Sukstanskii, Solid State Commun. **50**, 523 (1984).
[28] I. V. Baryakhtar and B. A. Ivanov, Zh. Eksp. Teor. Fiz. **87**, 328 (1983).