Universality and logarithmic corrections in two-dimensional random Ising ferromagnets

F. D. A. Aarão Reis, S. L. A. de Queiroz, and Raimundo R. dos Santos
Instituto de Física, Universidade Federal Fluminense, Avenida Litorânea s/n, 24210-340 Niterói RJ, Brazil
(Received 29 January 1997)

We address the question of weak versus strong universality scenarios for the random-bond Ising model in two dimensions. A finite-size scaling theory is proposed, which explicitly incorporates ln L corrections (L is the linear finite size of the system) to the temperature derivative of the correlation length. The predictions are tested by considering long, finite-width strips of Ising spins with randomly distributed ferromagnetic couplings, along which free energy, spin-spin correlation functions, and specific heats are calculated by transfer-matrix methods. The ratio γ/ν is calculated and has the same value as in the pure case; consequently conformal invariance predictions remain valid for this type of disorder. Semilogarithmic plots of correlation functions against distance yield average correlation lengths ξ^{av}, whose size dependence agrees very well with the proposed theory. We also examine the size dependence of the specific heat, which clearly suggests a divergency in the thermodynamic limit. Thus our data consistently favor the Dotsenko-Shalaev picture of logarithmic corrections (enhancements) to pure system singularities, as opposed to the weak universality scenario. [S0163-1829(97)05833-5]

I. INTRODUCTION

In the study of random magnetic systems, a frequently asked question is whether or not quenched disorder destroys a sharp phase transition and, in the latter case, whether critical exponents are the same as for the corresponding pure magnets.[1–3] The Harris criterion[4] provides useful guidance in a number of cases: if the exponent α, characterizing the divergence of the specific heat at the critical point of the pure system, is positive then randomness induces crossover to a different universality class; for negative α the disordered system is expected to exhibit the same critical behavior as the unperturbed one. However, such a rule is inconclusive for the subject of this work, the two-dimensional Ising model, where the specific heat of the pure system diverges logarithmically (that is, with $\alpha=0$) at the critical point. Further, the Harris approach is perturbative in the sense that only weak randomness is considered. Nonperturbative methods are thus required, especially when one wishes to investigate strongly disordered systems. A suitable way to deal with this sort of problem is through numerical calculations on finite systems. One then has to account for finite-size effects before extrapolating to the thermodynamic limit. This is done by testing specific hypotheses bearing upon the nature of asymptotic behavior.

In the present paper we investigate the theoretical prediction (see Refs. 2 and 3, and references therein) that disorder affects the phase transition of the two-dimensional Ising model only via a specific, well-defined set of logarithmic corrections to pure-system critical behavior; here we extend, and give further details of, the results preliminarily reported in Ref. 5. Such a prediction is in contrast to recent work,[6–8] according to which critical quantities such as the zero-field susceptibility and correlation length display power-law singularities, with the corresponding exponents γ and ν changing continuously with disorder so that the ratio γ/ν is kept constant at the pure system's value (the so-called *weak universality* scenario[9]).

Here we calculate free energies and spin-spin correlation functions on long, finite-width strips of two-dimensional disordered Ising systems. The main motivation for the use of this geometry is that strip calculations, together with finite-size scaling (FSS) concepts,[10,11] are among the most accurate techniques to extract critical points and exponents for nonrandom low-dimensional systems.[12,13] The rate of decay of correlation functions determines correlation lengths along the strip. These latter are, in turn, an essential piece of Nightingale's phenomenological renormalization scheme,[12,13] and have been given further relevance via the connection with critical exponents provided by conformal invariance concepts.[14] Early extensions of strip scaling to random systems[15] have since been pursued further[5,16,17] and put into a broader perspective. In particular, it has been shown that although in-sample fluctuations of correlation functions do not die out as strip length is increased, averaged values converge satisfactorily;[18] throughout the present paper we shall make use of this fact to calculate error bars of related quantities.

We consider the two-dimensional Ising model on a square lattice with bond randomness. The particular version of disorder studied in this work is a binary distribution of ferromagnetic interaction strengths for both vertical and horizontal bonds,

$$P(J_{ij})=\frac{1}{2}[\delta(J_{ij}-J_0)+\delta(J_{ij}-rJ_0)], \quad 0\leqslant r\leqslant 1, \quad (1)$$

which is the prototypical random-bond Ising system, and exhibits the unique advantage that its critical temperature $\beta_c=1/k_BT_c$ is exactly known[19,20] as a function of r through

$$\sinh(2\beta_cJ_0)\sinh(2\beta_crJ_0)=1. \quad (2)$$

For given r one can then sit at $T=T_c(r)$ and be sure that numerical errors due to imprecise knowledge of the critical point are absent. Also, a vast amount of simulational work

has been done on this same model (see Ref. 3), thus comparison is made easier when appropriate.

The layout of the paper is as follows. We first recall in Sec. II the main predictions[2,3] concerning logarithmic corrections to the singular behavior of bulk quantities for a disordered two-dimensional Ising model, and discuss how such corrections should show up in the corresponding finite-size quantities. In particular, we show that a logarithmic term is expected to be the leading correction to the finite-size behavior of the temperature derivative of $\xi^{\rm av}$. In Sec. III we outline numerical aspects of our calculational approach for the magnetic susceptibility, the correlation length, and the specific heat; also, the numerical results are presented and discussed. Section IV summarizes our findings.

II. LOGARITHMIC CORRECTIONS AND FINITE-SIZE SCALING

For infinite-system quantities close to the critical point, with $t \equiv (T-T_c)/T_c$, the following forms have been proposed (see Refs. 2 and 3, and references therein) for the correlation length $\xi_\infty(t)$ and initial susceptibility $\chi_\infty(t)$:

$$\xi_\infty \sim t^{-\nu}[1+C\ln(1/t)]^{\tilde{\nu}}, \tag{3}$$

$$\chi_\infty \sim t^{-\gamma}[1+C\ln(1/t)]^{\tilde{\gamma}}, \tag{4}$$

where $\nu=1$, $\tilde{\nu}=1/2$, $\gamma=7/4$, $\tilde{\gamma}=7/8$, and C is a disorder-dependent positive constant; for $C=0$ one recovers pure-system behavior. Corresponding expressions have been derived for magnetization and specific heat, which will not concern us for now. Thus theory predicts that the dominant power-law singularities (with the same indices as for the uniform system) will actually be *enhanced* by logarithmic divergences. We shall keep to current use in the field[2,3] and refer to these latter as *corrections*, though strictly speaking the term is inappropriate.

In searching for signatures of such diverging logarithmic corrections in systems of finite size, one must be careful about applying recipes used when the bulk singularity is purely of a power-law nature. For instance, a naive application, to Eq. (3), of the usual shortcut[11] $t \to L^{-1}$ to extract the size dependence at criticality, would yield a correlation length growing faster than L, which clearly cannot be true. Instead, one must consider the relationship between bulk quantities predicted by theory and exemplified by Eqs. (3) and (4), namely

$$\chi_\infty \sim (\xi_\infty)^{\gamma/\nu}. \tag{5}$$

To see what this implies, recall the FSS hyphotesis[10,11] for a generic quantity $\mathcal{Q}_L(t)$:

$$\mathcal{Q}_L(t) = f(L)\mathcal{G}(z), \quad z \equiv \frac{\xi_\infty(t)}{L}, \tag{6}$$

where L is the linear lattice size and one assumes small t, large L. As is well known, the L dependence must be removed as $z \to 0$. It is immediate that, whenever the relationship between $\mathcal{Q}_\infty(t)$ and $\xi_\infty(t)$ is a power law such as in Eq. (5) above, $f(L)$ will be a power law as well. This, together with the complementary condition that only the L dependence must remain asymptotically for $z \gg 1$, ensures in this case that the finite-size susceptibility at the critical point must be

$$\chi_L(0) \sim L^{\gamma/\nu} = L^{7/4}. \tag{7}$$

In other words, FSS implies that logarithmic corrections must *not* show up, and the finite-size susceptibility at T_c will exhibit pure power-law behavior against L, with the same power as in the homogeneous case. The same argument is, of course, valid for $\xi_L(t)$ which must then scale linearly with L at T_c. As shown below, numerical data bear out such predictions, for both $\chi_L(0)$ and $\xi_L(0)$.

This raises the question of how, on finite systems, to seek evidence for effects of the bulk corrections predicted in Eqs. (3) and (4). In the following we show that the proper quantities to consider are temperature derivatives of, e.g., $\xi_L(t)$. We apply standard FSS concepts to show that, although the dominant behavior of such quantities is in powers of L, the leading *corrections* to FSS must depend on $\ln L$. This is in contrast with the corresponding (non-diverging) corrections to FSS for, say, ξ which can be fitted by inverse power laws (see Ref. 17 and below).

First we recall that the FSS form for ξ is, from Eq. (6),

$$\xi_L(t) = L \quad \phi(z), \quad \phi(z) \to \begin{cases} z, & z \ll 1, \\ \text{const}, & z \gg 1. \end{cases} \tag{8}$$

The temperature derivative of ξ_L is then

$$\mu_L(t) \equiv \frac{d\xi_L(t)}{dt} = \mu_\infty(t)\phi'(z), \quad \phi'(z) \to \begin{cases} 1, & z \ll 1, \\ 0, & z \gg 1, \end{cases} \tag{9}$$

where $\mu_\infty(t) \equiv d\xi_\infty/dt$ and the prime denotes a derivative with respect to z. While the bulk limit $z \ll 1$ of Eq. (9) is a straightforward identity, the vanishing of $\phi'(z)$ for $z \gg 1$ [as implied by Eq. (8)] must be qualified. Indeed, $\mu_L(t)$ does not diverge in the latter regime, while $\mu_\infty(t)$ does when $t \to 0$. Thus $\phi'(z) \sim [\mu_\infty(t)]^{-1}$, in the sense that the dependence of ϕ' on t through ξ_∞ must be such as to cancel the diverging t dependence of μ_∞. Since the FSS ansatz predicts that t only arises through the ratio $\xi_\infty(t)/L$, one can deduce the L dependence of $\mu_L(t)$ for $z \gg 1$. Up to now, the argument is entirely general and variations of it have been commonly used in the FSS literature.

Turning to the two-dimensional random-bond Ising model, where the bulk quantities are expected to behave as in Eqs. (3) and (4), one has for $t \ll 1$ (consistent with our goal of deriving expressions suitable for the $z \gg 1$ regime):

$$\xi_\infty \sim t^{-\nu}(\ln 1/t)^{\tilde{\nu}}, \tag{10}$$

which can be iteratively inverted to give t as a function of ξ_∞:

$$t \sim \xi_\infty^{-1/\nu}(\ln \xi_\infty)^{\tilde{\nu}/\nu}. \tag{11}$$

The expression for $\mu_\infty(t)$ is

$$\mu_\infty(t) \sim t^{-(1+\nu)}[1+C\ln(1/t)]^{\tilde{\nu}}, \tag{12}$$

plus less-divergent terms, which for $t \ll 1$ can be put as

$$\mu_\infty(t)\sim t^{-(1+\nu)}(\ln 1/t)^{\tilde{\nu}}\simeq \xi_\infty/t=\xi_\infty^{1+1/\nu}(\ln\xi_\infty)^{-\tilde{\nu}/\nu}, \tag{13}$$

where Eq. (11) was used in the last step. It follows immediately that

$$\phi'(z)\sim z^{-(1+1/\nu)}(\ln z)^{\tilde{\nu}/\nu} \tag{14}$$

which, when plugged back into Eq. (9) together with Eq. (13), gives

$$\mu_L\sim L^{1+1/\nu}[1-\ln L/\ln\xi_\infty]^{\tilde{\nu}/\nu},\quad z\gg 1, \tag{15}$$

so all diverging factors related to ξ_∞ are removed, but a nondiverging ξ_∞-dependent term remains which eventually vanishes. Strictly speaking, Eq. (15) means that for both $t\ll 1$ *and* $L\gg 1$, but such that $z\gg 1$, one must observe essentially the leading power-law form $\mu_L\sim L^{1+1/\nu}$. However, even though Eq. (2) enables one to sit exactly at $t=0$, Eq. (15) suggests the existence of a regime in which the leading correction to power-law behavior is $\sim(1-A\ln L)^{\tilde{\nu}/\nu}$ for finite and not very large strip widths L, thus defining an effective (nondiverging) screening length $\xi_s\equiv e^{1/A}$. This heuristic procedure draws on ideas used to interpret experimental data for systems where a full divergence of the correlation length is hindered by percolation,[21] random field,[22] or frustration[23] effects. Defining the inverse correlation lengths κ (actually observed), $\kappa_0\sim t^\nu$ and $\kappa_s\equiv(\xi_s)^{-1}$ (representing the physical factor that smears the divergence, e.g., domain size), one writes

$$\kappa=\kappa_0+\kappa_s \tag{16}$$

with good results.[21–23] Here, κ_s does not originate from a physical feature of the infinite system; instead, it reflects the overall effect of higher-order corrections in such a preasymptotic region (strip widths $L\lesssim 15$). While ξ_s is of a different nature to the crossover length L_C setting the scale above which disorder effects are felt,[2,3] the two lengths vary similarly with disorder, as explained below.

We now describe the numerical procedures used to test the predictions given by Eqs. (7) and (15), and the respective results.

III. CALCULATIONAL METHOD AND RESULTS

We have used long strips of a square lattice, of width $4\leqslant L\leqslant 14$ sites with periodic boundary conditions. In order to provide samples that are sufficiently representative of disorder, we iterated the transfer matrix[13] typically along 10^7 lattice spacings.

At each step, the respective vertical and horizontal bonds between first-neighbor spins i and j were drawn from the probability distribution Eq. (1) above. We have mainly used three values of r in calculations: $r=0.5$, 0.25, and 0.1; the two smallest values have been chosen for the purpose of comparing with recent Monte Carlo simulations where ν and γ are evaluated.[24] The critical temperatures, from Eq. (2), are $T_c\ (0.5)/J_0=1.641\ldots$; $T_c\ (0.25)/J_0=1.239\ldots$; $T_c\ (0.1)/J_0=0.9059\ldots$ (to be compared with $T_c\ (1)/J_0=2.269\ldots$). We also evaluated critical correlation lengths and their derivatives for $r=0.01$ and 0.001, with respective critical temperatures $T_c/J_0=0.5089\ldots$ and $0.3426\ldots$.

A. Susceptibility

The calculation of finite-size susceptibility data and their extrapolation goes as follows. First, we include a uniform longitudinal field h in the Hamiltonian, and obtain the largest Lyapunov exponent Λ_L^0 for a strip of width L and length $N\gg 1$ in the usual way.[15,25] Starting from an arbitrary initial vector $\mathbf{v}_0$, one generates the transfer matrices $\mathcal{T}_i$ that connect columns i and $i+1$, drawing bonds from the distribution Eq. (1), and applies them successively, to obtain

$$\Lambda_L^0=\frac{1}{N}\ln\left\{\frac{\|\Pi_{i=1}^N\mathcal{T}_i\mathbf{v}_0\|}{\|\mathbf{v}_0\|}\right\}. \tag{17}$$

The average free energy per site is then $f_L^{\rm av}(T,h)=-(1/L)\,\Lambda_L^0$, in units of k_BT. The initial susceptibility of a strip, $\chi_L(T_c)$, is given by

$$\chi_L(T_c)=\left.\frac{\partial^2 f_L^{\rm av}(T,h)}{\partial h^2}\right|_{T=T_c,h=0}=L^{\gamma/\nu}Q(0), \tag{18}$$

where, according to the discussion in the preceding section, we assume a pure power-law dependence on L at $T=T_c$.

As $f_L^{\rm av}(T,h)$ is expected to have a normal distribution,[26,25] so will χ_L. Thus the fluctuations are Gaussian, and relative errors must die down with sample size (strip length) N as $1/\sqrt{N}$. Typical strip lengths varied from $N=2\times 10^6$ (for $r=0.5$) to $N=2\times 10^7$ (for $r=0.1$), which are much longer than those used in Ref. 17; they provide estimates for the free energy with an accuracy of 0.01%, which is crucial to compute reliable numerical derivatives. In order to get rid of startup effects, the first $N_0=10^5$ iterations were discarded. The intervals (of external field values, in this case) used in obtaining finite differences for the calculation of numerical derivatives must be strictly controlled, so as not to be an important additional source of errors. We have managed to minimize these latter effects by using δh typically of order 10^{-4} in units of J when calculating $f_L^{\rm av}(T_c;h=0,\pm\delta h)$ for the derivative in Eq. (18). We estimated the first Lyapunov exponent at $(T=T_c,h=0)$ and $(T=T_c,h=\pm\delta h)$ with four different realizations of the impurity distribution, each one giving a separate estimate of the initial susceptibility. From them the average $\chi_L(T_c)$ and the error bars (twice the standard deviation among the four overall averages) are taken.

A succession of estimates, $(\gamma/\nu)_L$, for the ratio γ/ν, is then obtained from Eq. (18) as follows:

$$\left(\frac{\gamma}{\nu}\right)_L=\frac{\ln[\chi_L(T_c)/\chi_{L-1}(T_c)]}{\ln[L/(L-1)]}. \tag{19}$$

The respective error bars follow from those of the corresponding finite-size susceptibilities. In order to extrapolate this sequence, we refer to early work on the eigenvalue spectrum of the transfer matrix for pure systems with a marginal operator in the Hamiltonian.[27] There, it is shown that the critical free energy per site is affected only by an additive logarithmic term in the coefficient of the leading, L^{-2}-dependent, finite-size correction (proportional to the conformal anomaly,[28] c): $f(L)-f(\infty)=-(\pi/6L^2)[c+B(\ln L)^{-3}+\cdots]$. Since disorder is expected to be marginally relevant in the present case, and assuming that the field

derivatives commute with the L dependence (at least as dominant terms are concerned), we expect a similar picture to hold here. Of course, with the imprecisions introduced by randomness one can only expect to see the leading power-law dependence (see, e.g., Ref. 17 for further illustrations of this point).

Least-squares fits for plots of $(\gamma/\nu)_L$ against $1/L^2$ provide the following extrapolations: $\gamma/\nu = 1.748 \pm 0.012$, 1.749 ± 0.008, and 1.746 ± 0.013, respectively, for $r = 0.50$, 0.25, and 0.10; the latter two estimates agree with 1.74 ± 0.03 and 1.73 ± 0.05, obtained in Ref. 24.

The overall picture is thus consistent with the prediction of Eq. (7), that is $\gamma/\nu = 7/4$, the same as for the pure system, for all degrees of disorder. Recalling the Introduction, this still is not enough to distinguish between weak- and strong-universality scenarios, as both coincide in their predictions for the *ratio* of exponents. One needs to try and isolate a single exponent, which will be done in the next subsection through investigation of correlation lengths.

Taken together with the results of Ref. 17 where η was found to be 1/4 through an analysis of averaged correlation lengths, and using the scaling relation $\gamma/\nu = 2 - \eta$, the present analysis of finite-size susceptibilities gives independent support to the view that (1) the conformal invariance relation[14] $\eta = L/\pi\xi_L(T_c)$ still holds for disordered systems, provided that an *averaged*—as opposed to *typical*, see next subsection—correlation length is used; and that (2) the appropriate correlation length to be used is that coming from the slope of semilog plots of correlation functions against distance.[17] Interestingly, the connection with the conformal invariance prediction also rules out any explicit diverging logarithmic L dependence on ξ_L.

B. Correlation lengths

The aim of this subsection is to check on the validity of Eq. (15), or rather, its predicted consequences in the preasymptotic region within our reach, $t \ll 1$, $L \lesssim 15$.

The first difference to the free energy calculation described above is that the correlation functions are expected to have a distribution close to *log*-normal[29,26] rather than a normal one. This has been thoroughly checked recently.[18] Thus self-averaging is not present, and fluctuations for a given sample do *not* die down with increasing sample size. However, it has been numerically verified that the spread among overall averages (i.e., central estimates) from different samples does shrink (approximately as $N^{-1/2}$) as the samples' size (N) increases (see Fig. 2 of Ref. 18). Accordingly, in what follows the error bars quoted arise from fluctuations among four central estimates, each obtained from a different impurity distribution. Similar procedures seem to have been followed in Monte Carlo calculations of correlation functions in finite $(L \times L)$ systems.[30]

The direct calculation of correlation functions, $\langle \sigma_0 \sigma_R \rangle$, follows the lines of Sec. 1.4 of Ref. 13, with standard adaptations for an inhomogeneous system.[17] For fixed distances up to $R = 100$, and for strips with the same length as those used for averaging the free energy, the correlation functions are averaged over an ensemble of 10^4–10^5 different estimates to yield $\overline{\langle \sigma_0 \sigma_R \rangle}$.

The average correlation length, $\xi^{\rm av}$, is defined by

$$\overline{\langle \sigma_0 \sigma_R \rangle} \sim \exp(-R/\xi^{\rm av}), \tag{20}$$

and is calculated from least-squares fits of straight lines to semilog plots of the average correlation function as a function of distance, in the range $10 \leqslant R \leqslant 100$. And, finally, $\xi^{\rm av}$ is in turn averaged over the different realizations of impurity distributions.

Recall that, as explained in Ref. 17, the inverse of $\xi^{\rm av}$ is not the same as the difference between the two leading Lyapunov exponents, which gives the decay of the *most probable*, or typical (as opposed to averaged), correlation function.[17,25,31] It has been predicted[32] that typical correlations in bulk two-dimensional random Ising magnets decay as $\langle \sigma_0 \sigma_R \rangle \sim R^{-1/4} (\ln R)^{-1/8}$, while for averaged ones as in Eq. (20) logarithmic corrections are washed away, resulting in a simple power-law dependence. For strips one could expect, in analogy with the case of pure systems with marginal operators,[27] additive logarithmic corrections to the leading L^{-1} behavior of typical correlations: $\Lambda_L^1 - \Lambda_L^0 = (\pi/L)[\eta + D(\ln L)^{-1} + \cdots]$ with $\eta = 1/4$.

It has been conjectured that the averaged correlation functions at criticality of the random-bond Ising model are identical to those of the pure case;[30] numerically the two quantities are indeed very close,[18,30] while most-probable and pure-system correlation functions do no fit each other so well, though their L dependence is similar.[18] Given the exact result[33] that, for strips of pure Ising spins the corrections to the leading L^{-1} behavior of $(\xi^{\rm av})^{-1}$ as given by Eq. (20) depend on L^{-2}, it seems reasonable to expect L^{-x} (i.e., faster than inverse logarithmic) terms also in the present case. This has been shown to work well, with the same $x = 2$, in Ref. 17.

We now proceed to testing Eq. (15). We calculate μ_L at T_c [see Eq. (9)] numerically, from values of $\xi_L^{\rm av}$ evaluated at $T_c \pm \delta T$, with $\delta T/T_c = 10^{-3}$.

Assuming a simple power-law divergence $\xi_\infty \sim t^{-\nu}$—i.e., ignoring, for the time being, less-divergent terms such as logarithmic corrections—we obtain the estimates for systems of sizes L and $L-1$:

$$\frac{1}{\nu_L} = \frac{\ln(\mu_L/\mu_{L-1})_{T=T_c}}{\ln(L/L-1)} - 1. \tag{21}$$

This is slightly different from the usual fixed-point calculation,[13] and is more convenient in the present case where the exact critical temperature is known. Our data for each pair of $(L, L-1)$ strips have appeared in Ref. 5, and we quote here, for completeness, just the extrapolated (against $1/L^2$) values: $\nu = 1.032 \pm 0.031$ (for $r = 0.5$; here we have extended the previous calculations up to $L = 14$), $\nu = 1.083 \pm 0.014$ ($r = 0.25$), and $\nu = 1.14 \pm 0.06$ ($r = 0.10$). Taken at face value, these data show a systematic trend towards values of ν slightly larger than the pure-system value of 1, though the variation is smaller than that shown in Ref. 24.

Before accepting this trend as an indication of the weak-universality scenario, we must test for corrections to pure-system behavior caused by less-divergent terms, as being responsible for the apparent change of ν with disorder. We

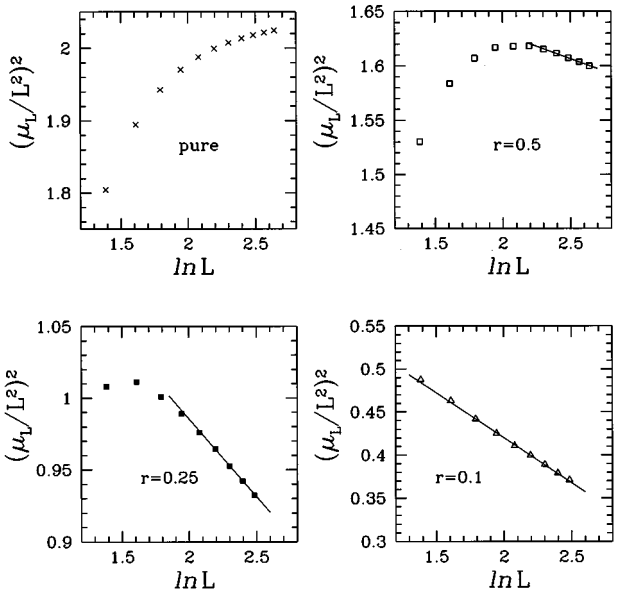

FIG. 1. Finite-size scaling plots of logarithmic corrections [Eq. (22)]. Straight lines are least-squares fits of data, respectively, for $L=9-14$ ($r=0.5$), $7-12$ ($r=0.25$), and $4-12$ ($r=0.1$). The error bars are smaller than the data points.

then try to check whether our data fit a form inspired by Eq. (15) with $\nu=1$ (the pure-system value) and $\tilde{\nu}=1/2$, namely

$$\frac{\mu_L}{L^2}\sim(1-A\ln L)^{1/2}. \tag{22}$$

Prior to displaying our results, we recall that the influence of randomness is expected to show on scales larger than a disorder-dependent characteristic length L_C.[2,3] For $L<L_C$ one should have apparent pure-system behavior.

A plot of $(\mu_L/L^2)^2$ as a function of $\ln L$ for different values of r, including $r=1$, is shown in Fig. 1. The pure-system behavior consists of a monotonic approach to a horizontal line, with an ever-decreasing slope. For $r=0.50$ and 0.25 we can see the pure-system trend for small L, followed by a clearly marked crossover towards a form consistent with Eq. (22). In each case, log-corrected behavior sets in for suitably large L, exactly in the manner predicted by theory: the data stabilize onto a straight line with negative slope only for $L\gtrsim L_C$, which decreases with increasing disorder.[2,3] One may assume, admittedly with some arbitrariness, L_C for each r to be approximately the location of the maximum of the respective curve in Fig. 1. This gives $L_C\simeq 8$, 5, and 2, respectively, for $r=0.50$, 0.25, and 0.10 (for $r=0.10$ data for $L=2$ and 3, not shown in the figure, were used as well).

An order-of-magnitude guide to the size of the preasymptotic region where Eq. (22) is expected to hold, such that for larger L the pure power-law behavior predicted by Eq. (15) at $t=0$ takes over, is the ''screening length'' $\xi_s\equiv e^{1/A}$ of Eq. (22). For $r=0.50$, 0.25, and 0.10 one has the approximate values $\xi_s\sim 4\times10^{16}$, 7×10^4 and 4×10^2, respectively. Though any of these is far beyond the largest strip width within reach of calculations, the trend against disorder is clearly similar to that of L_C.

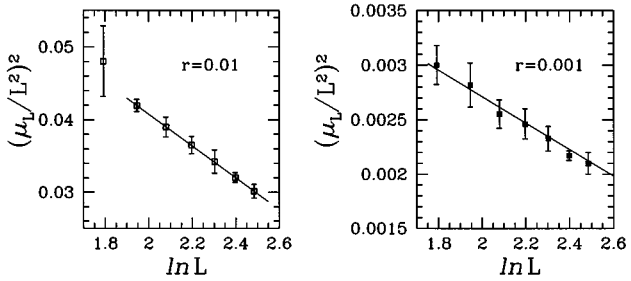

FIG. 2. Finite-size scaling plots of logarithmic corrections [Eq. (22)] for strong disorder. Straight lines are least-squares fits of data, respectively, for $L=6-12$ ($r=0.001$) and $7-12$ ($r=0.01$).

We thus tried stronger disorder (smaller values of r), in order to look for a signature of pure power-law behavior at a feasible $L\lesssim 15$. In Fig. 2 we show $(\mu_L/L^2)^2$ as a function of $\ln L$ for $r=0.01$ and 0.001. Proximity to the percolation threshold is reflected in the large error bars, which render our central estimates virtually meaningless for $L\lesssim 5$; for larger L, fluctuations are reduced, owing to the exponential growth in the number of intracolumn configurations, so we still can manage reasonable fits in that range. Unfortunately, no clear sign can be seen of a trend towards a horizontal line. We believe that a conjunction of (i) smaller r, (ii) larger L, and (iii) longer strip length N would eventually unearth the expected stabilization, though we do not feel secure to venture numerical guesses at this point.

The above correlation length analysis thus provides us with an interpretation of the numerical data which, it should be stressed, is backed by theory,[2,3] without resorting to disorder-dependent exponents. Nevertheless, we have found that the general statistical quality of the data does not allow one to distinguish clearly in favor of either possibility, in terms, e.g., of least-squares fits. We therefore seek complementary quantitative information through the analysis of specific heat data.

C. Specific heats

The same theory[2,3] that gives rise to Eqs. (3) and (4) predicts that the singular part of the bulk specific heat per particle for the disordered Ising model, near the critical point, is given by

$$C_\infty(t)\simeq(1/C_0)\ln[1+C_0\ln(1/t)], \tag{23}$$

where again C_0 is proportional to the strength of disorder, and the pure-system simple logarithmic divergence is recovered as $C_0\to 0$. For $C_0\neq 0$ and $t\ll 1$ a double-logarithmic singularity arises, whose amplitude Eq. (23) predicts to decrease as disorder increases. The bulk specific heat cannot then be put as a simple function of the correlation length given in Eq. (3), and one cannot predict pure-system behavior against L for finite systems, as was the case for the susceptibility and correlation length above. Instead, theory gives[3]

$$C_L(t=0)\simeq C_1+a\ln(1+b\ln L), \tag{24}$$

where, similarly to Eq. (23), $b\to 0$ for vanishing disorder. In this latter limit the product ab must remain finite, but it is

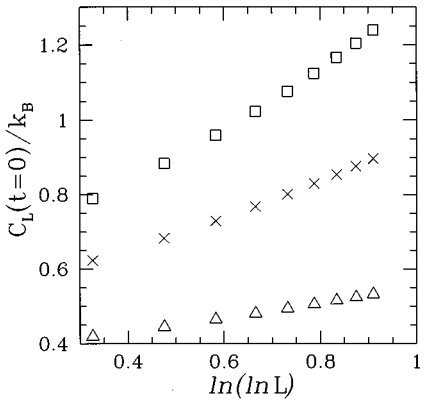

FIG. 3. Specific heat per site at criticality for $L=4-12$ and $r=0.50$ (squares), 0.25 (crosses), and 0.1 (triangles), against $\ln\ln L$ [Eq. (24)].

not *a priori* obvious from theory whether $a \propto b^{-1}$ away from that. In fact, the form Eq. (24) has been verified by Monte Carlo simulations of $L\times L$ systems[34] with the result that the slope of plots of C_L against $\ln\ln L$ *decreases* for increasing disorder. This shows that in this case the simple FSS recipe $t \to L^{-1}$ seems to work satisfactorily.

An investigation of the specific heat on strips is clearly of interest, in order to check the consistency of our own correlation-length data, and also to provide a comparison with the trends found for the specific heat in $L\times L$ systems, both as described above and in recent work[6] where a nondiverging behavior is apparently found in the thermodynamic limit.

Our results are displayed in Fig. 3, where one can see that the fit to a double-logarithmic form is reasonable; for small disorder $r=0.5$ we get an overall better fit to a pure logarithmic divergence, similarly to the result for $L\times L$ lattices.[34] This is again because, as disorder decreases one gets apparent pure-system behavior for relatively large L.

The slope of the plots turns smaller for higher disorder, again in agreement with the trend found for $L\times L$ lattices;[34] however, no sign of an eventual trend towards nondivergence[6] can be distinguished.

The recent claims that for strongly disordered Ising systems in two dimensions, the specific heat is finite at T_c, have been made on the basis of numerical simulations of site-diluted models.[6] Specific heats were plotted against t $(t>0)$ for system sizes and temperatures such that $L/\xi_\infty(t)>1$ (thus excluding the very close vicinity of the transition); see Fig. 1 of Ref. 6. While for impurity concentration $c=1/9$ a divergence was clearly seen, data for $c=1/4$ and 1/3 were interpreted as signaling a finite bulk specific heat at the transition. Such findings have been criticized.[35] At this point it is worth recalling experimental data. First, in bulk systems the specific heat exhibits a broad regular background against which the singular part must be singled out. Early experiments on the two-dimensional site-diluted Ising system $Rb_2Co_xMg_{1-x}F_4$ showed that the amplitude of the singular part of the magnetic specific heat *decreases* as dilution $1-x$ increases.[36] However, owing to experimental difficulties, chief among them the smearing of T_c due to sample inhomogeneities, clear peaks could be found only for $1-x\lesssim 0.11$. Later, results from the more accurate technique of birefringence[37] and with a presumably higher-quality sample confirmed[38] that the specific heat diverges for $1-x=0.15$, apparently with a single logarithmic dependence identical to that of the pure system; this was ascribed to an extreme narrowness of the region where disordered (double-logarithmic) behavior would show up (in agreeement with theory[3,38]). Though, to our knowledge, a systematic study of the variation of specific heat of two-dimensional Ising systems against dilution by, e.g., birefringence techniques, has not been done, useful hints may be taken from the corresponding three-dimensional case of $Fe_xZn_{1-x}F_2$. There, birefringence experiments[39] show that, as dilution increases, the relative position of the (narrow) peak at T_c against that of the (broad) maximum of a short-range order background contribution switches from higher to lower temperatures. This fact is not directly related to the particular three-dimensional features which are used to explain the dilution dependence of the specific heat amplitude.[39] Thus, it is not unlikely that for two dimensions too the apparent nondiverging behavior seen, for $T>T_c$ at $c=1/4$ and 1/3, represents only the background. To see the actual (probably small) peak one would have to go closer to T_c; imprecisions in the knowledge of T_c for site-diluted systems (see, e.g., Ref. 16) may be of capital importance then.

In contrast to this, here and in Ref. 34 one sits right *at* the exactly known T_c. Further, according to the discussion of finite-size specific heats above, the amplitude of the peak at the bulk transition translates directly into the slope of the plot of C_L against $\ln\ln L$, so the regular background is easily dealt with.

In short, the evidence presented here clearly indicates that the specific heat diverges at the transition, with a double-logarithmic behavior. Thus the critical exponent α is nonnegative. Through hyperscaling arguments, this ties in with our findings for the correlation length, as shown in the following. For weak enough disorder, there should be no question about the dimensionality of the system, as opposed to near the percolation threshold in the *diluted* case [corresponding to $r=0$ for the bond distribution Eq. (1)], where one might argue in favor of substituting the *fractal* dimension for the actual lattice dimensionality. Therefore, hyperscaling should be fully applicable with $d=2$, which yields

$$\frac{\alpha}{2}=1-\nu. \tag{25}$$

Since our specific heat data implies $\alpha \geqslant 0$ (most likely $\alpha=0$), one must have $\nu \leqslant 1$, thus excluding the disorder-varying exponents given in Sec. III B and in previous works.[6–8,24]

IV. CONCLUSIONS

We have addressed the question of strong versus weak universality in the two-dimensional random-bond (i.e., exchange couplings being either J or rJ with equal probability; $0<r\leqslant 1$ measures the degree of disorder) Ising model, through extensive transfer-matrix calculations. A key ingredient in the analysis of our data has been the consideration of subtle finite-size scaling (FSS) effects; these come about as a result of constraints imposed by the Dotsenko-Shalaev

theory[2] for logarithmic corrections in the thermodynamic limit. We have established that while the correlation length (and the susceptibility) itself should display no signature of size-dependent logarithmic corrections, its temperature derivative, $\mu_L \equiv d\xi_L/dt$, shows a $\ln L$ dependence (L is the strip width) over a wide range of system sizes. Actually, at the (exactly known) critical temperature for the infinite system and for constant disorder (i.e., fixed r) the behavior with linear size is as follows. For $L < L_C$, with L_C being a crossover length, the system behaves as in the pure case; L_C decreases monotonically with disorder and $2 \leqslant L_C \leqslant 8$ for the values of r we considered. Above L_C, μ_L is dominated by a $\ln L$ enhancement *over the usual pure system power law*; that is, the numerical data can be explained through consistent theories, without resorting to disorder-varying critical exponents. The FSS theory developed here also suggests that as L increases, beyond a (heuristically introduced) screening length ξ_s, one will eventually reach an asymptotic regime where the logarithmic enhancements will vanish, leaving only pure power-law (pure-system-like) behavior; see Sec. II. This coherence length tracks the crossover length, in the sense of decreasing with increasing disorder, but its order of magnitude is way beyond the reach of our numerical capabilities ($\xi_s \gtrsim 10^2$) for us to venture a more refined analysis of this issue. Note, however, that when $t \to 0$ *after* the thermodynamic limit has been taken (which is an entirely different matter) it is expected that $\ln(1/t)$ corrections, as predicted by the Dotsenko-Shalaev theory, should manifest themselves. Also, our data independently confirm that the conformal invariance result $\xi^{\mathrm{av}} = L/\pi\eta$ is still valid for the two-dimensional random-bond Ising model, with $\eta = 1/4$ as in the pure case.

As a further test of the consistency of the proposed scenario, we have examined the size dependence of the specific heat for this system. Consistently with the above findings, the specific heat was seen to be clearly divergent in the thermodynamic limit. Since there are no physical grounds to invoke a mechanism leading to changes in the hyperscaling relation, the case for weak universality cannot be supported by our data. Further, it must be noted that a variety of studies of this problem, both theoretical[40,41] and experimental[36,38,42] concurs with the idea that the leading singularities remain the same as in the pure case, though they have not dealt with the detection of logarithmic corrections.

As regards works whose conclusion is that weak-universality holds instead,[6,7,24] though $\xi_L(T)$ and the susceptibility $\chi_L(T)$ were calculated, no attempt seems to have been made to fit the corresponding data to a form similar to Eq. (22). Thus it remains to be checked whether they would also be consistent with suitable FSS expressions based on strong-universality concepts.

ACKNOWLEDGMENTS

We thank Laboratório Nacional de Computação Científica (LNCC) for use of their computational facilities, and Brazilian agencies CNPq and FINEP for financial support. Special thanks are due to R. B. Stinchcombe for invaluable discussions, and to D. Stauffer for useful suggestions.

[1] R. B. Stinchcombe, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, New York, 1983), Vol. 7.
[2] B. N. Shalaev, Phys. Rep. **237**, 129 (1994).
[3] W. Selke, L. N. Shchur, and A. L. Talapov, in *Annual Reviews of Computational Physics*, edited by D. Stauffer (World Scientific, Singapore, 1994), Vol. 1.
[4] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[5] F. D. A. Aarão Reis, S. L. A. de Queiroz, and R. R. dos Santos, Phys. Rev. B **54**, R9616 (1996).
[6] J. -K. Kim and A. Patrascioiu, Phys. Rev. Lett. **72**, 2785 (1994).
[7] R. Kühn, Phys. Rev. Lett. **73**, 2268 (1994).
[8] M. Fähnle, T. Holey, and J. Eckert, J. Magn. Magn. Mater. **104-107**, 195 (1992).
[9] M. Suzuki, Prog. Theor. Phys. **51**, 1992 (1974).
[10] M. E. Fisher, in *Critical Phenomena*, Proceedings of the International School of Physics ''Enrico Fermi,'' Course LI, Varenna, 1970, edited by M. S. Green (Academic, New York, 1971).
[11] M. N. Barber, in *Phase Transitions and Critical Phenomena* (Ref. 1), Vol. 8.
[12] M. P. Nightingale, J. Appl. Phys. **53**, 7927 (1982).
[13] M. P. Nightingale, in *Finite Size Scaling and Numerical Simulations of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).
[14] J. L. Cardy, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1987), Vol. 11.
[15] H. -F. Cheung and W. L. McMillan, J. Phys. C **16**, 7027 (1983); **16**, 7033 (1983); D. A. Huse and I. M. Morgenstern, Phys. Rev. B **32**, 3032 (1985); U. Glaus, *ibid.* **34**, 3203 (1986); J. Phys. A **20**, L595 (1987).
[16] S. L. A. de Queiroz and R. B. Stinchcombe, Phys. Rev. B **46**, 6635 (1992); **50**, 9976 (1994).
[17] S. L. A. de Queiroz, Phys. Rev. E **51**, 1030 (1995).
[18] S. L. A. de Queiroz and R. B. Stinchcombe, Phys. Rev. E **54**, 190 (1996).
[19] R. Fisch, J. Stat. Phys. **18**, 111 (1978).
[20] W. Kinzel and E. Domany, Phys. Rev. B **23**, 3421 (1981).
[21] R. A. Cowley, R. J. Birgeneau, G. Shirane, H. J. Guggenheim, and H. Ikeda, Phys. Rev. B **21**, 4038 (1980).
[22] R. J. Birgeneau, H. Yoshizawa, R. A. Cowley, G. Shirane, and H. Ikeda, Phys. Rev. B **28**, 1438 (1983).
[23] A. G. Schins, M. Nielsen, A. F. M. Arts, and H. W. de Wijn, Phys. Rev. B **49**, 8911 (1994).
[24] J.-K. Kim (unpublished).
[25] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices in Statistical Physics*, Springer Series in Solid State Sciences Vol. 104, edited by Helmut K. Lotsch (Springer, Berlin, 1993).
[26] B. Derrida, Phys. Rep. **103**, 29 (1984).
[27] J. L. Cardy, J. Phys. A **19**, L1093 (1986).
[28] H. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, *ibid.* **56**, 746 (1986).
[29] B. Derrida and H. Hilhorst, J. Phys. C **14**, L539 (1981).

[30] A. L. Talapov and L. N. Shchur, Europhys. Lett. **27**, 193 (1994).
[31] A. Crisanti, S. Nicolis, G. Paladin, and A. Vulpiani, J. Phys. A **23**, 3083 (1990).
[32] A. W. W. Ludwig, Nucl. Phys. B **330**, 639 (1990).
[33] B. Derrida and L. de Seze, J. Phys. (France) **43**, 475 (1982).
[34] J. -S. Wang, W. Selke, V. S. Dotsenko, and V. B. Andreichenko, Physica A **164**, 221 (1990).
[35] W. Selke, Phys. Rev. Lett. **73**, 3487 (1994); J. -K. Kim and A. Patrascioiu, *ibid.* **73**, 3489 (1994).
[36] M. Suzuki and H. Ikeda, J. Phys. C **11**, 3679 (1978).
[37] P. Nordblad, D. P. Belanger, A. R. King, V. Jaccarino, and H. Ikeda, Phys. Rev. B **28**, 278 (1983).
[38] I. B. Ferreira, A. R. King, V. Jaccarino, J. L. Cardy, and H. J. Guggenheim, Phys. Rev. B **28**, 5192 (1983).
[39] I. B. Ferreira, J. L. Cardy, A. R. King, and V. Jaccarino, J. Appl. Phys. **69**, 5075 (1991).
[40] B. Derrida *et al.*, J. Phys. (France) **48**, 335 (1987).
[41] H. -O. Heuer, Europhys. Lett. **16**, 503 (1991).
[42] M. Hagen, R. A. Cowley, R. M. Nicklow, and H. Ikeda, Phys. Rev. B **36**, 401 (1987).