Ferrimagnetism in a one-dimensional Heisenberg model

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In this paper, we show that for a one-dimensional antiferromagnetic Heisenberg model with unequal spins, its absolute ground state has both antiferromagnetic and ferromagnetic long-range orders.
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In recent years, the study of one-dimensional (1D) quantum spin chains has attracted many physicists' interest. Various powerful techniques, such as the Bethe ansatz solutions, bosonization method,[1,2] and the so-called matrix product states technique[3,4] have been introduced to reveal the rich physical properties of these models. It should be mentioned that these systems are also directly related to some real magnetic materials.[1,5]

As is well known by the Mermin-Wagner theorem,[6] a one-dimensional quantum spin system cannot possess a magnetic long-range order when temperature $T\neq 0$. On the other hand, $T=0$ may be a critical point. However, in neither spin-$\frac{1}{2}$ nor spin-1 antiferromagnetic Heisenberg models on a simple chain has a magnetic long-range order been found. Very recently, Mikeska *et al.* considered a Heisenberg ferrimagnetic spin chain.[7,8] By using either the spin-wave theory[7] or the matrix product states approach,[8] they have shown clearly that the absolute ground state of this model has a ferrimagnetic long-range order.[9] Then they carried out the calculations on the low-lying excitations.

In this paper, we study this ferrimagnetic model by a different approach and prove the existence of the ferrimagnetic long-range order in its absolute ground state in a mathematically rigorous way. Our conclusions confirm the previous results of Mikeska *et al.*[7,8]

To begin with, we would like to introduce some useful definitions and terminologies. Take a finite simple lattice chain Λ with lattice constant $a=1$ and impose the periodic boundary condition on it. Let $N_\Lambda=2N$ be the number of lattice sites in Λ. Then the Hamiltonian of the ferrimagnetic Heisenberg model studied in Refs. 7 and 8 can be written as

$$H_\Lambda=\sum_{j=1}^{N}\mathbf{s}_{2j}\cdot\boldsymbol{\tau}_{2j+1}, \tag{1}$$

where $\mathbf{s}_{2j}$ and $\boldsymbol{\tau}_{2j+1}$ are spin-$\frac{1}{2}$ and spin-1 operators located at sites $2j$ and $2j+1$, respectively. Apparently, with respect to Hamiltonian (1), lattice Λ is bipartite and each sublattice has the same number of sites. As pointed out in Ref. 8, some real magnetic materials belonging to the family of Cu(II)Ni(II) complexes can be well described by this model.[5]

This model can be slightly generalized as follows. We take an arbitrary lattice Λ and divide it into two separate sublattices A and B. Correspondingly, we let each site of Λ be occupied by a spin $\tilde{\mathbf{s}}$, where $\tilde{\mathbf{s}}_{\mathbf{i}}=\mathbf{s}_{\mathbf{i}}$ if $\mathbf{i}\in A$ and $\tilde{\mathbf{s}}_{\mathbf{i}}=\boldsymbol{\tau}_{\mathbf{i}}$ if $\mathbf{i}\in B$. We define

$$\tilde{H}_\Lambda=\sum_{\mathbf{i}\in A}\sum_{\mathbf{j}\in B}J_{\mathbf{ij}}\tilde{\mathbf{s}}_{\mathbf{i}}\cdot\tilde{\mathbf{s}}_{\mathbf{j}}, \tag{2}$$

where $\{J_{\mathbf{ij}}\geqslant 0\}$ is a set of antiferromagnetic coupling constants and they are allowed to be site dependent. We shall require that lattice Λ be connected by $\{J_{\mathbf{ij}}\}$.

Let $\tilde{\Psi}$ be an eigenstate of $\tilde{H}_\Lambda$. When Λ is a simple cubic lattice, which is a special case of the bipartite lattices, the *transverse* and *longitudinal* spin-correlation functions can be simply defined by

$$g_T(\mathbf{q})\equiv\langle\tilde{\Psi}|\tilde{S}_+(-\mathbf{q})\tilde{S}_-(\mathbf{q})|\tilde{\Psi}\rangle,$$

$$g_L(\mathbf{q})\equiv\langle\tilde{\Psi}|\tilde{S}_z(-\mathbf{q})\tilde{S}_z(\mathbf{q})|\tilde{\Psi}\rangle, \tag{3}$$

where

$$\tilde{S}_\alpha(\mathbf{q})\equiv\frac{1}{\sqrt{N_\Lambda}}\sum_{\mathbf{i}\in\Lambda}\tilde{s}_{\mathbf{i}\alpha}\exp(i\mathbf{q}\cdot\mathbf{i}),\quad \alpha=+,-,z \tag{4}$$

and $\mathbf{q}$ is a reciprocal vector of the simple cubic lattice. If inequality $g_T(\mathbf{q})\geqslant\beta N_\Lambda$ ($g_L(\mathbf{q})\geqslant\beta N_\Lambda$), where $\beta>0$ is a constant independent of N_Λ, holds for some reciprocal vector $\mathbf{q}$ as $N_\Lambda\to\infty$, we say that $\tilde{\Psi}$ has a momentum-$\mathbf{q}$ transverse (longitudinal) magnetic long-range order (MLRO). In particular, the momentum-0 MLRO is the ferromagnetic long-range order and the momentum-$\mathbf{Q}$ [$\mathbf{Q}=(\pi,\pi,\cdots,\pi)$] MLRO represents the antiferromagnetic long-range order.

For an arbitrary bipartite lattice Λ, the above definition is not suitable. To extend the definition of MLRO to such a lattice, we introduce

Definition 1: A complex function $f(\mathbf{i})$ defined on lattice Λ is called admissible if $|f(\mathbf{i})|=1$ for any $\mathbf{i}\in\Lambda$. For a specific admissible function $f(\mathbf{i})$, we define

$$g_T(f)\equiv\langle\tilde{\Psi}|\tilde{S}_+(f^*)\tilde{S}_-(f)|\tilde{\Psi}\rangle,$$

$$g_L(f)\equiv\langle\tilde{\Psi}|\tilde{S}_z(f^*)\tilde{S}_z(f)|\tilde{\Psi}\rangle \tag{5}$$

to be the transverse and longitudinal momentum-f spin-correlation functions of $\tilde{\Psi}$, respectively. In formula (5), $\tilde{S}_\alpha(f)\equiv(1/\sqrt{N_\Lambda})\Sigma_{\mathbf{i}\in\Lambda}f(\mathbf{i})\tilde{s}_{\mathbf{i}\alpha},\alpha=+,-,z$.

Obviously, the correlation functions defined above coincide with their counterpart on a simple cubic lattice if we choose $f(\mathbf{i})=\exp(i\mathbf{q}\cdot\mathbf{i})$. In particular, letting

$$\epsilon(\mathbf{i})=\begin{cases}1, & \text{for } \mathbf{i}\in A,\\ -1, & \text{for } \mathbf{i}\in B,\end{cases} \tag{6}$$

we see that the momentum-ϵ correlation functions, $g_T(\epsilon)$ and $g_L(\epsilon)$, are the transverse and longitudinal antiferromagnetic spin-correlation functions of $\tilde{\Psi}$. Similarly, the ferromagnetic correlation functions can be written as $g_T(I)$ and $g_L(I)$ with $I(\mathbf{i})=1$ for any $\mathbf{i}\in\Lambda$.

Now, we are ready to state our result. It can be summarized in the following theorem.

Theorem: Let N_A and N_B be the numbers of sites in sublattice A and B, respectively. When $N_A=N_B=N$, the absolute ground state $\tilde{\Psi}_0$ of $\tilde{H}_\Lambda$ has total spin $\tilde{S}=|s-\tau|N$. Furthermore, it supports both antiferromagnetic and ferromagnetic long-range orders. In other words, the system is a ferrimagnet.

The proof of the first statement in our theorem is almost identical to the proof of the well-known Lieb-Mattis theorem.[10] For completeness, we shall briefly repeat it in the following. The technique in proving the second statement of the theorem was developed in some of our previous papers.[11,12]

Proof of the theorem: First, we notice that the total spin operator $\tilde{S}^2=(\Sigma_{\mathbf{i}\in\Lambda}\tilde{\mathbf{s}}_{\mathbf{i}})^2$ and the total spin Z-component operator $\tilde{S}_z=\Sigma_{\mathbf{i}\in\Lambda}\tilde{\mathbf{s}}_{\mathbf{i}z}$ commute with $\tilde{H}_\Lambda$. Consequently, they are conserved quantities and the Hilbert space of $\tilde{H}_\Lambda$ can be divided into numerous subspaces $\{V(M)\}$. Each of them is characterized by a quantum number $\tilde{S}_z=M$. Now, we show that in each subspace $V(M)$, the ground state $\tilde{\Psi}_0(M)$ of $\tilde{H}_\Lambda$ is unique.

Since Λ is bipartite, it is possible to introduce a unitary transformation $\hat{U}_0$, which transforms the spin operator $\tilde{\mathbf{s}}_{\mathbf{i}}=\mathbf{s}_{\mathbf{i}}$ at a site of A by

$$\hat{U}_0^\dagger s_{\mathbf{i}x}\hat{U}_0=-s_{\mathbf{i}x}; \quad \hat{U}_0^\dagger s_{\mathbf{i}y}\hat{U}_0=-s_{\mathbf{i}y}; \quad \hat{U}_0^\dagger s_{\mathbf{i}z}\hat{U}_0=s_{\mathbf{i}z}, \tag{7}$$

and leaves the spin operator at each site of B unchanged. Formally, $\hat{U}_0$ can be written as

$$\hat{U}_0=\exp\left(i\pi\sum_{\mathbf{i}\in A} s_{\mathbf{i}z}\right). \tag{8}$$

Under this transformation, Hamiltonian $\tilde{H}_\Lambda$ is transformed into

$$\tilde{H}'_\Lambda=\hat{U}_0^\dagger\tilde{H}_\Lambda\hat{U}_0=\sum_{\mathbf{i}\in A}\sum_{\mathbf{j}\in B}J_{\mathbf{ij}}(-\tilde{\mathbf{s}}_{\mathbf{i}x}\tilde{\mathbf{s}}_{\mathbf{j}x}-\tilde{\mathbf{s}}_{\mathbf{i}y}\tilde{\mathbf{s}}_{\mathbf{j}y}+\tilde{\mathbf{s}}_{\mathbf{i}z}\tilde{\mathbf{s}}_{\mathbf{j}z})$$
$$=\sum_{\mathbf{i}\in A}\sum_{\mathbf{j}\in B}J_{\mathbf{ij}}\left[-\frac{1}{2}(\tilde{\mathbf{s}}_{\mathbf{i}+}\tilde{\mathbf{s}}_{\mathbf{j}-}+\tilde{\mathbf{s}}_{\mathbf{i}-}\tilde{\mathbf{s}}_{\mathbf{j}+})+\tilde{\mathbf{s}}_{\mathbf{i}z}\tilde{\mathbf{s}}_{\mathbf{j}z}\right], \tag{9}$$

and subspace $V(M)$ is mapped into itself.

Choosing the natural basis of $V(M)$

$$\phi(\{\tilde{s}_{\mathbf{i}z}\})=|\tilde{s}_{1z},\tilde{s}_{2z},\cdots,\tilde{s}_{N_\Lambda z}\rangle \tag{10}$$

with $\Sigma_{\mathbf{i}\in\Lambda}\tilde{s}_{\mathbf{i}z}=M$, we are able to write $\tilde{H}'_\Lambda$ in a matrix whose off-diagonal elements are nonpositive. By the connectivity condition on $\{J_{\mathbf{ij}}\}$, it can be shown that $\tilde{H}'_\Lambda$ is irreducible in the sense that, for any pair of basis vectors ϕ' and ϕ'', there is an integer $M\geqslant 0$ such that

$$\langle\phi'|(\tilde{H}'_\Lambda)^M|\phi''\rangle\neq 0. \tag{11}$$

For such a matrix, the well-known Perron-Fröbenius theorem applies.[13] The theorem tells us that

(i) The ground state $\tilde{\Psi}'_0(M)$ of $\tilde{H}'_\Lambda$ in $V(M)$ is unique;

(ii) The expansion coefficients $\{a_\phi\}$ of $\tilde{\Psi}'_0(M)$ in terms of the basis vectors $\{\phi(\{\tilde{s}_{\mathbf{i}z}\})\}$ are positive.

As a consequence of (i), the ground state $\tilde{\Psi}_0(M)$ of $\tilde{H}_\Lambda$ in $V(M)$ is also nondegenerate since the Hamiltonian is unitarily equivalent to $\tilde{H}'_\Lambda$. By following the proof of the Lieb-Mattis theorem[10] and using conclusion (ii), one can easily show that the ground state $\tilde{\Psi}_0(M)$ of $\tilde{H}_\Lambda$ has a nonzero overlap with the ground state $\Phi_0(M)$ of the so-called Lieb-Mattis Hamiltonian

$$\mathcal{H}_\Lambda=\sum_{\mathbf{i}\in A}\sum_{\mathbf{j}\in B}\tilde{\mathbf{s}}_{\mathbf{i}}\cdot\tilde{\mathbf{s}}_{\mathbf{j}}. \tag{12}$$

In Hamiltonian $\mathcal{H}_\Lambda$, $J_{\mathbf{ij}}=1$ for any pair of sites $\mathbf{i}\in A$ and $\mathbf{j}\in B$. Therefore, $\tilde{\Psi}_0(M)$ and $\Phi_0(M)$ must have the same total spin and total spin Z component. In particular, the *absolute* ground states $\tilde{\Psi}_0$ of $\tilde{H}_\Lambda$ and the *absolute* ground states Φ_0 of $\mathcal{H}_\Lambda$ should have the same total spin, which can be easily calculated for Φ_0. In our case, it is equal to

$$\tilde{S}_0=|s-\tau|N. \tag{13}$$

Next, we show the coexistence of the antiferromagnetic and ferromagnetic long-range orders in the absolute ground state $\tilde{\Psi}_0$ of $\tilde{H}_\Lambda$. For this purpose, we shall exploit the positivity of the expansion coefficients $\{a_\phi\}$ of $\tilde{\Psi}'_0(M)$, the ground states of $\tilde{H}'_\Lambda$ (not $\tilde{H}_\Lambda$), again.

Take two arbitrary lattice points $\mathbf{h}$ and $\mathbf{l}$ of lattice Λ and consider the expectation value of operator $\tilde{s}_{\mathbf{h}+}\tilde{s}_{\mathbf{l}-}$ in $\tilde{\Psi}'_0(M)$. Since

$$\tilde{s}_{\mathbf{h}+}|\mathbf{h},\tilde{s}_z\rangle=\sqrt{\tilde{s}(\tilde{s}+1)-\tilde{s}_z(\tilde{s}_z+1)}|\mathbf{h},\tilde{s}_z+1\rangle,$$
$$\tilde{s}_{\mathbf{l}-}|\mathbf{l},\tilde{s}_z\rangle=\sqrt{\tilde{s}(\tilde{s}+1)-\tilde{s}_z(\tilde{s}_z-1)}|\mathbf{l},\tilde{s}_z-1\rangle, \tag{14}$$

we should have

$$\langle\tilde{\Psi}'_0(M)|\tilde{s}_{\mathbf{h}+}\tilde{s}_{\mathbf{l}-}|\tilde{\Psi}'_0(M)\rangle\geqslant 0 \tag{15}$$

by the positivity of the expansion coefficients $\{a_\phi\}$.

We now perform the inverse unitary transformation $\hat{U}_0^{-1}$ in the subspace $V(M)$. $\tilde{H}'_\Lambda$ is mapped back onto $\tilde{H}_\Lambda$ and $\tilde{\Psi}'_0(M)$ onto $\tilde{\Psi}_0(M)$. However, inequality (15) now reads

$$\langle\tilde{\Psi}_0(M)|\tilde{\mathbf{s}}_{\mathbf{h}+}\tilde{\mathbf{s}}_{\mathbf{l}-}|\tilde{\Psi}_0(M)\rangle\begin{cases}\geqslant 0, & \text{for } \mathbf{h},\ \mathbf{l}\in A \text{ or } B;\\ \leqslant 0, & \text{otherwise.}\end{cases} \tag{16}$$

according to Eq. (7). Consequently, the *transverse* spin-correlation function of the nondegenerate ground state of $\tilde{H}_\Lambda$ in each subspace $V(M)$ satisfies

$$g_T(\epsilon)\geq g_T(f) \tag{17}$$

for any admissible function $f(\mathbf{i})$. In particular, the inequality holds for $I(\mathbf{i})=1$. Applying inequality (17) to the $2\tilde{S}_0+1$ absolute ground states $\{\tilde{\Psi}_0(M),-\tilde{S}_0\leq M\leq\tilde{S}_0\}$ of $\tilde{H}_\Lambda$, we see that, if $\tilde{\Psi}_0$ has the transverse ferromagnetic long-range order, it must also support the transverse antiferromagnetic long-range order.

Next, we would like to extend inequality (17) to the longitudinal spin-correlation function $g_L(f)$. At the first glance, this extension seems straightforward since $\tilde{H}_\Lambda$ has SU(2) spin symmetry. However, as we show in the following, this problem demands a more careful thinking due to the high-spin degeneracy of the absolute ground state of $\tilde{H}_\Lambda$.

We notice that when the external fields are absent, the $2\tilde{S}_0+1$ degenerate absolute ground states of $\tilde{H}_\Lambda$ have the same statistical mechanics weight. In other words, they are experimentally indistinguishable. Therefore, if one tries to detect MLRO in these states by some means (such as the neutron-scattering technique), one can only obtain averaged data. This fact leads us to introduce the following definition.

Definition 2: Let $f(\mathbf{i})$ be an admissible function defined on lattice Λ. We define the averaged spin-correlation functions by

$$G_T(f)\equiv\frac{1}{2\tilde{S}_0+1}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M)|\tilde{S}_+(f^*)\times\tilde{S}_-(f)|\tilde{\Psi}_0(M)\rangle,$$

$$G_L(f)\equiv\frac{1}{2\tilde{S}_0+1}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M)|\tilde{S}_z(f^*)\times\tilde{S}_z(f)|\tilde{\Psi}_0(M)\rangle. \tag{18}$$

Since inequality (17) holds for each $\tilde{\Psi}_0(M)$, we immediately obtain

$$G_T(\epsilon)\geq G_T(I). \tag{19}$$

Furthermore, we have the following lemma.

Lemma: Identity

$$G_T(f)=2G_L(f) \tag{20}$$

holds for any admissible function $f(\mathbf{i})$ defined on Λ.

Proof: By the definitions of $\tilde{S}_+(f^*)$ and $\tilde{S}_-(f)$, $g_T(f)$ can be rewritten as

$$g_T(f)=\langle\tilde{\Psi}_0(M)|\tilde{S}_x(f^*)\tilde{S}_x(f)+\tilde{S}_y(f^*)\tilde{S}_y(f)|\tilde{\Psi}_0(M)\rangle + \frac{i}{N_\Lambda}\langle\tilde{\Psi}_0(M)|\sum_{\mathbf{h},\mathbf{l}\in\Lambda}f^*(\mathbf{h})f(\mathbf{l})[\tilde{s}_{\mathbf{h}y}\tilde{s}_{\mathbf{l}x} - \tilde{s}_{\mathbf{h}x}\tilde{s}_{\mathbf{l}y}]|\tilde{\Psi}_0(M)\rangle. \tag{21}$$

First, we simplify the last sum on the right-hand side of Eq. (21). When $\mathbf{h}\neq\mathbf{l}$, we have $[\tilde{s}_{\mathbf{h}x},\tilde{s}_{\mathbf{l}y}]=[\tilde{s}_{\mathbf{h}y},\tilde{s}_{\mathbf{l}x}]=0$. Therefore, $\tilde{s}_{\mathbf{h}y}\tilde{s}_{\mathbf{l}x}-\tilde{s}_{\mathbf{h}x}\tilde{s}_{\mathbf{l}y}$ is a Hermitian operator and hence, its expectation value in any state is a real quantity. On the other hand, since $\tilde{H}_\Lambda$ is a real matrix, its absolute ground states $\{\tilde{\Psi}_0(M),-\tilde{S}_0\leq M\leq\tilde{S}_0\}$ can be chosen as real state vectors. Consequently, the expectation value F of $\tilde{s}_{\mathbf{h}y}\tilde{s}_{\mathbf{l}x}-\tilde{s}_{\mathbf{h}x}\tilde{s}_{\mathbf{l}y}$ in $\tilde{\Psi}_0(M)$ must be a pure imaginary quantity because the operator is an imaginary matrix. This implies that $F\equiv 0$. Therefore, the sum on the right-hand side of Eq. (21) is reduced to

$$\frac{i}{N_\Lambda}\langle\tilde{\Psi}_0(M)|\sum_{\mathbf{h}\in\Lambda}|f(\mathbf{h})|^2[\tilde{s}_{\mathbf{h}y}\tilde{s}_{\mathbf{h}x}-\tilde{s}_{\mathbf{h}x}\tilde{s}_{\mathbf{h}y}]|\tilde{\Psi}_0(M)\rangle=\frac{M}{N_\Lambda}. \tag{22}$$

Next, we apply the unitary operator $\hat{U}_1=\exp((i\pi/2)\Sigma_{\mathbf{i}\in\Lambda}\tilde{s}_{\mathbf{i}z})$, which rotates each localized spin about the $\tilde{s}_{\mathbf{i}z}$ axis by an angle $\pi/2$, to rewrite the expectation value of $\tilde{S}_y(f^*)\tilde{S}_y(f)$ in $\tilde{\Psi}_0(M)$. We obtain

$$\begin{aligned}\langle\tilde{\Psi}_0(M)|\tilde{S}_y(f^*)\tilde{S}_y(f)|\tilde{\Psi}_0(M)\rangle &= \langle\tilde{\Psi}_0(M)|\hat{U}_1(\hat{U}_1^\dagger\tilde{S}_y(f^*)\hat{U}_1)(\hat{U}_1^\dagger\tilde{S}_y(f)\hat{U}_1)\hat{U}_1^\dagger|\tilde{\Psi}_0(M)\rangle\\ &=\langle\tilde{\Psi}_0(M)|\exp\left(-\frac{i\pi}{2}M\right)\tilde{S}_x(f^*)\tilde{S}_x(f)\exp\left(\frac{i\pi}{2}M\right)|\tilde{\Psi}_0(M)\rangle\\ &=\langle\tilde{\Psi}_0(M)|\tilde{S}_x(f^*)\tilde{S}_x(f)|\tilde{\Psi}_0(M)\rangle.\end{aligned} \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (21), we obtain

$$g_T(f)=2\langle\tilde{\Psi}_0(M)|\tilde{S}_x(f^*)\tilde{S}_x(f)|\tilde{\Psi}_0(M)\rangle+\frac{M}{N_\Lambda} \tag{24}$$

and

$$G_T(f)=\frac{2}{2\tilde{S}_0+1}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M)|\tilde{S}_x(f^*)\tilde{S}_x(f)|\tilde{\Psi}_0(M)\rangle. \tag{25}$$

Next, we apply the unitary operator $\hat{U}_2=\exp[(i\pi/2)\Sigma_{\mathbf{i}\in\Lambda}\tilde{S}_{\mathbf{i}y}]$ to the left-hand side of Eq. (25). Since $\hat{U}_2^\dagger\tilde{H}_\Lambda\hat{U}_2=\tilde{H}_\Lambda$, under $\hat{U}_2$, the $2\tilde{S}_0+1$ degenerate absolute ground states $\{\tilde{\Psi}_0(M),-\tilde{S}_0\leqslant M\leqslant\tilde{S}_0\}$ of $\tilde{H}_\Lambda$ will be transformed in terms of a $2\tilde{S}_0+1$ dimensional irreducible unitary representation $\underline{U}=(U_{mn})$ of the SU(2) group.[14] Consequently, we obtain

$$
\begin{aligned}
G_T(f)&=\frac{2}{2\tilde{S}_0+1}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M)|\hat{U}_2(\hat{U}_2^\dagger\tilde{S}_x(f^*)\hat{U}_2)(\hat{U}_2^\dagger\tilde{S}_x(f)\hat{U}_2)\hat{U}_2^\dagger|\tilde{\Psi}_0(M)\rangle\\
&=\frac{2}{2\tilde{S}_0+1}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\sum_{M_1=-\tilde{S}_0}^{\tilde{S}_0}\sum_{M_2=-\tilde{S}_0}^{\tilde{S}_0}\overline{u}_{M_1M}u_{M_2M}\langle\tilde{\Psi}_0(M_1)|\tilde{S}_z(f^*)\tilde{S}_z(f)|\tilde{\Psi}_0(M_2)\rangle\\
&=\frac{2}{2\tilde{S}_0+1}\sum_{M_1=-\tilde{S}_0}^{\tilde{S}_0}\sum_{M_2=-\tilde{S}_0}^{\tilde{S}_0}\delta_{M_1,M_2}\langle\tilde{\Psi}_0(M_1)|\tilde{S}_z(f^*)\tilde{S}_z(f)|\tilde{\Psi}_0(M_2)\rangle\\
&=\frac{2}{2\tilde{S}_0+1}\sum_{M_1=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M_1)|\tilde{S}_z(f^*)\tilde{S}_z(f)|\tilde{\Psi}_0(M_1)\rangle\equiv 2G_L(f).
\end{aligned}\tag{26}
$$

In the above derivation, we used the fact that $\underline{U}=(u_{mn})$ is a unitary matrix.

Combining identity (20) and inequality (19), we obtain

$$G_L(\epsilon)=\frac{1}{2}G_T(\epsilon)\geqslant\frac{1}{2}G_T(I)=G_L(I).\tag{27}$$

A little algebra yields

$$
\begin{aligned}
G_L(I)&=\frac{1}{(2\tilde{S}_0+1)N_\Lambda}\sum_{M=-\tilde{S}_0}^{\tilde{S}_0}\langle\tilde{\Psi}_0(M)|\tilde{S}_z\tilde{S}_z|\tilde{\Psi}_0(M)\rangle\\
&=\frac{2}{(2\tilde{S}_0+1)N_\Lambda}[\tilde{S}_0^2+(\tilde{S}_0-1)^2+\cdots+1^2]\geqslant\frac{\tilde{S}^2}{3N_\Lambda}.
\end{aligned}\tag{28}
$$

Therefore, as $\tilde{S}_0=N|s-\tau|=N/2=N_\Lambda/4$, we have

$$G_L(\epsilon)\geqslant G_L(I)\geqslant\beta N_\Lambda\tag{29}$$

with $\beta=\frac{1}{48}$. It implies that the absolute ground states of $\tilde{H}_\Lambda$, as well as Hamiltonian (1) defined on the one-dimensional chain, have both longitudinal ferromagnetic and antiferromagnetic long-range orders. Therefore, they are ferrimagnets.

Our proof is accomplished. **QED**.

In summary, we showed in this article that the global ground states of some ferrimagnetic Heisenberg model on a bipartite lattice Λ, in particular, the one-dimensional ferrimagnetic model studied in Refs. 7 and 8, have both ferromagnetic and antiferromagnetic long-range orders, with the antiferromagnetic long-range order being predominant. It is interesting to see that although the ''parents'' models, which have either spin-1 or spin-$\frac{1}{2}$ localized at each site, do not have the magnetic long-range orders, their mixture, the one-dimensional ferrimagnetic model, does support a ferrimagnetic MLRO.

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