Mesoscopic conductance and its fluctuations at a nonzero Hall angle

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We consider the bilocal conductivity tensor and the two-probe conductance and its fluctuations for a disordered phase-coherent two-dimensional system of noninteracting electrons in the presence of a magnetic field, including correctly the edge effects. Analytical results are obtained by perturbation theory in the limit $\sigma_{xx} \gg 1$. For mesoscopic systems the conduction process is dominated by diffusion, but we show that, due to the lack of time-reversal symmetry, the boundary condition for diffusion is altered at the reflecting edges. Instead of the usual condition that the derivative along the direction normal to the wall of the diffusing variable vanishes, the derivative at the Hall angle to the normal vanishes. We demonstrate the origin of this boundary condition in several approaches. Within the standard diagrammatic perturbation expansion, we evaluate the bilocal conductivity tensor to leading order in $1/\sigma_{xx}$, exhibiting the edge currents and the boundary condition. We show how to calculate conductivity and conductance using the nonlinear σ model with the topological term, to all orders in $1/\sigma_{xx}$. Edge effects are related to the topological term, and there are higher-order corrections to the boundary condition. We discuss the general form of the current-conservation conditions. We evaluate explicitly the mean and variance of the conductance, to leading order in $1/\sigma_{xx}$ and to order $(\sigma_{xy}/\sigma_{xx})^2$, and find that the variance of the conductance increases with the Hall ratio. Thus the conductance fluctuations are no longer simply described by the unitary universality class of the $\sigma_{xy}=0$ case, but instead there is a one-parameter family of probability distributions. Our results differ from previous calculations, which neglected σ_{xy}-dependent effects other than the leading-order boundary condition. In the quasi-one-dimensional limit, the usual universal result for the conductance fluctuations of the unitary ensemble is recovered, in contrast to results of previous authors. [S0163-1829(97)08128-9]

I. INTRODUCTION

In the past decade much attention has been given to the statistical properties of quantum conductors with complete phase coherence (with size L smaller than the phase coherence length L_{in}). A notable feature of such systems is the lack of self-averaging in their transport properties. In mesoscopic systems for which L is less than the localization length ξ, the conductance fluctuation amplitude (the standard deviation of the conductance), is found, at low magnetic field, to be of order 1, and to be independent of system size and the degree of disorder (but dependent on the dimensionality, shape, and overall symmetry properties of the system).[1–7] (Note that, in the present paper, all conductivities σ and conductances g have the factor e^2/h removed, so they are dimensionless in two dimensions; their dimensionful analogs are recovered by multiplying by e^2/h.) These universal conductance fluctuations (UCF's) have been well understood within the framework of perturbation theory[3,4,7] and the one-parameter scaling theory of quantum conductance.[8,9] The physics underlying the UCF is the long-ranged spatial correlation among the wave functions of the conduction electrons in the diffusive regime. The universality of the phenomenon has also stimulated the formulation of a random-matrix theory description of quasi-one-dimensional (quasi-1D) Refs. 10–13 and quasi-zero-dimensional (quasi-0D) conductors,[14,15] which reproduce quantitatively the results of diagrammatic perturbation theory. In earlier perturbative work the effect of a magnetic field was only included through the introduction of an appropriate Aharonov-Bohm phase in the zero-field propagator. This leads to the elimination of Cooperon contributions to conductance fluctuations, causing a crossover from the so-called orthogonal to the unitary ensemble, and a consequent reduction of the variance by a factor of two.[4,7] In the quasi-1D case, this result is also recovered by the random-matrix approaches.[12,13]

In two dimensions, the interplay of quantum interference effects and magnetic field leads to the quantum Hall effect in high magnetic field B.[16] In mesoscopic samples, conductance fluctuations persist in fairly high magnetic field $\omega_c\tau_0 > 1$, where ω_c is the cyclotron frequency, τ_0 is the elastic scattering time in zero magnetic field, and the fluctuation amplitude remains comparable to the low-field limit.[17–23] For B fields sufficiently high that quantization of the Hall conductance sets in, the fluctuations are strongly suppressed in the plateau regions, but reappear in both longitudinal and Hall resistance in the transition regions between plateaus. It is therefore of theoretical interest to generalize the theory for conductance fluctuations to all fields. At $\omega_c\tau_0 > 1$, the trajectories of the electrons are significantly influenced by the Lorentz force between successive scattering events, and the dynamical effect of the magnetic field must be treated. The diffusion at high field occurs by a different mechanism from the low-field regime. For the short-ranged random potential model, the center of the cyclotron orbit hops a length of order the cyclotron radius R_c whenever it encounters a scattering center; therefore R_c plays the role of the mean free path l. The bare conductivity σ_{xx}^0 in the middle of the Nth Landau level is of order $(2N+1)/\pi$ (Ref. 24) ($N=0, 1, \ldots$). Despite the altered nature of the micro-

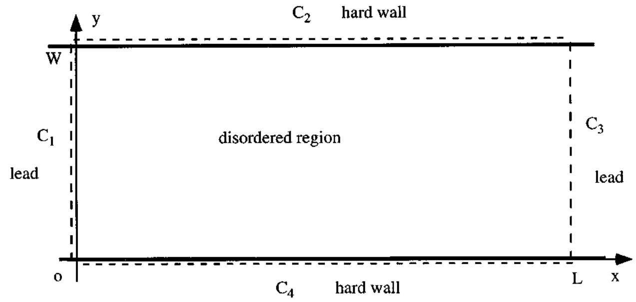

FIG. 1. The two-probe geometry.

scopic diffusion, a unified treatment of mesoscopic conductance fluctuations in relatively high fields is possible because, even at $\omega_c\tau_0>1$, there exists a perturbative regime where the transport process is dominated by diffusion. As long as there are many Landau levels occupied, $1/\sigma^0_{xx}$ or $1/k_F l$ serves as a small parameter, and perturbation theory is still useful. Previous perturbation theories have shown that the weak localization correction to σ_{xx} for the unitary class is of order $-(\sigma^0_{xx})^{-1}\ln(L/l)$.[25–27] The localization correction is relatively small for systems with L less than the crossover length $\xi_{\rm pert}=le^{(\sigma^0_{xx})^2}$ (for σ^0_{xx} large). For $l<L<\xi_{\rm pert}$, the conductance fluctuations are expected to be similar to the UCF. At $L>\xi_{\rm pert}$, the renormalization-group flows for the system, driven by nonperturbative effects,[28–31] carry it either to one of the localization fixed points where $\sigma_{xx}\to 0$ and σ_{xy} becomes quantized, or to one of the nontrivial fixed points where σ_{xy} is a half-integer and σ_{xx} approaches a universal value. Numerical work finds that the conductance fluctuations in the critical regimes[20–23] have a different distribution from the UCF, and one would expect this distribution to be beyond the scope of a perturbative treatment. We comment further on the critical regime in the conclusion, Sec. VI.

In this paper we study the conductance and its variance for a two-probe geometry in two dimensions in the presence of disorder and magnetic field, and in the perturbative regime ($l\ll L\ll\xi_{\rm pert}$). As shown in Fig. 1, the edges of the sample are defined by a hard-wall confinement potential and the two ends of the sample are connected to highly conducting leads. The first analytic work on this problem was by two of the authors (Xiong and Stone[32]), in which they generalized the previous diagrammatic perturbative techniques[27] to treat the conductance fluctuations. At the level of the self-consistent Born approximation, the only effect they found of the magnetic field B was a field-dependent diffusion constant $D(B)$. Since the value of the diffusion constant cancels from the conductance fluctuations, they found no effect of the magnetic field on the amplitude of the two-probe conductance fluctuations (other than the well-known factor of two reduction associated with the crossover to the unitary ensemble), although the correlation field B_c, which determines the spacing of the fluctuations in magnetic field, was found to increase with increasing field in a manner consistent with experiment.[18] The reason for the increase is that for systems with $L>L_{\rm in}$, $B_c\sim\phi_0/L^2_{\rm in}$, where $L^2_{\rm in}=D(B)\tau_{\rm in}$ ($\phi_0=h/e$, and $\tau_{\rm in}$ is the inelastic scattering time). Since the diffusion constant $D(B)$ decreases with increasing magnetic field (a result reviewed in Sec. II), B_c increases. Although this conclusion about the correlation field is basically correct, the conclusion concerning the fluctuation amplitude is now understood to be correct only for a periodic boundary condition in the transverse direction and must be revised for the case of a system with reflecting edges. As discovered independently by Khmel'nitskii and Yosefin (KY), Maslov and Loss (ML), and one of the present authors (Read),[33–35] the boundary condition is modified from the vanishing of the normal derivative of the diffusing variable to the vanishing of the derivative at an angle to the normal. KY and Read further showed that this angle is the Hall angle $\theta_H=\tan^{-1}\sigma^0_{xy}/\sigma^0_{xx}$, where σ^0_{xy} is the bare Hall conductivity. These authors pointed out the possibility that the mesoscopic conductance fluctuations may depend on magnetic field due to the boundary condition. KY and ML attempted to evaluate this dependence both numerically and analytically. Simulations performed for the two-probe conductance of small systems in the nonquantized regime[34] show that the maximum fluctuation amplitude appears toward the bottom of the Landau levels where the Hall ratio is large, indicating some dependence on the Hall ratio. However, the analytic calculations by KY and ML (Refs. 33 and 34) do not agree with our present results since these authors merely modified the diffusion propagator in previous expressions for UCF diagrams. Like KY, we find that σ^0_{xy} enters not only the boundary condition but also the current vertex. Moreover the altered boundary condition permits new diagrams to occur which, roughly speaking, describe interference effects associated with a σ^0_{xy}-dependent ''interaction'' of the diffusion modes. These diagrams, which were not considered by ML and KY, must be included when edges are present. We evaluate the two-probe conductance and its variance to leading order in $1/\sigma^0_{xx}$ and to order γ^2, where $\gamma=\sigma^0_{xy}/\sigma^0_{xx}$. For wide samples with $W\sim L$, where L and W are now the length and width of the sample, we find that the variance does not depend on σ^0_{xx} and σ^0_{xy} individually, but does depend on γ. The variance increases as γ^2 for small γ, and hence is no longer independent of magnetic field (although it is still independent of size in this order in $1/\sigma^0_{xx}$, and has no direct dependence on the mean free path). Interestingly, however, in the quasi-1D limit ($W\ll L$), the Hall ratio is absorbed into an effective 1D conductivity which cancels in all diagrams. Therefore in this limit the UCF result of the unitary class is recovered (in contrast to the claims by KY and ML that this is modified for $\gamma\neq 0$). The implication is that quasi-1D conductance fluctuations are still described by the standard random-matrix theory of disordered conductors, but that the 2D fluctuations, even in perturbation theory, define a family of random-matrix ensembles parametrized by the Hall ratio γ.

In this paper, we use the disorder-averaged diagrammatic approach and the field-theoretical approach in a complementary way. To study transport properties of a system with phase coherence, the appropriate starting point is the bilocal conductivity tensor $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')$. In Sec. II, we evaluate the mean $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')$ to leading order in $1/\sigma^0_{xx}$ using the diagrammatic approach, and demonstrate the microscopic origin of the edge contributions. In Sec. III, we set up the field-theoretical formalism for the evaluation of linear-response functions. We discuss the connection between the tilted boundary condition and the nonlinear σ-model action with a topological term proportional to σ^0_{xy}.[28,30] Previously it was known that this topological term is crucial to the critical transition of the quantum Hall effect at large length scale ($L \gg \xi_{\text{pert}}$).[29,30] For a system with reflecting edges, this term is a nonvanishing surface term, which *does* influence transport properties in perturbation theory, valid when $L \ll \xi_{\text{pert}}$, through various σ^0_{xy}-dependent boundary contributions. We show how to incorporate an external source field so that the moments of the bilocal conductivity and conductance can be calculated, and show the extent to which current conservation is maintained in this system. In Secs. IV and V, the conductance and its variance are calculated by expanding in power series in $1/\sigma^0_{xx}$ and γ. In the remainder of Sec. I we begin the discussion of the main ideas and summarize our results. The details of the calculations, and further discussions, are given in later sections.

A. Local conductivity parameters, two-probe conductance, and edge states

In this paper we focus on calculations of the two-probe conductance in a magnetic field. It is, however, possible to generalize our calculations to treat the conductance matrix of a multiprobe conductor, as has been done previously for zero (or weak) field.[36,37] Two-probe conductance describes an experimental setup in which voltage measurements are made only between the current source and sink and not between distinct voltage probes as in a typical Hall measurement (two-probe measurements are not uncommon for mesoscopic conductors, because of the difficulty of making multiple contacts). Therefore such a measurement cannot separately determine $\sigma_{xx}(B)$ and $\sigma_{xy}(B)$. In this subsection, we show how the assumption of a local form for the conductivity in a system with edges leads to the result that the two-probe conductance is approximately proportional to σ_{xx} when $\sigma_{xx} \gg |\sigma_{xy}|$, and to $|\sigma_{xy}|$ when $|\sigma_{xy}| \gg \sigma_{xx}$. Interestingly, such an argument already indicates the appearance of the tilted boundary condition which affects the conductance fluctuations as well.

We wish to find the current produced in linear response to an applied electric field. We will consider the two-probe conductance which results from assuming a local form for the conductivity,

$$j_\mu(\mathbf{r}) = \frac{e^2}{h}\sigma^0_{\mu\nu}\mathcal{E}_\nu(\mathbf{r}), \tag{1.1}$$

where, due to the macroscopic homogeneity of the sample and Onsager relations, the conductivity parameters obey $\sigma^0_{xx} = \sigma^0_{yy}$ and $\sigma^0_{xy} = -\sigma^0_{yx}$. It is a common misconception that $\boldsymbol{\mathcal{E}}$ in this formula is the same as the applied electric field **E**. In fact, in general, $\boldsymbol{\mathcal{E}}$ in this formula should be interpeted as the *electromotive* field, that is, as minus the gradient of the electrochemical potential $V = \phi - \mu/e$, where ϕ is the electric potential, and μ the chemical potential (more generally, $\boldsymbol{\mathcal{E}} = \mathbf{E} + \nabla\mu/e$). In this paper, we will neglect electron-electron interactions of all kinds, so ϕ can be viewed as the externally applied electric potential. (If it is desired to include Coulomb interactions through a self-consistent potential, then ϕ is the total electric potential, the sum of the externally applied potential and the potential produced by the electrons in the system; the latter potential is determined by the change in the expectation value of the density, in response to the external field, through the 3D Poisson equation. This is distinct from similar-looking equations below, which have the form of the 2D Laplace equation. Other short-range interaction effects would contribute to μ.) The chemical potential in the above expressions is defined in terms of a local quasiequilibrium (in the presence of a nonzero response current), which must be established by inelastic effects. Hence this formulation is only valid on scales greater than the inelastic length L_{in} (this of course requires that there be some interaction between the electrons). It is only in this sense, which implies the absence of effects due to single-particle phase coherence, that Equation (1.1) is a classical formula; it does not require that all effects of quantum mechanics be neglected. Eq. (1.1) (when valid) is the most convenient form for expressing the linear response, since a voltage measurement determines electrochemical potential differences. It does not imply that a local relation exists between the electromotive and electric fields. As the chemical potential is determined by the local conditions, in particular by the local density, and that density is affected by the transport, which in our case will be diffusive, this relation is not local. Thus the current response to the *electric* field is actually nonlocal, as in Ref. 38, even in this ''classical'' case. We will return to this in Sec. I D.

We will now calculate the two-probe conductance of a rectangular sample with insulating edges connected to conducting leads at each end. The potential in the leads is assumed to be held at constant values V_1 and V_2. At every point on the insulating edges the normal current must vanish. Using Eq. (1.1) and $\boldsymbol{\mathcal{E}} = -\nabla V$, it follows that $(\partial_y - \gamma\partial_x)V(\mathbf{r}) = 0$, i.e., the electromotive field at the Hall angle to the normal must vanish. Thus in this case the appearance of a tilted boundary condition on the electrochemical potential follows simply from the fact that the field is tilted from the current by the Hall angle. From the continuity equation, $\nabla\cdot\mathbf{j} = 0$, and one finds that $\nabla^2 V = 0$ in bulk. Solving the Laplace equation for $V(\mathbf{r})$ with fixed voltages at the two ends and the tilted boundary conditions at the edges is not a simple exercise, but has been done for this 2D rectangular geometry by conformal mapping[39] (we give a solution in another form in Sec. I D). From this solution one can obtain the two-probe conductance for an arbitrary Hall ratio; however, here we analyze only its limiting behavior. The conductance g^0 can be found by integrating the current over a cross section perpendicular to the current flow:

$$g^0 = -\sigma^0_{xx}\int_0^W dy(\partial_x + \gamma\partial_y)V(\mathbf{r})/(V_2 - V_1). \tag{1.2}$$

First consider the case $\gamma \to 0$; it is then convenient to integrate over all transverse cross sections in the sample and divide by the sample length L. The integral then just evaluates the voltage at the two ends where it is fixed, yielding the familiar Ohmic result

$$\lim_{\gamma\to 0} g^0 = \sigma^0_{xx}W/L. \tag{1.3}$$

In the opposite limit $\sigma^0_{xx}\to 0$, $|\gamma|\to\infty$, the boundary condition implies that everywhere along the edge $\mathcal{E}_x=0$, i.e., there is no voltage drop along the edge, and the voltage must remain equal to V_1 along one edge and V_2 at the other edge except at singularities at diagonally opposite corners (where it jumps between V_1 and V_2). The transverse potential drop is equal to $(V_1-V_2)\mathrm{sgn}\gamma$, therefore

$$\lim_{|\gamma|\to\infty} g^0 = -\lim_{|\gamma|\to\infty}\sigma^0_{xx}\int_0^W dy\ \gamma\partial_y V(\mathbf{r})/(V_2-V_1)=|\sigma^0_{xy}|. \tag{1.4}$$

So, using the classical local conductivity, the two-probe conductance changes from being dominated by the longitudinal conductance at small γ to being dominated by the Hall conductance at large γ. We can also solve the limit $L/W\to\infty$ with γ fixed; for a fixed current, the voltage drop is dominated by the part of the geometry a distance greater than W from the ends, in which the current distribution is essentially independent of y, and one finds that $g^0\to[\{(\sigma^0_{xx})^2+(\sigma^0_{xy})^2\}/\sigma^0_{xx}]W/L\equiv(1/\rho^0_{xx})W/L$. Note that the crossover to this behavior occurs at a value of L/W that depends on γ. In contrast to the above results for a rectangular sample with edges, for periodic boundary conditions in the transverse direction, which is equivalent to transport along a cylinder, the conductance is always $\sigma^0_{xx}W/L$, for any value of γ. We note that this geometry is equivalent, through a conformal mapping, to the Corbino disk geometry, in which the voltage drop is radial, and, since the equations are conformally invariant, results for the cylinder also apply to the disk. Although the local formulation cannot be used in the fully phase-coherent (‘‘quantum’’) case where $L_{\mathrm{in}}\gg L$, we expect that the physics illustrated by this argument will be relevant to the average quantum conductance.

Indeed, in the *quantized* Hall regime $L\gg\xi$, previous arguments based on the Landauer formula for two-probe conductance in terms of transmission coefficients have noted the relation between two-probe conductance and Hall conductance. These approaches assume that the incident and outgoing channels are N-edge states.[40,41] These edge states are analogous to classical skipping orbits advancing in one sense along each edge and occasionally being scattered into the bulk. Any actual calculation of these transmission coefficients will be equivalent to evaluating the bilocal conductivity between the two ends;[42] however, physical arguments are made that in high field the backscattering of edge states will be suppressed giving perfect edge transmission and $g=N=|\sigma_{xy}|$; i.e., the two-probe conductance is equal to the Hall conductance which takes its quantized value. (Since $\sigma_{xx}=0$ this is consistent with the classical result above). Although physically appealing, such arguments assume that the interaction of bulk and edge states can be ignored. However, in a bulk 2D sample it is known that it is only localization effects which prevent edge states from backscattering through the bulk states; they thus require that $W\gg\xi$, the localization length. We note that away from the critical values E_{cN} ($N=0, 1, \ldots$) of the Fermi energy, which lie near the center of the Nth Landau level, $\xi\simeq\xi_{\mathrm{pert}}$, while $\xi\to\infty$ as $E_F\to E_{cN}$ for each N. We cannot analytically calculate the conductance in this nonperturbative regime L, $W>\xi_{\mathrm{pert}}$; however, our results below do describe samples for which

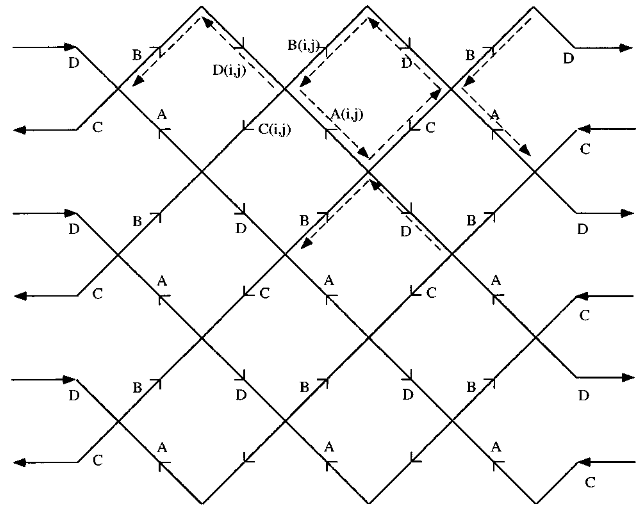

FIG. 2. The Chalker-Coddington network model. Each unit cell contains four distinct links, A, B, C, and D. The tilted boundary condition arises from the fact that along each link, the random walk is along only one direction.

L, $W\ll\xi_{\mathrm{pert}}$, and localization effects have not inhibited backscattering of edge states. The key ingredient to describe the edge-bulk coupling in the perturbative regime is the tilted boundary condition for diffusion which we derive in the next subsections. In the fully phase-coherent case, the boundary condition is, however, modified further.

B. Classical network model, edge states, and tilted boundary condition on diffusion

Khmel'nitskii and Yosefin[33] (KY), Maslov and Loss[34] (ML), and Read[35] have previously obtained the boundary condition on the diffusion process in high magnetic field. KY's argument appears to be related in part to the classical conductivity formulas reviewed in Sec. I A. ML considered the effect of the edge in high field on the microscopic diffusion process using a Boltzmann equation approach. They found that the tendency to skip in only one direction when colliding with the edge does lead to the tilted boundary condition on the diffusion equation. They expressed the tilt angle in terms of the ratio of the mean free path along the edge and the bulk mean free path. Read[35] used the nonlinear sigma model approach, which will be described later in this paper. KY and Read were able to identify the tilt angle as the Hall angle. In this subsection we rederive the boundary condition in a particularly transparent manner using a classical version of the network model for high-field transport introduced by Chalker and Coddington.[43] In this case one can also see immediately that the tilt angle is the Hall angle.

The original Chalker-Coddington model[43,44] describes the quantum tunneling between the semiclassical orbits along the equipotential contours of the smooth random potential (see Fig. 2). To derive the diffusive behavior of the probability density in this model we will neglect interference effects and describe each node by the probability that a walker approaching it makes a step to the right (R) or left (T); $T+R=1$. (This simplified model has been considered by several earlier authors;[45–47] it is essentially classical, and could serve as a lattice realization of the classical behavior discussed in Sec. I A.) The links of the lattice can be divided into four sublattices $\alpha=A$, B, C, and D (see Fig. 2), and each unit cell of the lattice contains one of each of the four classes of links. The nearest-neighbor separation is a. We use $\rho_\alpha(i,j,t)$ to denote the probability of being at link α, site i,j at time t. Assuming that it takes time τ for a particle to move from one link to the next, one can define a random-walk problem on the network and write down probability evolution equations:

$$\rho_B(i,j,t+\tau)=\rho_D(i,j,t)T+\rho_A(i,j,t)R,$$

$$\rho_C(i,j,t+\tau)=\rho_A(i,j,t)T+\rho_D(i,j,t)R,$$

$$\rho_A(i,j,t+\tau)=\rho_B(i,j-1,t)T+\rho_C(i+1,j,t)R,$$

$$\rho_D(i,j,t+\tau)=\rho_C(i,j+1,t)T+\rho_B(i-1,j,t)R. \quad (1.5)$$

This problem differs from the usual random walk in the respect that on each link the walk is in only one direction, hence breaking time-reversal symmetry.

The above equation can be diagonalized in Fourier space. One can show that the long-time large-distance modes have a diffusive spectrum

$$-i\omega_k=Dk^2$$

for $\omega\ll 1/\tau$ and $k\ll 1/a$, where D, the diffusion constant, is given by $D=\frac{1}{4}a^2RT/(R^2+T^2)\tau$. The associated eigenmodes are of the following form in Fourier space:

$$\begin{pmatrix}\rho_A(k)\\ \rho_B(k)\\ \rho_C(k)\\ \rho_D(k)\end{pmatrix}=\begin{pmatrix}1+\sigma_{xx}^0 ik_xa+\sigma_{xy}^0 ik_ya\\ 1\\ 1+(\sigma_{xx}^0-\sigma_{xy}^0-1)ik_xa+(\sigma_{xx}^0+\sigma_{xy}^0)ik_ya\\ 1+(-\sigma_{xy}^0-1)ik_xa+\sigma_{xx}^0 ik_ya\end{pmatrix},$$

where, in anticipation of our discussion below, we identify the two constants $\sigma_{xx}^0=RT/(R^2+T^2)$ and $\sigma_{xy}^0=-T^2/(T^2+R^2)$ as the the bare longitudinal and Hall conductivities of this model. Note that these conductivities satisfy the ''semicircle relation''

$$\sigma_{xx}^2+(\sigma_{xy}+N+1/2)^2=\tfrac{1}{4}, \quad (1.6)$$

with $N=0$ here in the case of the lowest Landau level, which has been claimed to be a general, exact result in the quantum Hall effect.[47] In real space, all four components of the probability distribution satisfy the same coarse-grained equation:

$$-D\nabla^2\overline{\rho(\mathbf{r},t)}=-\partial_t\overline{\rho(\mathbf{r},t)}. \quad (1.7)$$

At the absorbing ends, the particle moves away from the tunneling region with constant velocity. The fact that the re-entry probability is zero gives the boundary condition at the leads,

$$\overline{\rho}=0 \quad \text{in the lead.} \quad (1.8)$$

Since the particles always move in the direction of the arrow on a link, the density on a link can also be considered as its current. The differences among the four components define the diffusion current densities, which are suitable for coarse graining. For instance, we can define $j_y=[\rho_B(i,j)-\rho_D(i+1,j)]/a$ and $j_x=[\rho_D(i,j)-\rho_C(i,j)]/a$. For the low-frequency and long-wavelength modes, we can show that $j_y=-(\sigma_{xx}^0\partial_y-\sigma_{xy}^0\partial_x)\overline{\rho(\mathbf{r},t)}$. Along the reflecting walls, at zero frequency, the normal current is zero, e.g., $\rho_B(i,j)-\rho_D(i+1,j)=0$ at top edge, which gives the boundary condition

$$(\partial_n-\gamma\partial_t)\overline{\rho}=0, \quad (1.9)$$

where $\hat{\mathbf{n}}$ is the outward normal, $\hat{\mathbf{t}}=\hat{\mathbf{n}}\times\hat{\mathbf{z}}$ is the tangential direction of the boundary, and $\gamma=-T/R=\sigma_{xy}^0/\sigma_{xx}^0$ with our above identification of the bare conductivities in the model. These expressions for the current density are of the form used by KY and in Sec. I A.

It is interesting to note that, in the network model, the definition for the local diffusion current density is not unique. One can also define, e.g., $j_x'=[\rho_D(i,j)-\rho_C(i,j+1)]/a$, $j_y'=[\rho_B(i,j)-\rho_D(i,j)]/a$. j_y' and j_y differ by a total derivative term $\partial_x\overline{\rho}$, and the corresponding x components differ by $-\partial_y\overline{\rho}$ and by a δ-function boundary term $\overline{\rho(x,W)}\,\delta(y-W)-\overline{\rho(x,0)}\,\delta(y)$. In the presence of such an edge current, the boundary condition that ensures ''current conservation,'' which in the interior is the requirement $\nabla\cdot\mathbf{j}'=0$, becomes

$$\partial_x\int_{W-0^+}^{W+0^+}dy'\,j_x'(x,y')-j_y'(x,W)=0 \quad (1.10)$$

at the top edge, which is still equivalent to Eq. (1.9). These forms for the current density are similar to those of ML. The edge contribution ensures that the total current through a cross section transverse to the x direction is the same, whichever definition of current is used.

One can see that in the long-time, large-distance limit the lack of time-reversal symmetry affects the diffusion process only through the tilted boundary condition, which is present only because of the edges. As we will show explicitly below, these boundary conditions, although derived from a lattice here for convenience, are quite general for conduction with broken time-reversal symmetry. As one might hope, in the long-time, large-distance limit the microscopic details of different models cease to matter.

As noted just above, the boundary condition derived in Eq. (1.7) applies for $\gamma=-T/R$, the single-node transmission and reflection coefficients of the classical network model. We now must further justify identifying this ratio as the bare Hall ratio. There are two approaches to this. There is no applied electric field or electric potential in this problem so far, so one may simply define the chemical potential on each link as proportional to the current (or density) there (in analogy to the Landauer approach for the entire sample). This was done by Kucera and Strěda,[45] and leads to the formulas for σ_{xx}^0 and σ_{xy}^0 given above. In our view a somewhat more satisfactory method is to calculate the steady-state current for the network under periodic transverse boundary conditions (i.e., a cylindrical system), when current is injected at only one end of the network, with unit current on each incoming link at that end. This is just the appropriate time-independent solution of Eq. (1.5); in the absence of edges the solution is the linear $k=0$ mode:

$$\rho_B(x,y)=b_1x+b_0,$$

where $b_1=1/(L/a+\sigma_{xx}^0)$, $b_0=1-[\sigma_{xx}^0/(L/a+\sigma_{xx}^0)]$, and

$$\begin{pmatrix} \rho_A(x,y)-\rho_B(x,y) \\ \rho_C(k)-\rho_B(x,y) \\ \rho_D(x,y)-\rho_B(x,y) \end{pmatrix} = \begin{pmatrix} \sigma_{xx}^0 b_1 \\ (\sigma_{xx}^0-\sigma_{xy}^0-1)b_1 \\ (-\sigma_{xy}^0-1)b_1 \end{pmatrix},$$

where the constants σ_{xx}^0 and σ_{xy}^0 are as defined above Eq. (1.6). The above solution has a uniform current distribution with a total longitudinal current, for $L \gg a$,

$$I_x=g=\frac{W}{L}\frac{\sigma_{xx}^0}{1+(\sigma_{xx}^0 a/L)}\simeq\frac{W}{L}\sigma_{xx}^0,$$

and a total transverse current, circulating around the cylinder,

$$I_y\simeq-\sigma_{xy}^0,$$

justifying the interpretation we have assumed.

To solve for the current in the presence of edges is much more difficult, even in this classical model. We will see, however, that on large scales the problem is equivalent to that solved by Rendell and Girvin[39] and discussed in Sec. I A. The reason for the similarity is already clear; in the classical network model, we assumed that the chemical potential was proportional to the density, and, when the external electric field is zero, the coarse-grained current density was related to this potential in the same way in both cases.

C. Continuum action and propagator

The diffusive behavior generated by the classical network model of Sec. I B can be reproduced in a simple continuum field theory. Consider the action

$$S_0=-\frac{\sigma_{xx}^0}{4}\int d^2r\,\partial_\mu z\,\partial_\mu\bar{z}-\frac{\sigma_{xy}^0}{4}\int d^2r\,\epsilon_{\mu\nu}\partial_\mu z\,\partial_\nu\bar{z}. \tag{1.11}$$

Here $z(x)$ is a complex scalar field, $\bar{z}$ is its complex conjugate, and the geometry is the same as in Sec. I B. The second term is clearly a total derivative. To obtain the equations of motion and boundary conditions in this model, we first rewrite S_0 as

$$\frac{\sigma_{xx}^0}{4}\int d^2r[z\nabla^2\bar{z}]-\frac{\sigma_{xx}^0}{4}\int d^2r[\partial_\mu\{z(\partial_\mu+\gamma\epsilon_{\mu\nu}\partial_\nu)\bar{z}\}]. \tag{1.12}$$

Then one can see that the second term is again a total derivative, and that it can be written as a boundary term. Taking the functional derivative with respect to z, we obtain the equation of motion for $\bar{z}$:

$$\begin{aligned} -\frac{\sigma_{xx}^0}{4}\nabla^2\bar{z}(\mathbf{r})&=0 \quad \text{in the bulk,} \\ (\partial_n-\gamma\partial_t)\bar{z}(\mathbf{r})&=0 \quad \text{at the reflecting walls.} \end{aligned} \tag{1.13}$$

Similarly for z, we obtain

$$\begin{aligned} -\frac{\sigma_{xx}^0}{4}\nabla^2 z(\mathbf{r})&=0 \quad \text{in the bulk,} \\ (\partial_n+\gamma\partial_t)z(\mathbf{r})&=0 \,\text{at the reflecting walls.} \end{aligned} \tag{1.14}$$

At the absorbing boundaries, we simply impose $z=\bar{z}=0$. These equations are equivalent to the zero-frequency limit of those in Sec. I B. We observe that the boundary conditions on z and on $\bar{z}$ are not consistent, which means we cannot find any nonzero configurations $z(\mathbf{r})$ that satisfy both these conditions simultaneously. This is due to the fact that the differential operator, that appears in Eq. (1.12) between z and $\bar{z}$, is not self-adjoint. As noted by KY and ML, equations similar to Eqs. (1.13) and (1.14) define the right and left eigenfunctions of this operator; these eigenfunctions are not complex conjugates of each other. The eigenfunctions are not simple to obtain in our geometry, because the x and y dependence does not separate, when $\gamma\neq0$. The analysis of the eigenfunctions in KY and ML ignores the boundary conditions at $x=0$, L, and is appropriate only for $L/W\to\infty$.

To obtain a (zero-frequency) diffusion propagator, we use S_0 as the action in a functional integral, and define $d(\mathbf{r},\mathbf{r}')$ by

$$\begin{aligned} d(\mathbf{r},\mathbf{r}')&\equiv\frac{\sigma_{xx}^0}{4}\langle\langle\bar{z}(\mathbf{r})z(\mathbf{r}')\rangle\rangle_0 \\ &\equiv\frac{\sigma_{xx}^0}{4}\int D[z,\bar{z}]\,\bar{z}(\mathbf{r})z(\mathbf{r}')e^{S_0(z,\bar{z})}/Z_0. \end{aligned} \tag{1.15}$$

[Here Z_0 is the same functional integral without $z(\mathbf{r})$ and $\bar{z}(\mathbf{r})$ inserted.] Then $d(\mathbf{r},\mathbf{r}')$ satisfies

$$\begin{aligned} -\nabla^2 d(\mathbf{r},\mathbf{r}')=-\nabla'^2 d(\mathbf{r},\mathbf{r}')&=\delta(\mathbf{r}-\mathbf{r}') \quad \text{in the bulk,} \\ (\partial_n-\gamma\partial_t)d(\mathbf{r},\mathbf{r}')&=0, \quad \mathbf{r} \text{ at the reflecting walls,} \\ (\partial_n'+\gamma\partial_t')d(\mathbf{r},\mathbf{r}')&=0, \quad \mathbf{r}' \text{ at the reflecting walls.} \end{aligned} \tag{1.16}$$

The propagator exists, and can be shown to satisfy the stated conditions. One notices that the propagator is not symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$, since the boundary conditions for $\mathbf{r}$ and $\mathbf{r}'$ at the edges differ by a sign. In principle, the propagator can be evaluated by expanding in the right and left eigenfunctions, as in KY and ML; however, as these are not readily available for our geometry, we will just define it by Eqs. (1.16) (see also Sec. III D below).

The action S_0 has also appeared in the literature in connection with an open string with opposite electric charges attached to the ends, in a uniform magnetic field.[48] The boundary conditions have also appeared there, along with explicit results for the diffusion propagator d in some geometries simpler than ours. We will see later (in Sec. I E and in Sec. III) that Eq. (1.11) also arises as the lowest-order part of the nonlinear sigma model action. In fact, the full Chalker-Coddington model with phase coherence[43] is related to the nonlinear sigma model[49] in a manner closely analogous to the relation between the models discussed in Sec. I B and here, which are just the linearized versions. In Sec. III, we will also discuss the expressions for the currents, like those in Sec. I B, from the point of view of the nonlinear sigma model action.

D. Bilocal conductivity tensor and conductance

In Sec. I A above we used a classical formulation of the conductivity, and worked out some of the consequences for the two-probe conductance in a high magnetic field. In this subsection we introduce a full quantum formulation for the bilocal conductivity tensor and the conductance in a disordered phase-coherent system in order to treat both the average quantum conductance and its variance.

For a quantum conductor with phase coherence, that is, when $L_{\rm in}$ is larger than the sample size, the electron wave function is sensitive to the external field in the entire space. Equation (1.1) cannot be used, as there is no definition of the chemical potential within the sample. We apply standard linear-response theory to a finite disordered region (denoted by A) with Fermi energy E_F, connected to perfect leads held at fixed voltages, which induce a local electric field in the disordered region the detailed form of which is not relevant. (For the two-probe case, the sample occupies the region $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant W$; see Fig. 1.) Following the treatment of Baranger and Stone,[42] one finds that there is a nonlocal relation between the current response and the applied electric field,

$$j_\mu(\mathbf{r}) = \frac{e^2}{h}\int_A d^2r'\,\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')E_\nu(\mathbf{r}'), \tag{1.17}$$

where the bilocal conductivity tensor $\sigma(\mathbf{r},\mathbf{r}')$ (which has dimensions of inverse length squared) at $T=0$ can be expressed in terms of a pair of Green's functions:[42]

$$\begin{aligned}\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}') = {}& \frac{\hbar^4}{4m_e^2}[G^+(\mathbf{r},\mathbf{r}';E_F)\overleftrightarrow{D}^*_\mu \overleftrightarrow{D}'_\nu G^-(\mathbf{r}',\mathbf{r};E_F)] \\ & - \frac{\hbar^4}{4m_e^2}\int_{-\infty}^{E_F} dE'\Bigg[\frac{d}{dE'}G^+(\mathbf{r},\mathbf{r}',E')\overleftrightarrow{D}^*_\mu \overleftrightarrow{D}'_\nu G^+ \\ & \times(\mathbf{r}',\mathbf{r},E') + G^-(\mathbf{r},\mathbf{r}',E')\overleftrightarrow{D}^*_\mu \overleftrightarrow{D}'_\nu \\ & \times \frac{d}{dE'}G^-(\mathbf{r}',\mathbf{r},E')\Bigg],\end{aligned} \tag{1.18}$$

where

$$G^\pm(E) = \frac{1}{E-H\pm i\eta}$$

and

$$\begin{aligned}G(\mathbf{r}',\mathbf{r})\overleftrightarrow{\mathbf{D}}G(\mathbf{r},\mathbf{r}') = {}& G(\mathbf{r}',\mathbf{r})[\nabla - i(e/\hbar)\mathbf{A}_0(\mathbf{r})]G(\mathbf{r},\mathbf{r}') \\ & - G(\mathbf{r},\mathbf{r}')[\nabla + i(e/\hbar)\mathbf{A}_0(\mathbf{r})]G(\mathbf{r}',\mathbf{r}).\end{aligned}$$

Here H is the Hamiltonian, discussed further in Sec. II, $\mathbf{A}_0$ is the vector potential representing the background magnetic field, and we will also use $-i\hbar\mathbf{D} = -i\hbar\nabla - e\mathbf{A}_0$. In the presence of the magnetic field, the bilocal conductivity tensor is not entirely a Fermi-energy quantity. Even at $T=0$, the complete current response function, in the presence of magnetic field, contains not only terms involving G^+G^- at the Fermi energy, but also terms involving G^+G^+ and G^-G^- integrated over all energies E up to the Fermi energy. We denote the G^+G^-, G^+G^+ and G^-G^- terms as $\sigma^{+-}_{\mu\nu}$, $\sigma^{++}_{\mu\nu}$, and $\sigma^{--}_{\mu\nu}$. In disordered systems, products of Green's functions that are both retarded or both advanced are generally short ranged because of the amplitude cancellations among different wave fronts (they typically only extend over the range of the mean free path), so we can treat σ^{++} and σ^{--} as contact terms:

$$\sigma^{aa}(\mathbf{r},\mathbf{r}') = \sigma^{aa}\delta(\mathbf{r}-\mathbf{r}'), \tag{1.19}$$

where $a=+,-$, $\sigma^{aa} = \int_A d^2r\,\sigma^{aa}(\mathbf{r},\mathbf{r}')$.

In the presence of a magnetic field B, the current given by Eq. (1.17) does not necessarily satisfy $\nabla\cdot\mathbf{j}=0$ even when $\mathbf{E}$ is time-independent, unless we also require $\nabla\times\mathbf{E}=0$ (i.e., that the component B of the magnetic field perpendicular to the 2D layer is also time independent). As shown by Baranger and Stone,[42] under this condition "current conservation" $\nabla\cdot\mathbf{j}=0$ is satisfied, and from it we can derive conditions on the bilocal conductivity. Writing $\mathbf{E}(\mathbf{r}') = -\nabla'\phi(\mathbf{r}')$ (where ϕ is the electric potential), we have

$$\begin{aligned}\mathbf{j}(\mathbf{r}) = {}& \frac{e^2}{h}\int d^2r'\,\sigma(\mathbf{r},\mathbf{r}')\cdot\overleftarrow{\nabla}'\phi(\mathbf{r}') \\ & - \frac{e^2}{h}\sum_n \phi_i\int_{C_i}\sigma(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}',\end{aligned} \tag{1.20}$$

where ϕ_i is the (constant) potential in the ith lead, C_i is a surface across the ith lead, and the boundary terms at the reflecting walls vanish because the normal component $\sigma_{n\nu}(\mathbf{r},\mathbf{r}')$ vanishes for $\mathbf{r}$ at the wall (and similarly for $\nu=n$, $\mathbf{r}'$ at the wall). Thus $\nabla\cdot\mathbf{j}=0$ implies that the following conditions are satisfied:

$$\begin{gathered}\overrightarrow{\nabla}\cdot\sigma(\mathbf{r},\mathbf{r}')\cdot\overleftarrow{\nabla}'=0, \\ \nabla\cdot\int_{C_i}\sigma(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}'=0 \quad \text{for all} i.\end{gathered} \tag{1.21}$$

The above identities have been verified in Ref. 42 for the exact bilocal conductivity tensor.

Using the second identity in Eq. (1.21), one can transform the linear response equation (1.17) into a different form. Assuming, without loss of generality, that $\mu=E_F$ in all the leads (we always view E_F as a position-independent constant), the electric potentials there differ from the voltages (electrochemical potentials) only by a constant, and the total current in the ith lead can be written as a function of only the voltages in the leads:

$$I_i = \frac{e^2}{h}\sum_j g_{ij}V_j, \tag{1.22}$$

where g_{ij}'s are conductance coefficients. The g_{ij}'s are related to the bilocal conductivity tensor by

$$g_{ij} = -\int\int d\mathbf{S}_i\cdot\sigma(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}'_j\,. \tag{1.23}$$

where S_i and S_j are cross sections in the ith and jth leads, and $d\mathbf{S}_i$ and $d\mathbf{S}_j$ are differentials of outward-pointing normals. [In Eq. (1.22), we used the relation $\Sigma_j g_{ij}=0$, which follows from Eq. (1.23), and the second of Eqs. (1.21), and implies that a constant V produces no current in any lead.] For cross sections S_i not intersecting S_j, for $i\neq j$, the off-Fermi-energy terms $\sigma^{++}(\mathbf{r},\mathbf{r}')$ and $\sigma^{--}(\mathbf{r},\mathbf{r}')$ can be shown to make zero contribution,[42] and therefore g_{ij} can be expressed as a Fermi-energy quantity. It has been shown that g_{ij} is proportional to the total transmission coefficient of the scattering states at the Fermi energy from the ith lead to the jth lead.[41,50,42] In this paper we will consider only the simple case of the two-probe conductance, in which certain further simplifications are possible.

Since the total currents at all cross sections are the same for a two-probe setup, we can average over all cross sections to obtain a volume-integral form for the two-probe conductance:

$$g=\frac{1}{L^2}\int_A d^2r\int_A d^2r'\,\sigma_{xx}(\mathbf{r},\mathbf{r}'). \tag{1.24}$$

Here we must be careful to include the off-Fermi-energy terms, since they are needed to preserve the uniformity of the current across each section (from the technical point of view, since x and x' can now coincide, we are no longer justified in dropping the σ^{aa} terms). Despite this disadvantage the volume-integral form of g is often more convenient to use in actual calculation than the surface-integral form, since volume averaging still eliminates many diagrams which would be nonzero in the surface-integral approach. Thus Eqs. (1.18), (1.23), and (1.24) will serve as the starting points for the evaluation of quantum conductance and conductance fluctuations.

We may gain further insight into the meaning of the current conservation conditions, and the relation to the classical case, by use of the self-consistent Born approximation (SCBA) results for the disorder average of the bilocal conductivity tensor. As we will show in Sec. II, in this approximation $\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$ (the single angle brackets will always denote the average over the disorder) is of the following form when $\mathbf{r}$ and $\mathbf{r}'$ are more than a mean free path from the edges:

$$\begin{aligned}\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}=&[\sigma^0_{xx}\delta_{\mu\nu}+(\sigma^{I,0}_{xy}+\sigma^{II,0}_{xy})\epsilon_{\mu\nu}]\delta(\mathbf{r}-\mathbf{r}')\\&-\frac{1}{\sigma^0_{xx}}[\sigma^0_{xx}\partial_\mu+\sigma^{I,0}_{xy}\epsilon_{\mu\mu'}\partial_{\mu'}]\\&\times[\sigma^0_{xx}\partial'_\nu-\sigma^{I,0}_{xy}\epsilon_{\nu\nu'}\partial'_{\nu'}]d(\mathbf{r},\mathbf{r}'),\end{aligned} \tag{1.25}$$

where σ^0_{xx}, $\sigma^{I,0}_{xy}$, $\sigma^{II,0}_{xy}$, and $\sigma^{I,0}_{xy}+\sigma^{II,0}_{xy}=\sigma^0_{xy}$ are the SCBA conductivity parameters,[28] $\sigma^{II,0}_{xy}$ comes from the σ^{++} and σ^{--} parts of Eq. (1.18), and d is the diffusion propagator discussed in Sec. I C (with σ^0_{xy} appearing in the boundary conditions). In the zero-field limit, $\sigma^{I,0}_{xy}=\sigma^{II,0}_{xy}=0$, the above reduces to

$$\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}=\sigma^0_{xx}[\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')-\partial_\mu\partial'_\nu d^0(\mathbf{r},\mathbf{r}')]$$

(in which d^0 is the diffusion propagator for $\sigma^0_{xy}=0$). To our knowledge this basic result first appeared in the mesoscopic physics literature in Ref. 38, although it may well have been known earlier. It or Eq. (1.25) shows that the current response to an electric field has a nonlocal part, the term containing d^0 or d in the formulas, due to diffusion. For nonzero magnetic field, one finds from Eq. (1.25) that

$$\partial_\mu\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}=\epsilon_{\mu\nu}\sigma^{II,0}_{xy}\partial_\mu\delta(\mathbf{r}-\mathbf{r}'). \tag{1.26}$$

The divergence of the response current is therefore

$$\begin{aligned}\nabla\cdot\langle\mathbf{j}(\mathbf{r})\rangle&=\frac{e^2}{h}\int d^2r'\,\nabla\cdot\langle\sigma(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}\cdot\mathbf{E}(\mathbf{r}')\\&=\frac{e^2}{h}\sigma^{II,0}_{xy}\nabla\times\mathbf{E}(\mathbf{r}).\end{aligned} \tag{1.27}$$

In the presence of a magnetic field, the current is divergenceless only when $\nabla\times\mathbf{E}=-\partial B/\partial t=0$; otherwise there is a time dependence in the density, $\partial\rho/\partial t=(e^2/h)\sigma^{II,0}_{xy}\partial B/\partial t$. To obtain a truly static response, we would have to impose $\nabla\times\mathbf{E}=0$. (There is also an edge-current contribution involving σ^{II}_{xy}, which will be described in the SCBA case in Sec. II.) This behavior is typical of the quantum Hall effect, in which $\sigma^{II}_{xy}\epsilon_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')$ is the only part of $\langle\sigma(\mathbf{r},\mathbf{r}')\rangle$ that is nonzero in the interior of the system on scales larger than the localization length ξ, and σ^{II}_{xy} is quantized to integer values. It is the local expression of the gauge-invariance argument.[51,40]

The measured experimental quantity is the two-probe conductance. It is straightforward to show (see Sec. II), from an equation similar to Eq. (1.25), that the two-probe conductance within the SCBA, g^0, can be written in terms of the diffusion propagator as

$$g^0=-\sigma^0_{xx}\int_0^W dy'\int_0^W dy(\partial_x+\gamma\partial_y)(\partial'_x-\gamma\partial'_y)d(\mathbf{r},\mathbf{r}'), \tag{1.28}$$

where $x\neq x'$ are arbitrary. We note that although $\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$ in the bulk depends on $\sigma^{I,0}_{xy}$, the conductance depends only on the full σ^0_{xy}, due to additional edge current contributions to σ which we omitted in Eq. (1.25). These contributions are similar to those discussed in Sec. I B.

It is in fact possible to show that the mean bilocal conductivity and conductance obtained in SCBA are identical to those obtained from the ''classical'' formulas of Sec. I A, if $\sigma^{II,0}_{xy}$ is zero. To obtain the response to an arbitrary electric field $\mathbf{E}$, we write the conditions of Sec. I A on the current density, using $\mathcal{E}=\mathbf{E}+\nabla\mu/e$, as

$$\nabla^2\mu/e=-\partial_\mu(E_\mu+\gamma\epsilon_{\mu\nu}E_\nu), \tag{1.29}$$

and the boundary conditions

$$(\partial_n-\gamma\partial_t)\mu/e=-(E_n-\gamma E_t) \tag{1.30}$$

at the reflecting walls, $\mu/e=E_F/e$ in the leads. These inhomogeneous equations for μ/e are equivalent for $\gamma=0$ to a 2D electrostatics problem with a mixed Dirichlet-Neumann boundary condition,[52] and can be solved using a Green's-function technique. A slight generalization of the same technique works for $\gamma\neq 0$. The required Green's function is precisely $d(\mathbf{r},\mathbf{r}')$ as defined in Sec. I C, and one finds

$$\mu(\mathbf{r})/e = E_F/e - \int d^2r'[(\partial'_\nu - \gamma\epsilon_{\nu\nu'}\partial'_{\nu'})d(\mathbf{r},\mathbf{r}')]E_\nu(\mathbf{r}'). \tag{1.31}$$

Using Eqs. (1.1), the bilocal conductivity tensor that results is exactly of form (1.25), with $\sigma_{xy}^{II,0}=0$, and no additional edge contributions. Consequently, the two-probe conductances g^0 given by Eqs. (1.2) and (1.28) are the same, for the same values of σ_{xx}^0 and σ_{xy}^0, and since this involves only the total σ_{xy}^0, it remains true even if $\sigma_{xy}^{II,0}\neq 0$ is included as in the SCBA. This implies that the bilocal conductivity in SCBA has just the form which follows from a local relation between current and electromotive field, even though there is no sensible definition of a local chemical potential in the phase-coherent limit. Hence all the conclusions drawn in Sec. I A can also be applied within the SCBA.

In general, there is also an edge-current contribution, with coefficient $\sigma_{xy}^{II,0}$, to Eq. (1.25), which will be described in Sec. II. There is no reason why both the bulk and edge $\sigma_{xy}^{II,0}$ contributions should not also appear in the so-called classical formulation of Sec. I A, even though they were not included in Ref. 39, since terms of this form would presumably still be present even if there were inelastic scattering.

We may also connect the results of Secs. I A and I D with the field theory in Sec. I C (again for $\sigma_{xy}^{II,0}=0$). In Secs. I B and I C, the external electric field was zero. If we replace $\partial_\mu z$, $\partial_\mu \overline{z}$ by $\partial_\mu z - 2iA_\mu$, $\partial_\mu \overline{z} + 2i\overline{A_\mu}$ in the action S_0, Eq. (1.11) (this $\mathbf{A}$ should not be confused with $\mathbf{A}_0$ or any other ''physical'' vector potential) and then define $j_\mu(\mathbf{r}) = \delta S_0[\mathbf{A}]/\delta A_\mu(\mathbf{r})$ to be the current, we find $j_\mu = i\sigma_{\mu\nu}^0(\partial_\nu \overline{z} + 2i\overline{A_\nu})/2$, and the conditions for an extremum of the action (equations of motion) are $\partial_\mu j_\mu(\mathbf{r})=0$ in the interior and $j_n=0$ at the reflecting walls. Thus these equations have the same form as the local classical conductivity equations, with $i\overline{z}/2$, $-\overline{A_\mu}$ in place of $(\mu - E_F)/e$, E_μ, and since, for a quadratic action like S_0, the linear response $\delta\langle\langle j_\mu\rangle\rangle_0/\delta\overline{A_\nu}$ to the perturbation $\mathbf{A}$ can be equivalently obtained either from calculation of the correlation function [similar to that in the definition of d, Eq. (1.15)], or using the equations of motion as above, we also obtain the same bilocal conductivity in this approach. This calculation is done in detail in Secs. III and IV, so we refrain from giving further details here. Section III C also includes the corresponding ''classical'' equations for the case when $\sigma_{xy}^{II,0}\neq 0$.

The SCBA for the average conductance is the leading approximation in an expansion in powers of $1/\sigma_{xx}^0$, and it is not really surprising that this leading approximation behaves identically to the classical case. When evaluating conductance fluctuations, or weak-localization corrections, one must consider higher orders in $1/\sigma_{xx}^0$. In such calculations the tilted boundary condition is modified further. In the framework of diagrammatic perturbation theory, this can be alternatively viewed as the appearance of additional boundary vertices describing interference effects. These vertices are more easily obtained and evaluated in the nonlinear σ-model approach to which we now turn.

E. Nonlinear σ-model approach

The approach of Sec. I D, in which the self-consistent Born approximation is the leading contribution to conductivity and conductance, can be developed as a diagrammatic expansion (see Sec. II A). However, this approach becomes cumbersome in higher orders because all diagrams contain vertices which need to be evaluated in terms of the average single-particle Green's functions, and are dressed with disorder lines in all possible ways. However, when these vertices, which describe interference or ''interactions'' of diffusing modes, are calculated at small wave vectors, they are all found to be related to the same quantities σ_{xx}^0 and σ_{xy}^0. These complex and often redundant calculations can be avoided by using the nonlinear σ-model (NLσM) representation of the problem.

The NLσM approach starts by considering only Green's functions at the Fermi energy, which means that the non-Fermi-energy parts of the conductivity described in Sec. I D cannot necessarily be obtained, though the conductance and its fluctuations can. After introducing replicas, the disorder is integrated out, followed by the variables representing the electrons propagating at the Fermi energy. After a Hubbard-Stratonovich decoupling, and neglecting modes that have no long-range effects, one is led to the action[28,30] (more details will be given in Sec. III)

$$S = -\frac{\sigma_{xx}^0}{8}\int_A d^2r\, \mathrm{tr}[\partial_\mu Q \partial_\mu Q] - \frac{\sigma_{xy}^0}{8}\int_A d^2r\, \mathrm{tr}[\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q], \tag{1.32}$$

where Q is a $2n\times 2n$ Hermitian matrix obeying $Q^2 = I_{2n}$, (I_{2n} is the identity matrix); Q has n eigenvalues equal to $+1$, n equal to -1. Using a parametrization given explicitly in Sec. III, it can be shown that for small fluctuations about the maximum action configuration where Q is diagonal, S reduces at quadratic order to n^2 copies of the earlier action S_0 in Eq. (1.11).

Before discussing perturbation theory for this action, we wish to mention some general issues. The second term in S is the so-called topological term. It is the only possible term that can be added to the first term in two dimensions that is consistent with the symmetries of the problem and contains only two gradients (higher derivatives would be irrelevant at long length scales). On a compact, oriented manifold without boundary, such as a sphere or a torus (i.e., periodic boundary conditions), this term (with a factor of $2\pi i \sigma_{xy}^0$ removed) is a topological invariant, which takes integral values. This follows from the fact that the term is a total derivative, and the absence of boundaries. Consequently, only the value of σ_{xy}^0 modulo 1 is important. Moreover, because the term is a topological invariant, σ_{xy}^0 does not appear in perturbation theory at all, but only in nonperturbative effects involving configurations (''instantons'') for which the topological term is nonzero. However, in the integer quantum Hall effect, we expect to obtain plateaus at integral σ_{xy} and transitions between them at σ_{xy} half-odd-integral (σ_{xx}, σ_{xy} denote the renormalized, large-scale, parameters, as opposed to the bare values σ_{xx}^0 and σ_{xy}^0 at the cutoff scale l). Because of the periodicity in σ_{xy}^0, the NLσM predicts that all these plateaus and transitions will have identical universal properties. But, by the same token, it is also unable to predict the integral part of σ_{xy} that would be observed in a measurement; this information appears to be lost in going to the NLσM.

The apparent paradox is resolved on examining the action S for a system with reflecting boundaries. The ''topological'' term is a total derivative that can be rewritten as a boundary term, just as for the action S_0 above. The boundary term is not a topological invariant, so it can affect perturbation theory, and, since it takes arbitrary real values, the magnitude of σ_{xy}^0 is important, and not just its value modulo integers. Thus when the boundaries of the system are correctly taken into account, the value of σ_{xy} can be obtained within the NLσM formulation. Some additional remarks about this point are made in Sec. VI.

Since the leading-order part of the action is the same as S_0, the propagator for small fluctuations in Q is the same as the propagator d discussed earlier, and depends on σ_{xy}^0 through the boundary condition. In the work of KY and ML, this modified propagator was the only effect included, and was just inserted into the Xiong-Stone results for conductance fluctuations. However, the NLσM shows that the boundary term also contributes at higher order, producing new vertices for ''interactions'' between diffusons, which are boundary interactions with coefficient σ_{xy}^0, and which contribute to the fluctuations at leading order. These terms must be present in order to maintain the full U$(2n)$ symmetry of the NLσM, which essentially corresponds to preserving the continuity equation for the current. We have also obtained them in the diagrammatic approach, but only with much additional effort. (Pruisken[30] also discussed the relation of the topological term to edge states, but appeared to infer an incorrect boundary condition. His boundary condition is very useful in instanton calculations[29–31] but does not correctly represent the edge effects, unlike the conditions to be discussed in this paper.) In this paper, we evaluate the effects of these terms to leading order in $1/\sigma_{xx}^0$, and, to simplify the calculations, also to leading nontrivial order in $\gamma=\sigma_{xy}^0/\sigma_{xx}^0$. As well as calculating the mean and variance of the two-probe conductance, we show how the expression for the mean bilocal conductivity tensor in the SCBA can be recovered within the NLσM, including the non-Fermi-energy parts, by modifying the coupling of the NLσM to the external field, and we discuss the resulting form of current conservation conditions to all orders in perturbation theory.

One may wonder if the boundary conditions for a system with boundary invalidate the conclusions of the analysis of Pruisken and co-workers,[29–31] who studied effects of instantons in a system without a boundary. Strictly speaking, in a finite system with boundaries, there are no well-defined topologically distinct sectors. However, small instantons which are well localized inside the system boundary, so that Q approaches a constant outside the instanton core and satisfies the boundary conditions at the edges, while probably not exactly local minima of the action, are still nearly so when their size goes to zero compared with the system dimensions, and in this limit their topological charge will still be an integer, and they will make the same contribution to the action as they did for the other boundary conditions. Thus the effects on renormalization group flows for σ_{xx}^0 and σ_{xy}^0, obtained in the interior of the system, should be unchanged. These effects will not be considered further in this paper, which emphasizes perturbation theory.

In the one-dimensional case of the NLσM, no other term can be added to the basic gradient-squared term, and there are of course no sides on which the boundary condition could be modified. Therefore, the conductance fluctuations in the unitary ensemble should be universal, and must be recovered in the quasi-1D limit of the 2D system in a magnetic field which we are considering; this constitutes a strong check on the 2D calculations. We will show that in this limit the only effect of σ_{xy}^0 is to modify the 1D conductivity, which is known to cancel from the conductance fluctuations to leave a universal number. KY and ML claimed that σ_{xy}^0 does affect the quasi-1D limit; the present argument shows that their results must be incorrect. In Sec. VI, we will briefly mention the situation for dimensions higher than two.

II. DIAGRAMMATIC EXPANSION FOR $\langle \sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$

In this section, we evaluate the mean bilocal conductivity tensor for the short-ranged potential model using the diagrammatic impurity-averaging technique.[53] We will first review the self-consistent Born approximation (SCBA), which is the leading order of the perturbation expansion, and establish basic parameters such as the mean free path l, and the bare conductivities σ_{xx}^0 and σ_{xy}^0. A gradient expansion is used to treat the current vertex. Within the SCBA, $\langle \sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$ has a contact term as well as a long-ranged term which can be expressed in terms of the diffusion propagator. In the bulk, $\sigma_{xy}^{I,0}$ appears in the long-ranged term. Along the reflecting boundary, the edge currents give rise to δ-function contributions proportional to $\sigma_{xy}^{II,0}$. As a result, it is $\sigma_{xy}^0=\sigma_{xy}^{I,0}+\sigma_{xy}^{II,0}$ that appears in the boundary condition and the two-probe conductance. We will also check that current conservation is respected within the SCBA.

A. Model, edge current, and ideal leads

An electron in a system with edges, in a random potential and subject to a perpendicular magnetic field, is described by the Hamiltonian (we neglect spin throughout this paper)

$$H=H_0+V(\mathbf{r}),\quad H_0=\frac{1}{2m_e}(-i\hbar\mathbf{D})^2+U(y),\quad (2.1)$$

where $V(\mathbf{r})$ is the random potential and is confined in the region of $0\leqslant x\leqslant L$, and $U(y)$ is the confinement potential (see Fig. 1). The uniform magnetic field is in the z direction, and we choose the gauge $\mathbf{A}_0=-By\hat{\mathbf{x}}$. For simplicity, let us assume the confinement $U(y)$ to be the hard-wall potential with $U(y)=0$ for $0\leqslant y\leqslant W$, and $U(y)=\infty$ for $y<0$ and $y>W$. The infinite potential barrier requires the wave function to vanish at $y=0$, W. The system is infinite in the x direction, but the disorder is present only in the region $0\leqslant x\leqslant L$. For the random potential, we will take the simplest among all short-ranged models, which has the statistics of the Gaussian white noise with zero mean (angle brackets denote the disorder average),

$$\langle V(\mathbf{r})\rangle=0,\quad \langle V(\mathbf{r})V(\mathbf{r}')\rangle=u\,\delta(\mathbf{r}-\mathbf{r}'),$$
$$\langle V(\mathbf{r}_1)V(\mathbf{r}_2)\cdots V(\mathbf{r}_n)\rangle_{\text{connected}}=0,\quad n>2, \tag{2.2}$$

where u describes the degree of disorder.

In the absence of the random potential, the unperturbed wave functions can be found by separating the variables.[40] The wave functions $\psi(\mathbf{r})$ are labeled by the wave vector k in the x direction and by N in the transverse direction; N turns out to be the Landau-level index. We have

$$\psi_{Nk}(\mathbf{r})=\frac{1}{\sqrt{L}}e^{ikx}\phi_{N,k}(y),$$

and $\phi_{N,k}(y)$ satisfies

$$\{\hbar^2[(-i\partial_y)^2+(y-l_B^2k)^2l_B^{-4}]/2m_e+U(y)\}\phi_{N,k}(y)$$
$$=E_{N,k}\phi_{N,k}(y), \tag{2.3}$$

where $l_B^2=\hbar/eB$. Without the confinement potential, the Hamiltonian is simply that of a harmonic oscillator, with the harmonic potential centering at $y_k=kl_B^2$. We have[40] $\phi_{N,k}(y)=\chi_N(y-l_B^2k)$ and $E_{N,k}=E_N=(N+1/2)\hbar\omega_c$ for $W\gg y_k\gg 0$, where χ_N is the Nth wave function of the harmonic oscillator, and $N=0,1,2,\ldots$. The wave functions spread an extent $R_c=\sqrt{2N+1}\,l_B$ around y_k. We can see that for wave functions which center at a distance more than R_c away from the walls, the presence of the walls is hardly felt, but for those which reside within a distance R_c from the walls, their eigenenergies are raised above E_N, because the wave functions are forced to zero at the boundaries and thus made to oscillate more rapidly near the walls. Only the states within R_c of the edges have a nonzero group velocity along the walls, i.e., the expectation value of the velocity operator $\langle N,k|v_x|N,k\rangle\neq 0$. From now on, R_c plays the role of the short length scale of the problem, and we treat the edges as having zero width. We will later show that, in this sense, the inhomogeneity at the edges gives rise to δ-function contributions to the current (see also Ref. 54).

Although the above description of H_0 is very convenient for finding an explicit solution for the energy eigenfunctions when the system is infinitely extended in the x direction, it does not provide a convenient description of ideal leads in the presence of a nonzero magnetic field. For our purposes ideal leads should be perfect absorbers of all incident current, i.e., they should behave as if they have essentially infinite conductivity compared to the sample. The problem is that in the leads, where $x<0$ or $x>L$, the states at the Fermi energy generically consist of a certain number of edge channels moving in each direction, and this number is equal to the number of Landau levels below the Fermi energy in the bulk, which is $N+1$ when the Fermi energy lies above the Nth Landau level. Thus in the relevant sense, the leads have zero bulk conductivity and only absorb and inject current at the corners. There are several ways in which we could modify our model to remedy this problem. One way would be to let the magnetic field drop to zero at the ends of the sample, so the leads are like the usual 2D metallic leads in zero magnetic field. Another way, which corresponds roughly to the Ohmic metallic contacts used in real experimental systems, would be to ''thicken'' the system outside the sample, so that it becomes three dimensional, thus increasing its conductance. There is, however, a third way, which most closely conforms to the perfect leads used in the network model (see Fig. 2) and has some convenient properties. The links of the network can be viewed as edge channels, and outside the sample there are many of them, running parallel, alternately right and left moving, without backscattering. A similar setup can be produced in a 2D Hamiltonian model with a uniform magnetic field, by replacing $U(y)$, *only* in the leads, by a potential $U_1(y)$, which $=+\infty$ for $y>W$ or $y<0$, and has a sinusoidal form in $0<y<W$. If the Fermi energy lies between the maxima and minima of U_1, there will be many ''internal'' edge channels at the Fermi energy, consisting alternately of $N+1$ channels moving in one direction and $N+1$ moving in the other. For an infinitely long, translationally invariant system there will be no backscattering among these modes. In effect, we have many narrow leads in parallel, all connected to a single reservoir at $-\infty$ and to another at $+\infty$. Then the number of right-moving channels can be proportional to the width of the system, or arbitrarily large, and the current can be injected uniformly across the end of the sample. From now on, it is this model that we will implicitly use.

B. Self-consistent Born approximation (SCBA)

In order to calculate $\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$, we first need to evaluate the disorder-averaged single-particle Green's function and two-particle Green's function, $\langle G\rangle$ and $\langle GG\rangle$, which can be expanded in power series in $1/(k_Fl)$. The SCBA takes into account all the non-crossing diagrams (Fig. 3) and has been shown to be the leading contribution in $1/(k_Fl)$.[27]

Within the SCBA, the single-particle Green's function [see Fig. 3(a)] satisfies

$$[E-H_0(\mathbf{r})-\Sigma(\mathbf{r})]\overline{G}(\mathbf{r},\mathbf{r}';E)=\delta(\mathbf{r}-\mathbf{r}'), \tag{2.4}$$

where the self-energy in turn depends on $\overline{G}$ (we use the overbar to denote the SCBA Green's function):

$$\Sigma(\mathbf{r},E)=u\overline{G}(\mathbf{r},\mathbf{r};E).$$

The above equation can be solved analytically in the low-field limit ($\omega_c\tau_0\ll 1$) and the high-field limit ($\omega_c\tau_0\gg 1$).[24] In the intermediate-field range, it can be solved numerically. For our purpose, we do not need the explicit solutions. We use the SCBA Green's function to define the effective scattering rate $1/\tau$ and the effective mean free path l at the Fermi energy:

$$\frac{1}{\tau}=\frac{\Sigma^+-\Sigma^-}{i\hbar}=\begin{cases}1/\tau_0, & \omega_c\tau_0\ll 1\\ 2\Gamma\sin\theta(E_F)/\hbar, & \omega_c\tau_0\gg 1,\end{cases} \tag{2.5}$$

where $\Gamma=\sqrt{u/(2\pi l_B^2)}\sim\hbar\sqrt{\omega_c/\tau_0}$ is the width of broadened Landau levels, $\theta(E)=\cos^{-1}[(E-E_N)/2\Gamma]$, $0\leqslant\theta(E)\leqslant\pi$, and

$$l^2=u\int d^2r_1(x-x_1)^2\overline{G^+}(\mathbf{r},\mathbf{r}_1,E_F)\overline{G^-}(\mathbf{r}_1,\mathbf{r};E_F)$$
$$=\begin{cases}l_0^2, & \omega_c\tau_0\ll 1\\ R_c^2, & \omega_c\tau_0\gg 1.\end{cases} \tag{2.6}$$

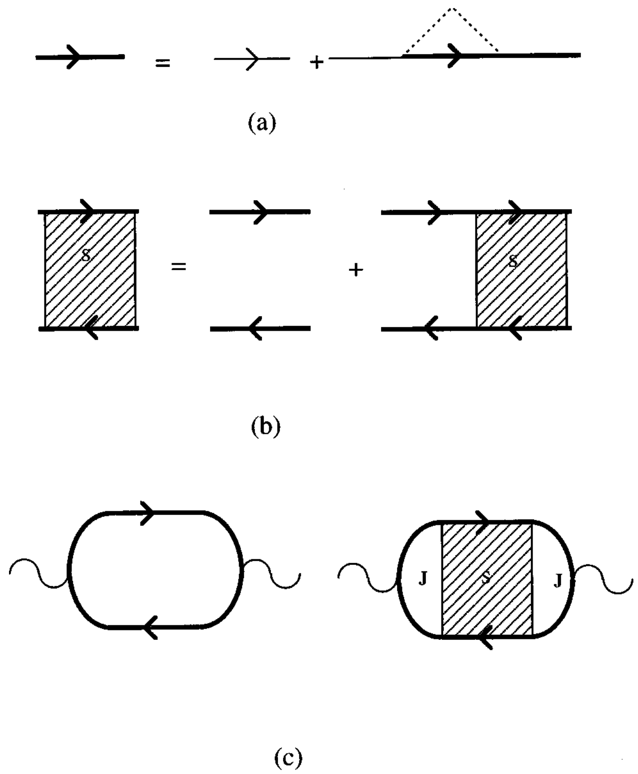

FIG. 3. (a) The SCBA single-particle Green's function. It sums up all the noncrossing diagrams. The thin line denotes G^0, the Green's function in the absence of disorder. The thick line denotes the SCBA Green's function $\overline{G}$. (b) The ladder sum for the SCBA two-particle Green's function. (c) The diagrams for the SCBA bilocal conductivity tensor.

Here l_0 is the mean free path in zero magnetic field, $l_0 = \hbar k_F \tau_0 / m_e$.

Within the SCBA, the two-particle Green's function $S^{+-}(\mathbf{r},\mathbf{r}') = u\overline{G^+(\mathbf{r},\mathbf{r}')G^-(\mathbf{r}',\mathbf{r})}$ amounts to adding up all the ladder diagrams [Fig. 3(b)]. The sum can be written in the form of an integral equation

$$
\begin{aligned}
S^{+-}(\mathbf{r},\mathbf{r}',E,E') &= u\overline{G^+}(\mathbf{r},\mathbf{r}',E)\overline{G^-}(\mathbf{r}',\mathbf{r},E') \\
&\quad + \int d^2r_1 u\overline{G^+}(\mathbf{r},\mathbf{r}_1,E)\overline{G^-}(\mathbf{r}_1,\mathbf{r},E') \\
&\quad \times S^{+-}(\mathbf{r}_1,\mathbf{r}',E,E'). \qquad (2.7)
\end{aligned}
$$

We make use of the fact that $\overline{G^+}(\mathbf{r},\mathbf{r}')\overline{G^-}(\mathbf{r}',\mathbf{r})$ is short ranged, and expand $S^{+-}(\mathbf{r}_1,\mathbf{r}')$ in the vicinity of $\mathbf{r}_1 = \mathbf{r}$. For E, E' close to E_F, we obtain

$$
\begin{aligned}
&\int d^2r_1 u\overline{G^+}(\mathbf{r},\mathbf{r}_1)\overline{G^-}(\mathbf{r}_1,\mathbf{r})S^{+-}(\mathbf{r}_1,\mathbf{r}') \\
&\quad = [C_0(\mathbf{r};E,E') + \mathbf{C}_1(\mathbf{r};E,E')\cdot\nabla \\
&\qquad + \tfrac{1}{2}C_2(\mathbf{r};E,E')\nabla^2 + \cdots]S^{+-}(\mathbf{r},\mathbf{r}') \qquad (2.8)
\end{aligned}
$$

where

$$
\begin{aligned}
C_0(\mathbf{r};E,E') &= u\int d^2r_1 \overline{G^+}(\mathbf{r},\mathbf{r}_1,E)\overline{G^-}(\mathbf{r}_1,\mathbf{r};E') \\
&\simeq 1 + i(E-E')\tau/\hbar,
\end{aligned}
$$

$$
\begin{aligned}
C_1(\mathbf{r};E,E') &= u\int d^2r_1 (\mathbf{r}-\mathbf{r}_1)\overline{G^+}(\mathbf{r},\mathbf{r}_1;E)\overline{G^-}(\mathbf{r}_1,\mathbf{r};E') \\
&\simeq 0,
\end{aligned}
$$

$$
\begin{aligned}
C_2(\mathbf{r};E,E') &= u\int d^2r_1 (x-x_1)^2\overline{G^+}(\mathbf{r},\mathbf{r}_1,E)\overline{G^-}(\mathbf{r}_1,\mathbf{r};E') \\
&= l^2.
\end{aligned}
$$

It follows that $S^{+-}(\mathbf{r},\mathbf{r}',E-E')$ satisfies the diffusion equation

$$
[-\mathcal{D}\tau\nabla^2 - i(E-E')\tau/\hbar]S^{+-}(\mathbf{r},\mathbf{r}';E-E') \simeq \delta(\mathbf{r}-\mathbf{r}'), \qquad (2.9)
$$

where $\mathcal{D}$ is the diffusion constant: $\mathcal{D}(E) = l^2/(2\tau)$. One can see that S^{+-} at $E=E'=E_F$, which is all that will be required in this paper, is proportional to the dimensionless diffusion propagator $d(\mathbf{r},\mathbf{r}')$ we defined in Eq. (1.16):

$$
\frac{l^2}{2}S^{+-}(\mathbf{r},\mathbf{r}';0) = d(\mathbf{r},\mathbf{r}').
$$

We postpone derivation of the boundary conditions on d until after we have discussed the conductivity tensor. The ladder sum for $S^{++} = u\overline{G^+G^+}$ and $S^{--} = u\overline{G^-G^-}$ can also be carried out in similar fashion. It is easy to see that $S^{++}(\mathbf{r},\mathbf{r}')$ and $S^{--}(\mathbf{r},\mathbf{r}')$ are generally short ranged.

Now we are ready to treat the mean bilocal conductivity tensor. Within the SCBA, $\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle$ has two contributions: the simple bubble diagram and the sum of the ladder series [Fig. 3(c)]:

$$
\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}} = \sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{bubble}} + \sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{ladder}}. \qquad (2.10)
$$

The bubble diagram has the range of the mean free path l and we treat it as a δ function:

$$
\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{bubble}} = \sigma^0_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}').
$$

$\sigma^0_{\mu\nu}$ are the SCBA conductivity parameters, which are essentially constant inside the sample:

$$
\begin{aligned}
\sigma^0_{\mu\nu} &= \int_A d^2r\,\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{bubble}} \\
&= \frac{1}{LW}\int_A d^2r\int_A d^2r'\,\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{bubble}}. \qquad (2.11)
\end{aligned}
$$

It follows from the definitions [cf. Eq. (1.18)] (Refs. 42 and 55) that

$$\sigma^0_{xx}=\frac{\hbar^2}{LW}\mathrm{Tr}\,[v_x\overline{G^+}(E_F)v_x\overline{G^-}(E_F)]$$
$$-\frac{\hbar^2}{LW}\int_\infty^{E_F}dE\left\{\mathrm{Tr}\left[v_x\overline{\left(\frac{d}{dE}G^+\right)}v_x\overline{G^+}\right]\right.$$
$$\left.+\mathrm{Tr}\left[v_x\overline{G^-}v_x\overline{\left(\frac{d}{dE}G^-\right)}\right]\right\}$$
$$=-\frac{\hbar^2}{2LW}\mathrm{Tr}\{v_x\Delta\overline{G(E_F)}v_x\Delta\overline{G(E_F)}\},\tag{2.12}$$

where $\Delta\overline{G}=\overline{G^+}-\overline{G^-}$, $\mathbf{v}=-i\hbar\mathbf{D}/m_e$, and Tr denotes the trace of the matrix product, in which $\overline{G^\pm}(\mathbf{r},\mathbf{r}')$ are viewed as $\mathbf{r}$, $\mathbf{r}'$ matrix elements. For σ^0_{xy},

$$\sigma^0_{xy}=\sigma^{I,0}_{xy}+\sigma^{II,0}_{xy},\quad \sigma^{I,0}_{xy}=\frac{\hbar^2}{LW}\mathrm{Tr}[v_x\overline{G^+}(E_F)v_y\overline{G^-}(E_F)],$$

$$\sigma^{II,0}_{xy}=-\frac{\hbar^2}{LW}\int_\infty^{E_F}dE'\left\{\mathrm{Tr}\left[v_x\overline{\left(\frac{d}{dE'}G^+\right)}v_y\overline{G^+}\right]\right.$$
$$\left.+\mathrm{Tr}\left[v_x\overline{G^-}v_y\overline{\left(\frac{d}{dE'}G^-\right)}\right]\right\}.\tag{2.13}$$

The above expression for σ^0_{xy} can be put in the same Fermi-energy form as in Ref. 55. σ^0_{xx} and σ^0_{xy} have the following limiting behavior:[24,28,30]

$$\sigma^0_{xx}=h\mathcal{D}(E_F)\rho(E_F)=\begin{cases}hn_e\tau_0/m_e, & \omega_c\tau_0\ll 1\\ (2N+1)\pi^{-1}\sin^2\theta(E_F), & \omega_c\tau_0\gg 1,\end{cases}$$

$$\sigma^0_{xy}=\begin{cases}\sigma^0_{xx}\omega_c\tau_0, & \omega_c\tau_0\ll 1\\ N+\nu, & \omega_c\tau_0\gg 1,\end{cases}\tag{2.14}$$

where $\rho(E)$ is the SCBA local density of states, n_e is the electron density, N is the highest Landau-level index, $N=0,\ 1,\ldots$, and $\nu=1-\theta/\pi$ is the filling fraction of the highest Landau level, which is well defined since there is vanishing local density of states in the bulk in between the Landau levels in the regime $\omega_c\tau_0\gg 1$, within the SCBA. Often, only the peak value $(2N+1)/\pi$ of σ^0_{xx} is quoted in the literature; we emphasize that σ^0_{xx} has oscillations and goes to zero when the Fermi energy lies in one of the gaps in the bulk density of states.

Now we treat the ladder diagrams. Since S^{++} and S^{--} are short ranged, σ^{++} and σ^{--} are also short ranged. One can further show that the ladder series for σ^{++}_{xx} and σ^{--}_{xx} do not make additional contributions to the δ-function term. The long-ranged term of $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')$ comes from

$$\sigma^{+-}_{\mu\nu}(\mathbf{r},\mathbf{r}')_{\mathrm{ladder}}=\hbar^2u\int d^2r_1\int d^2r_2J^{+-}_\mu(\mathbf{r},\mathbf{r}_1)$$
$$\times S^{+-}(\mathbf{r}_1,\mathbf{r}_2)J^{-+}_\nu(\mathbf{r}',\mathbf{r}_2).$$

The current vertex $\mathbf{J}(\mathbf{r};\mathbf{r}_1)$ is short ranged,

$$\mathbf{J}^{ab}(\mathbf{r},\mathbf{r}_1)=\frac{-i\hbar}{2m_e}\overline{G^b}(\mathbf{r}_1,\mathbf{r})\overleftrightarrow{\mathbf{D}}\overline{G^a}(\mathbf{r},\mathbf{r}_1),\tag{2.15}$$

where $a,\ b=+,\ -$, and $\overleftrightarrow{\mathbf{D}}$ was defined in Sec. I C. We can carry out the integral by expanding $S^{+-}(\mathbf{r}_1,\mathbf{r}')$ in the neighborhood of $\mathbf{r}$:

$$\int d^2r_1\mathbf{J}(\mathbf{r},\mathbf{r}_1)S^{+-}(\mathbf{r}_1,\mathbf{r}_2)$$
$$=[\mathbf{J_0}(\mathbf{r})+J_1(\mathbf{r})\cdot\nabla+\cdots]S^{+-}(\mathbf{r},\mathbf{r}_2).\tag{2.16}$$

The first term is

$$\mathbf{J}^{+-}_0(\mathbf{r})=\left(\frac{-i\hbar}{2m_e}\right)\int d^2r_1\overline{G^-}(\mathbf{r}_1,\mathbf{r})\overleftrightarrow{\mathbf{D}}\overline{G^+}(\mathbf{r},\mathbf{r}_1)$$
$$\simeq\frac{1}{\Sigma^+-\Sigma^-}\langle\mathbf{r}|\mathbf{v}(\overline{G^+}-\overline{G^-})|\mathbf{r}\rangle.\tag{2.17}$$

In the last step, we used the identity $\overline{G^+}\,\overline{G^-}=(\overline{G^+}-\overline{G^-})/(\Sigma^+-\Sigma^-)$. Rewriting

$$\langle\mathbf{r}|\mathbf{v}(\overline{G^+}-\overline{G^-})|\mathbf{r}\rangle\ \text{ as }\ 2\pi i\overline{\sum_\alpha\delta(E_F-E_\alpha)\psi_\alpha(\mathbf{r})\mathbf{v}\psi^*_\alpha(\mathbf{r})},$$

it is easy to recognize that $\mathbf{J}^{+-}_0(\mathbf{r})$ is proportional to the impurity-averaged equilibrium current density at position $\mathbf{r}$ and at the Fermi energy E_F. In the bulk, this should be zero by isotropy; however, at the edges isotropy is broken, and $\mathbf{J}^{+-}_0$ parallel to the boundary is not zero. Hence we can write

$$J^{+,-}_{0,x}(\mathbf{r})=\frac{2\pi i}{e(\Sigma^+-\Sigma^-)}\left[\left.\frac{\partial I_e(E)}{\partial E}\right|_{E_F}\delta(y-W)\right.$$
$$\left.-\left.\frac{\partial I_e(E)}{\partial E}\right|_{E_F}\delta(y)\right],\tag{2.18}$$

where $I_e(E_F)$ is the total edge current. It has been shown in Ref. 30 that

$$\left.\frac{\partial I_e}{\partial E}\right|_{E_F}=\left.\frac{\partial M}{\partial E}\right|_{E_F}=-\frac{e\sigma^{II,0}_{xy}}{h},\tag{2.19}$$

where $M(E)$ is the total magnetization.

The coefficients for the first-derivative terms are

$$J^{+-}_{1,\mu\nu}=\frac{1}{LW}\left\{\mathrm{Tr}[v_\mu\overline{G^+}r_\nu\overline{G^-}]-\mathrm{Tr}[r^\nu v^\mu\overline{G^+}\,\overline{G^-}]\right.$$
$$\left.+\frac{i\hbar}{2m}\mathrm{Tr}[\overline{G^+}\,\overline{G^-}]\delta_{\mu\nu}\right\},$$

$$J^{-+}_{1,\mu\nu}=\frac{1}{LW}\left\{\mathrm{Tr}[v_\mu\overline{G^-}r_\nu\overline{G^+}]-\mathrm{Tr}[r^\nu v^\mu\overline{G^-}\,\overline{G^+}]\right.$$
$$\left.+\frac{i\hbar}{2m}\mathrm{Tr}[\overline{G^-}\,\overline{G^+}]\delta_{\mu\nu}\right\}.$$

Expressing the coefficients in terms of the SCBA conductivities and the mean free path, we have

$$u\hbar^2J^{+-}_{1,xx}J^{+-}_{1,xx}=\sigma^0_{xx}l^2/2,\quad u\hbar^2J^{+-}_{1,xx}J^{+-}_{1,xy}=\sigma^{I,0}_{xy}l^2/2.$$

Putting all the pieces together, for the full bilocal large-scale conductivity tensor within the SCBA, including the edge effects, we obtain

$$\begin{aligned}\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}=&[\sigma^0_{xx}\delta_{\mu\nu}+(\sigma^{I,0}_{xy}+\sigma^{II,0}_{xy})\epsilon_{\mu\nu}]\delta(\mathbf{r}-\mathbf{r}')\\&-\frac{1}{\sigma^0_{xx}}\{\sigma^0_{xx}\partial_\mu+\sigma^{I,0}_{xy}\epsilon_{\mu\mu'}\partial_{\mu'}\\&+\sigma^{II,0}_{xy}\delta_{\mu x}[\delta(y-W)-\delta(y)]\}\{\sigma^0_{xx}\partial'_\nu\\&-\sigma^{I,0}_{xy}\epsilon_{\nu\nu'}\partial'_{\nu'}-\sigma^{II,0}_{xy}\delta_{\nu x}[\delta(y'-W)\\&-\delta(y')]\}d(\mathbf{r},\mathbf{r}').\end{aligned}\tag{2.20}$$

Within the SCBA, the conductivity tensor obeys $\langle\sigma_{y\nu}(\mathbf{r},\mathbf{r}')\rangle=0$, for $\mathbf{r}$ at the reflecting edges and $\mathbf{r}$ sufficiently far from $\mathbf{r}'$, as an exact relation *before* the long-wavelength approximation is made. Together with Eq. (2.20), this implies the large-scale boundary conditions (1.16) on the diffusion propagator d; see Appendix A. Because of the edge current, this is very similar to the discussion of the conservation of the $\mathbf{j}'$ current at the edge in Sec. I A (where $\sigma^{II,0}_{xy}$ was -1).

C. SCBA for the two-probe conductance

We are finally ready to express the two-probe conductance in terms of the surface integral of the diffusion propagator. For the integrated currents at any cross sections, the two opposite edge currents at the boundaries can be transformed to an additional bulk derivative in the transverse direction:

$$\int_0^W dy[\delta(y-W)-\delta(y)]d(\mathbf{r},\mathbf{r}')=\int_0^W dy\,\partial_y d(\mathbf{r},\mathbf{r}');$$

therefore, the second and third terms in the square brackets of Eq. (2.20) combine in this case to give σ^0_{xy}. We obtain the form for the SCBA conductance g^0, Eq. (1.28), expressed in terms of an integral over two cross sections. This can be further transformed into the following volume-integral form:

$$\begin{aligned}g^0=&\frac{1}{L^2}\int_A d^2r\int_A d^2r'\,\sigma^0_{xx}[\delta(\mathbf{r}-\mathbf{r}')\\&-(\partial_x+\gamma\partial_y)(\partial'_x-\gamma\partial'_y)d(\mathbf{r},\mathbf{r}')].\end{aligned}\tag{2.21}$$

In Sec. IV, we will see that the above expression for the conductance can also be obtained using the NLσM formalism which we develop below.

III. FIELD-THEORETICAL APPROACH

In this section, we first set up (in Sec. III A) a generating function from which we can obtain any expressions involving the Green's functions at the Fermi energy. As we saw in Sec. I,[42] this allows calculation of the mean and the variance of the two-probe conductance. In Sec. III B, we discuss the coupling to a U$(2n)$ vector potential, which can be used to generate the NLσM conductivity. For a certain form of coupling, we can in fact recover, at lowest order in $1/\sigma^0_{xx}$, the SCBA form of the bilocal conductivity tensor. We discuss the physical meaning of the various terms in relation to current conservation. The results show that the variance of the two-probe conductance can be calculated within the NL σM. We then turn to the perturbation expansion itself.

A. Nonlinear σ model

In setting up the partition function (or generating function for average Green's functions), we will use the replica method to perform the average over disorder. We define

$$\begin{aligned}Z=&\int D[V]P[V]\int D[\varphi,\overline{\varphi}]\exp\int d^2r\,\overline{\varphi(\mathbf{r})}\\&\times[E-H+i\eta\Lambda]\varphi(\mathbf{r}),\end{aligned}\tag{3.1}$$

where, as in Sec. II, $H=H_0+V(\mathbf{r})$, $\varphi=\ldots\varphi_i^a\ldots$ $(i=1,2,\ldots,n,\ a=+,-)$ is a $2n$-component vector of complex Grassmann numbers, $\eta=0^+$, and

$$\Lambda=\begin{pmatrix}I_n & 0\\ 0 & -I_n\end{pmatrix}$$

(I_n is the $n\times n$ identity matrix). Here we remark that the choice of Grassmann number (anticommuting) fields, which leads to the symmetry group being U$(2n)$, as will be discussed below, is not essential. If one uses bosonic (commuting) fields, the symmetry is the noncompact group U(n,n). This is usually not used in the Hall effect because there is no topological term in this case for $n>0$ integer, and so the σ_{xy} dependence found from nonperturbative instanton effects is not seen.[31] However, there is nonetheless a boundary term of the same structure as in the U$(2n)$ case, and for a system with boundaries this has effects even in perturbation theory, and these will be the same in the $n\to0$ limit for either choice of symmetry. We will continue to work with the choice that leads to the compact symmetry. The random potential $V(\mathbf{r})$ has the Gaussian distribution

$$P[V]\propto e^{-\int d^2r\,V^2(\mathbf{r})/2u},$$

as in Sec. II. For $\eta=0$, the action has global ($\mathbf{r}$ independent) U$(2n)$ symmetry, which acts on φ as $\varphi(\mathbf{r})\to U\varphi(\mathbf{r})$, where U is an element of U$(2n)$. For $\eta>0$, the symmetry is broken to U$(n)\times$U(n). In discussing the conducting properties, it is useful to introduce a source term that will generate current correlations. This is done by introducing the vector potential $\mathbf{A}(\mathbf{r})$, where $\mathbf{A}$ is a $2n\times2n$ Hermitian-matrix-valued vector field (not to be confused with the vector field $\mathbf{A}_0$ associated with the constant magnetic field $\mathbf{B}=\nabla\times\mathbf{A}_0$). It is introduced into Z by replacing the covariant derivative $\mathbf{D}$ (viewed as multiplied by I_{2n}) by $\mathbf{D}-i\mathbf{A}$. The generating functional $Z[\mathbf{A}]$ then has gauge invariance, since under a local U$(2n)$ gauge transformation $\varphi(\mathbf{r})\to U\varphi(\mathbf{r})$, $A_\mu\to UA_\mu U^{-1}+U\partial_\mu U^{-1}$, the action $S[\mathbf{A}]$ is invariant, and so is the integration measure $D[\varphi,\overline{\varphi}]$. Performing the functional integral over φ and $\overline{\varphi}$, we obtain

$$\begin{aligned}Z[\mathbf{A}]=&\int D[V]P[V]\exp\mathrm{tr\,Tr}\ln[E-H-i\delta\Lambda\\&-\hbar^2(\mathbf{A}^2+i\mathbf{D}\cdot\mathbf{A}+i\mathbf{A}\cdot\mathbf{D})/2m_e],\end{aligned}\tag{3.2}$$

where Tr denotes the trace over functions in real space as before, and tr is the trace over the $2n$ replicas.

From the above partition function, we cannot obtain the exact bilocal conductivity tensor $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')$ as given in Eq. (1.18), since we cannot generate the below-Fermi-energy contributions. However, on length scales greater than the mean free path, the latter simplify and can be reexpressed as Fermi-energy terms, as will be shown below, and these can be reproduced from our partition function. An exception to this is the $\sigma_{xy}^{II,0}$ term in the bulk, which therefore describes true non-Fermi-energy physics. We can, however, obtain the expression for the two-probe conductance, which can be written in terms of Fermi-energy quantities alone.[42] Let us assume that the source field $\mathbf{A}$ is independent of y, then using $\delta/\delta A(x)$ to denote a functional derivative for a y-independent variation, we can show that

$$-\lim_{\mathbf{A}\to 0}\frac{\delta^2 Z[\mathbf{A}]}{\delta A_{x,11}^{+-}(x)\,\delta A_{x,11}^{-+}(x')}$$
$$=\int D[V]P[V]\int dy\int dy'\Big\{e^{\mathrm{tr\,Tr}\ln(E-H-i\eta\Lambda)}$$
$$\times\Big[\hbar^2\frac{G^+(\mathbf{r},\mathbf{r}')+G^-(\mathbf{r},\mathbf{r}')}{2m_e}\delta(\mathbf{r}-\mathbf{r}')-\hbar^2 G^-(\mathbf{r}',\mathbf{r})$$
$$\times\left(\frac{-i\hbar}{2m_e}\overleftrightarrow{D}^*_{x'}\right)\left(\frac{-i\hbar}{2m_e}\overleftrightarrow{D}_x\right)G^+(\mathbf{r},\mathbf{r}')\Big]\Big\}. \tag{3.3}$$

The δ-function term is related to $\sigma_{xx}^{++}(\mathbf{r},\mathbf{r}')$ and $\sigma_{xx}^{--}(\mathbf{r},\mathbf{r}')$ (see Sec. I D). We argued in Sec. I D that $\sigma_{xx}^{aa}(\mathbf{r},\mathbf{r}')$ $(a=+,-)$ in disordered systems are short-ranged, and can be treated as contact terms $\sigma_{xx}^{aa}\delta(\mathbf{r}-\mathbf{r}')$. One can show using the commutation relations $v_x=i[\hat{H},x]/\hbar$ and $[v_x,x]=\hbar/im_e$ (Refs. 42, 55, and 32) that

$$\sigma_{xx}^{aa}=-\frac{\hbar^2}{2LW}\mathrm{Tr}[G^a(E_F)v_xG^a(E_F)v_x]=\frac{\hbar^2}{2m_e}G^a(\mathbf{r},\mathbf{r}).$$

Therefore, the expression in the square bracket in Eq. (3.3) gives an approximate version of the unaveraged $\sigma_{xx}(\mathbf{r},\mathbf{r}')$, Eq. (1.18), valid on scales greater than the mean free path l. The δ-function term drops out for two cross sections far apart. Taking the limit $n\to 0$, in which case $e^{\mathrm{tr\,Tr}\ln(E-H-i\eta\Lambda)}\to 1$, Eq. (3.3) gives the disorder-averaged two-probe conductance

$$\langle g(x,x')\rangle=-\lim_{n\to 0}\lim_{\mathbf{A}\to 0}\frac{\delta^2 Z[\mathbf{A}]}{\delta A_{x,11}^{+-}(x)\,\delta A_{x,11}^{-+}(x')}. \tag{3.4}$$

Similarly, the second moment of the conductance can be obtained by applying four derivatives to the partition function:

$$\langle g(x_1,x_1')g(x_2,x_2')\rangle$$
$$=\lim_{n\to 0}\lim_{\mathbf{A}\to 0}\frac{\delta^4 Z[\mathbf{A}]}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,11}^{-+}(x_1')\,\delta A_{x,22}^{+-}(x_2)\,\delta A_{x,22}^{-+}(x_2')}. \tag{3.5}$$

These expressions should be independent of $x_1,\ldots,x_2'$. This will be discussed in Sec. III B.

An effective NLσM action can be derived using the Hubbard-Stratonovich transformation;[28,30] a mean-field approximation then corresponds to the SCBA. The fluctuations about this mean-field theory produce all of the higher-order effects. An effective action for the long-range effects can then be derived; we obtain, retaining only terms with no more than two derivatives, and omitting the gauge field A until Sec. III B,

$$S=\int d^2r\{-\tfrac{1}{8}\sigma_{xx}^0\mathrm{tr}[\partial_\mu Q\partial_\mu Q]-\tfrac{1}{8}\sigma_{xy}^0\mathrm{tr}[\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q]$$
$$+\eta\ \mathrm{tr}[Q\Lambda]\}, \tag{3.6}$$

$$Z=\int D[Q]e^S. \tag{3.7}$$

The field Q is a $2n\times 2n$ Hermitian matrix obeying (at each $\mathbf{r}$) $Q^2=I_{2n}$, of which n eigenvalues are equal to $+1$, n to -1. This means that, at each $\mathbf{r}$, Q takes values in the coset manifold $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$. The action S has the same symmetry as the original action, $Q(\mathbf{r})\to UQ(\mathbf{r})U^{-1}$, but Q is invariant under the diagonal U(1) subgroup of $\mathrm{U}(2n)$. The remaining $\mathrm{SU}(2n)$ symmetry is again broken to $\mathrm{S}[\mathrm{U}(n)\times\mathrm{U}(n)]$ for nonzero η. The parameters σ_{xx}^0 and σ_{xy}^0 are bare conductivity parameters, like those resulting from the SCBA (but may differ by finite renormalizations, corresponding to short-range effects that are not included again by the NLσM), and describe the response of the system at the scale of the short-distance cutoff, which is of order the mean free path l. The measure $D[Q]=\Pi_{\mathbf{r}}dQ$ is the product over points $\mathbf{r}$ inside the sample of the unique $\mathrm{SU}(2n)$-invariant measures dQ on the space $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$ for each point $\mathbf{r}$. At the ends $x=0$, L, we impose $Q=\Lambda$ to represent the absorbing boundary condition.

B. Gauge invariance, current conservation, and boundary condition

In this subsection, we discuss the way in which the gauge potential $\mathbf{A}$ enters the NLσM action, and the related questions of current conservation, the equation of motion, and the tilted boundary condition. We begin by requiring that the action be gauge invariant. In Sec. III C, we will modify it to a non-gauge-invariant form to bring the conductivities into line with those discussed in the previous sections.

In view of the gauge invariance of the generating functional $Z[\mathbf{A}]$, the action $S[\mathbf{A}]$, including $\mathbf{A}$, should also be invariant (when $\eta=0$) under the local gauge transformation $Q\to U(\mathbf{r})Q(\mathbf{r})U^{-1}(\mathbf{r})$, $A_\mu\to UA_\mu U^{-1}-iU\partial_\mu U^{-1}$ (it is assumed that U respects $Q=\Lambda$ at the ends; the invariance of the functional integration measure implies that invariance of the action ensures invariance of $Z[\mathbf{A}]$). The simplest way to introduce the external source field $\mathbf{A}$ into the NLσM is to replace the partial derivative $\partial_\mu Q$ by the covariant derivative $D_\mu Q=\partial_\mu Q+i[A_\mu,Q]$ everywhere in S. This leads to a manifestly gauge-invariant action, which is given below as the σ_{xx}^0 and $\sigma_{xy}^{I,0}$ terms in Eq. (3.8). However, this is not the only way. The second way is less obvious and will follow a brief digression.

To obtain the other gauge-invariant coupling to **A**, let us first point out that the topological term, without any **A** dependence, is a total derivative, which is a function of the values of Q on the boundary and the homotopy class of Q in the interior for given boundary values, but not of the detailed form of Q in the interior. That is, it is possible to change Q in the interior, leaving its boundary values fixed, in such a way that the topological term *changes*. Now any change in Q can be viewed as the result of a gauge transformation, and a transformation U that leaves the boundary values of Q unchanged must reduce at each point on the boundary to some element in a certain $\mathrm{S}[\mathrm{U}(n)\times \mathrm{U}(n)]$ subgroup, determined by Q at that point. Such a gauge transformation is characterized by an integer-valued winding number q, say, which describes the winding of the gauge transformation U at the edge, and it changes the topological term by $2\pi i\sigma_{xy}^{0}q$ (for σ_{xy}^{0} equal to an integer, this has no effect on the exponential of the action). In particular, there is a continuously connected class of transformations for which $q=0$, so that there are continuously-connected classes of configurations Q with given boundary values throughout which the topological term takes the same value. Further discussion of the topological issues, related to edge states and quantization, is contained in Appendix C.

Now, because the topological term is (apart from the topological effects just discussed) a function of only the boundary values of Q, this suggests that we can attempt to compensate for a gauge transformation by including a coupling to **A** on only the reflecting (hard) walls. The form of this coupling can be easily obtained, and on including both forms of coupling with different coefficients $\sigma_{xy}^{I,0}$, and $\sigma_{xy}^{II,0}$, whose sum is $\sigma_{xy}^{0}=\sigma_{xy}^{I,0}+\sigma_{xy}^{II,0}$, in the respective forms of the topological term, we obtain the action which is gauge-invariant when $\eta=0$,

$$\begin{aligned}S[\mathbf{A}]=\int d^2r[&-\tfrac{1}{8}\sigma_{xx}^{0}\mathrm{tr}(D_\mu Q D_\mu Q)\\&-\tfrac{1}{8}\sigma_{xy}^{I,0}\mathrm{tr}\,(\epsilon_{\mu\nu}QD_\mu QD_\nu Q)\\&-\tfrac{1}{8}\sigma_{xy}^{II,0}\mathrm{tr}(\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q)+\eta\mathrm{tr}\,(Q\Lambda)]\\&+\frac{i}{2}\sigma_{xy}^{II,0}\int dx\{\mathrm{tr}[A_x(x,W)Q(x,W)]\\&-\mathrm{tr}[A_x(x,0)Q(x,0)]\}.\end{aligned}\tag{3.8}$$

The justification for identifying this split of σ_{xy}^{0} into two pieces $\sigma_{xy}^{I,0}$, and $\sigma_{xy}^{II,0}$, with that found in the previous sections, as implied by the choice of notation, will be given below. We note that the action is independent of the trace of **A**. To see this, we may split **A** into the sum of a traceless part and the trace multiplied by $I_{2n}/2n$. The latter part is the gauge potential corresponding to the diagonal U(1) subgroup generated by I_{2n}, and it does not contribute to $D_\mu Q$ (because $[I_{2n},Q]=0$), or to the $\sigma_{xy}^{II,0}$ edge coupling (because $\mathrm{tr}Q=0$). Hence $S[\mathbf{A}]$ is independent of it, but it may be left in for convenience. The action $S[\mathbf{A}]$ is easily verified to be invariant under any gauge transformation, including the topologically nontrivial ones that leave Q on the boundary unchanged.

Although we have not given a derivation of our gauge-invariant action $S[\mathbf{A}]$, Eq. (3.8), from the gauge-invariant generating functional (3.2) for averages of products of Green's functions at the Fermi energy, it is not very difficult to extend the existing derivations (see, e.g., Ref. 30) to include **A**, and obtain Eq. (3.8). It is almost self-evident that this will be obtained, by comparing (i) the diagrams for the response to **A** *at the Fermi energy* that are obtained from Eq. (3.2) with (ii) those studied in detail for the mean bilocal conductivity tensor within the SCBA in Sec. II, and (iii) those obtained in the perturbation theory for the NLσM constructed below.

We now use the gauge invariance of S in the $\eta\to 0$ limit to derive some current-conservation relations for the NLσM; only infinitesimal, topologically trivial gauge transformations are needed for this. The tilted boundary condition is one consequence of current conservation. We will recover the expression for the bilocal conductivity in the SCBA as the leading-order term in an expansion in $1/\sigma_{xx}^{0}$. We will also show that the conductance we obtained using Eq. (3.4) is independent of the positions of the cross sections so that the volume-integral form for the conductance can be used.

We begin by considering the equations of motion that follow from the action $S[\mathbf{A}]$. As their name implies, these are the equations of motion that are obtained if $S[\mathbf{A}]$ is used as the action of a classical nonlinear field theory (in one space and one imaginary time dimension). The canonical way to obtain these equations is to seek an extremum of S, such that $\delta S/\delta Q=0$, where the variation, which is the usual one that varies both the x and y dependence of Q, respects the restrictions $Q^2=I$, $Q=Q^\dagger$, and the boundary condition $Q=\Lambda$ at the open ends (we note that this is imposed on all configurations in the functional integral). The resulting equations also serve as operator relations in the quantum field theory that we take to be defined by the functional integral

$$Z[\mathbf{A}]=\int D[Q]e^{S[\mathbf{A}]}.$$

In functional integral language, the equations of motion become identities among correlation functions. They are obtained in general by the following argument: Consider a small change in Q, $Q\to Q'=Q+\delta Q$, as a change of variable in the functional integral. Since Q is integrated over, such a change can have no effect on $Z[\mathbf{A}]$. On the other hand, it changes S, and provided the change is such that the Jacobian resulting from the change in measure is 1, we obtain the identity

$$0=\int D[Q]\frac{\delta S}{\delta Q}e^{S[\mathbf{A}]},$$

which is the equation of motion.

A variation of Q that respects its form can be parametrized as $Q'=UQU^{-1}$, where $U=\exp iR$ and $R(\mathbf{r})$ is a $2n\times 2n$ hermitian matrix function of $\mathbf{r}$, and thus is a gauge transformation, which leaves the integration measure unchanged; however, we vary Q while leaving **A** fixed. If we view $S[\mathbf{A}]$ as a functional of Q as well as of **A**, thus writing $S[\mathbf{A}]\equiv S\{Q,\mathbf{A}\}$ (the curly brackets are used to avoid confusion of the two arguments of the functional with a commutator), then the equations of motion are equivalent to

$\delta S\{Q',\mathbf{A}\}/\delta R=0$, evaluated at $R=0$. This may be reexpressed by making use of the gauge invariance of $S\{Q,\mathbf{A}\}$. Gauge invariance tells us that if $Q'=UQU^{-1}$, $\mathbf{A}'=U\mathbf{A}U^{-1}-iU\nabla U^{-1}$, then

$$
\begin{aligned}
S\{Q,\mathbf{A}\}&=S\{Q',\mathbf{A}'\}\\
&=S\{Q+i[R,Q],\mathbf{A}\}+S\{Q,\mathbf{A}+i[R,\mathbf{A}]-\nabla R\}\\
&\quad -S\{Q,\mathbf{A}\}
\end{aligned}
\tag{3.9}
$$

to first order in R. It follows that, for $\mathbf{r}$ in the interior of the system, we obtain

$$
\begin{aligned}
\frac{\delta S}{\delta R_{\beta\alpha}}&=-\partial_\mu\frac{\delta S}{\delta A_{\mu,\beta\alpha}}-iA_{\mu,\alpha\gamma}\frac{\delta S}{\delta A_{\mu,\beta\gamma}}+iA_{\mu,\gamma\beta}\frac{\delta S}{\delta A_{\mu,\gamma\alpha}}\\
&=-(D_\mu j^\sigma_\mu)_{\alpha\beta}.
\end{aligned}
\tag{3.10}
$$

Here we have used $\alpha,\beta,\gamma,\ldots,$ for indices running from 1 to $2n$ (the first n values being $i,+,i=1,\ldots,n$, the remainder $i,\ -,\ i=1,\ldots,n$), with a summation convention, and introduced the definition of the current in the NLσM,

$$
j^\sigma_{\mu,\alpha\beta}(\mathbf{r})=\frac{\delta S[\mathbf{A}]}{\delta A_{\mu,\beta\alpha}(\mathbf{r})},
\tag{3.11}
$$

which is implicitly a local function of $Q(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$. The covariant derivative of $\mathbf{j}^\sigma$ is defined in the same way as that of Q. Therefore, using the notation

$$
\langle\langle\cdots\rangle\rangle=\int D[Q]\cdots e^{S[\mathbf{A}]}/Z[\mathbf{A}],
$$

where the $\cdots$ represent any functional of Q, and using the fact that the integration measure is invariant under U$(2n)$ gauge transformations, the equation of motion becomes the matrix equation

$$
\mathbf{D}\cdot\langle\langle\mathbf{j}^\sigma\rangle\rangle=0.
\tag{3.12}
$$

This can be read as the statement that the covariant divergence of the current is zero in the interior of the system. As such it is known as a Ward identity, and is a consequence of the U$(2n)$ symmetry of the original action S; the existence of a covariantly-conserved current as a consequence of a gauge-invariant coupling to a gauge potential is the essential content of Noether's theorem. A feature of the NLσM is that the Ward identities are not just consequences of, but are equivalent to, the equations of motion. There are also many similar Ward identities when the functional average contains $\mathbf{D}\cdot\mathbf{j}^\sigma$ times other functionals of Q. Particular cases of these, including all those that will be of interest in this paper, are those that contain other currents $\mathbf{j}^\sigma$. These may be obtained by taking functional derivatives of the basic Ward identity (3.12) with respect to $\mathbf{A}$, since the left-hand side is still a functional of $\mathbf{A}$. Functional derivatives of the action yield currents, however, $\mathbf{j}^\sigma$ contains $\mathbf{A}$, and so does the covariant derivative $\mathbf{D}$, so there are additional δ-function terms. The δ-function terms in the response functions that result from the $\mathbf{A}$ in $\mathbf{j}^\sigma$ will be referred to as contact terms, and correspond to those in earlier sections.

The above Ward identity, or equation of motion, (3.12), is only the bulk part of the system of equations. There are also boundary equations on the reflecting (hard) wall or edge. These come from two sources: (i) the boundary terms in the action $S[\mathbf{A}]$, and (ii) the boundary term that appears when integrating by parts to transfer the derivative from ∇R to $\delta S/\delta A$ in Eq. (3.9) when taking the functional derivative with respect to R. The boundary part of the equation of motion can be obtained in either of two equivalent ways. One way is to take the functional derivative with respect to the full dependence of R on the coordinates x and y, and obtain a single equation like Eq. (3.12) but containing δ-function terms at the edge. Since an equation that sets a sum of a finite function and a δ function to zero implies that each piece separately vanishes, we obtain the bulk equation of motion (or Ward identity) as in Eq. (3.12), together with a boundary condition that states that the function multiplying the δ function at the edge is zero. The other method is to separate the change in the action due to R, after integrating by parts to remove derivatives from R, into a bulk part that yields the bulk equation above, and a boundary part, then take a functional derivative with respect to the single coordinate x that describes the position on the boundary, to yield the boundary conditions. This was the method followed in Sec. I C for the quadratic action there, Eq. (1.11). With either method, it is straightforward to obtain the results below.

First, we should record the actual expression for the current density, obtained from definition (3.11). It is

$$
\begin{aligned}
j^\sigma_\mu(\mathbf{r})&=-i[\sigma^0_{xx}Q(\mathbf{r})D_\mu Q(\mathbf{r})-\sigma^{I,0}_{xy}\epsilon_{\mu\nu}D_\nu Q(\mathbf{r})]/2\\
&\quad +i\sigma^{II,0}_{xy}\delta_{\mu x}Q(\mathbf{r})[\delta(y-W)-\delta(y)]/2
\end{aligned}
\tag{3.13}
$$

(we used the identity $Q\partial_\mu Q=-\partial_\mu Q.Q$ which follows from $Q^2=I_{2n}$). Here we see explicitly the edge contribution with coefficient $\sigma^{II,0}_{xy}$. It is simply proportional to Q at the edge.

The boundary condition at the reflecting walls (assumed parallel to the x axis as always in this paper), that is obtained by either of the methods described above, can be written

$$
\sigma^0_{xx}QD_yQ+\sigma^{I,0}_{xy}D_xQ+\sigma^{II,0}_{xy}D_xQ=0.
\tag{3.14}
$$

As in the case of the bulk equation of motion, in the quantum field theory of the NLσM (defined by the functional integral), this is valid only when inserted in the average $\langle\langle\ \ldots\ \rangle\rangle$. This equation can be interpreted as stating the covariant conservation of the current at the edge, in a very similar way to that discussed in Appendix A within the SCBA. For $\sigma^{II,0}_{xy}=0$, we would have only the first two terms, and it would state simply that the normal component of the (bulk) current in Eq. (3.13) tends to zero at the edge. These terms originate from the edge term left after integrating the bulk part of the action by parts in order to take $\delta/\delta R$. In the presence of a nonzero $\sigma^{II,0}_{xy}$, this boundary condition is modified to include the last term, which is the covariant derivative (along the edge) of the edge part of the current in Eq. (3.13). Thus Eq. (3.14) is equivalent to

$$
j^\sigma_{y,\mathrm{bulk}}-D_x j^\sigma_{x,\mathrm{edge}}=0,
\tag{3.15}
$$

where the edge contribution is obtained as $j^\sigma_{x,\mathrm{edge}}(x,W)=\int_{W-0^+}^{W+0^+}dy\, j^\sigma_x(x,y)$, and similarly for the edge at $y=0$, as in Appendix A. The edge term originates, of course, from taking $\delta/\delta R$ on the edge term in the action itself, Eq. (3.8). In the boundary condition, the last two terms can be combined to leave

$$\sigma^0_{xx}QD_yQ+\sigma^0_{xy}D_xQ=0. \tag{3.16}$$

This is the analog of the tilted boundary condition discussed in Secs. I and II, generalized to the full *nonlinear* field theory, and including **A**; as in those discussions, only the total σ^0_{xy} enters the boundary condition. We will show, in Sec. III C below, that, in leading order in perturbation theory, this boundary condition reduces to exactly the one used in earlier sections.

To close this subsection, we obtain Ward identities that apply to moments of the two-probe conductance, in which the currents are integrated across sections parallel to the y axis. We have already seen that these should be calculable using only Green's functions at the Fermi energy, which can be obtained using our generating function or the NLσM. As in Sec. III A, we will therefore here specialize **A** to be in the x direction and to be independent of y. Functional derivatives of $Z[\mathbf{A}]$ of the form $\delta/\delta A_x(x)$ then produce the desired mean and variance of the two-probe conductance. If we specialize to such **A** in the NLσM action $S[\mathbf{A}]$ [noting that $A_x(x,W)=A_x(x,0)=A_x(x,y)$ for all x, y], then we see that the σ^0_{xy} terms can be combined, using an integration by parts, to leave

$$S[A_x]=\int d^2r[-\tfrac{1}{8}\sigma^0_{xx}\mathrm{tr}(D_\mu QD_\mu Q)$$
$$-\tfrac{1}{8}\sigma^0_{xy}\mathrm{tr}(\epsilon_{\mu\nu}QD_\mu QD_\nu Q)+\eta\ \mathrm{tr}(Q\Lambda)], \tag{3.17}$$

in which we emphasize that **A** takes the specialized form $\mathbf{A}(\mathbf{r})=[A_x(x),0]$, independent of y. This action is gauge invariant, and A_x remains y independent, for gauge transformations U that are independent of y. Using such a transformation, we can obtain, similarly to the above derivation, the identity

$$D_x\int dy\langle\langle j^\sigma_x(x,y)\rangle\rangle=0, \tag{3.18}$$

in which j^σ_x and the action $S[A_x]$ to be used in calculating the average still contain A_x. We note that, in $\int dy\, j^\sigma_x$, using Eq. (3.13), $\sigma^{I,0}_{xy}$ and $\sigma^{II,0}_{xy}$ terms can be combined into a single term, as implied by the action $S[A_x]$, in which the same is true. Thus from this point on, only the total σ^0_{xy} enters the calculations for the conductance and its moments.

On taking further functional derivatives $\delta/\delta A_x(x)$ of Eq. (3.18), we can obtain identities involving the mean and variance (or alternatively, second moment) of the conductance. From the discussion in earlier sections, we should have

$$\partial_x\langle g(x,x')\rangle=0, \tag{3.19}$$

for all x and x' inside the sample, including $x=x'$, which expresses the independence of the conductance on the location of the cross sections, and similar statements should hold for the higher moments and for the dependence on the other variables x', ... One may be concerned that D_x appears in Eq. (3.18), not ∂_x, and that this might lead to additional terms containing e.g., $\delta(x-x')$ when further functional derivatives are taken. However, we have verified that such terms vanish, to all orders in perturbation theory, for the replica components we require to produce the mean and variance of the conductance, as in Eqs. (3.4) and (3.5). Thus we have performed all the steps in a derivation showing that the calculation of the two-probe conductance and its variance, in the rectangular geometry, can be carried out within the NL σM, including the independence of the locations of the cross sections, which allows the use of an average over these locations. The explicit expressions for the mean and variance are given in Secs. IV and V, and evaluated to leading order in the perturbation expansion.

C. Perturbation expansion, current conservation, and bilocal conductivity

The approach given in Sec. III C is sufficient for the two-probe conductance, provided the cross sections used are parallel to the y axis. For more general cross sections, and to consider the non-Fermi-energy effects and the bilocal conductivity, a deeper analysis is required, which is contained in the present subsection, but which may be skipped (apart from the first part introducing the perturbation expansion) by readers interested only in the two-probe conductance.

Our goal here is to compare the consequences of the definition of the current and the Ward identities with the properties of the currents in our model of the original electron system. To provide motivation, we will compare results in the NLσM formulation with those in the SCBA in Sec. II, and for this purpose we will now introduce the perturbation expansion of the model. We then show how to modify the coupling of **A** to the NLσM so as to reproduce the properties found in Sec. II.

The matrix Q, which obeys $Q^\dagger=Q$, $Q^2=I_{2n}$, can be parametrized in the following way:

$$Q=\begin{pmatrix}\sqrt{1-zz^\dagger} & z\\ z^\dagger & -\sqrt{1-z^\dagger z}\end{pmatrix}, \tag{3.20}$$

where z is an $n\times n$ complex matrix. Expanding the NLσM action in terms of the z matrix, we obtain

$$S[\mathbf{A}]=S_0[\mathbf{A}]+S_1[\mathbf{A}], \tag{3.21}$$

where $S_0[\mathbf{A}]$ is the part quadratic in z, $z^\dagger$, and **A**, which is the same as in Eq. (1.11), except that there are now n^2 copies of z, and we include the gauge potential **A**. For **A** of the restricted form

$$\mathbf{A}=\begin{pmatrix}0 & \mathbf{A}^{+-}\\ \mathbf{A}^{-+} & 0\end{pmatrix}, \tag{3.22}$$

where $\mathbf{A}^{+-}$ is a complex $n\times n$-matrix-valued vector field, and $\mathbf{A}^{-+}$ is its adjoint (these are the only components of **A** that will be used below), we have

$$S_0[\mathbf{A}]=-\frac{\sigma_{xx}^0}{4}\int d^2r\,\mathrm{tr}[(\partial_\mu z-2iA_\mu^{+-})(\partial_\mu z^\dagger+2iA_\mu^{-+})]$$
$$-\frac{\sigma_{xy}^{I,0}}{4}\int d^2r\,\epsilon_{\mu\nu}\mathrm{tr}[(\partial_\mu z-2iA_\mu^{+-})(\partial_\nu z^\dagger+2iA_\nu^{-+})]$$
$$-\frac{\sigma_{xy}^{II,0}}{4}\int d^2r\,\epsilon_{\mu\nu}\mathrm{tr}(\partial_\mu z\partial_\nu z^\dagger)$$
$$-\frac{i\sigma_{xy}^{II,0}}{2}\oint dl_\mu\mathrm{tr}(A_\mu^{+-}z^\dagger-zA_\mu^{-+}). \tag{3.23}$$

The line integral $\oint dl_\mu$ is taken in the counterclockwise direction around the edge of the sample. Here and below, we use the symbol tr for a trace on the n-dimensional space, as well as for that on the $2n$-dimensional one; it should be clear from the context which is meant. S_1 describes the interaction between the diffusion modes caused by quantum interference effects. We give it here only for $\mathbf{A}=0$, and to the order $O[(zz^\dagger)^3]$ required for our later calculations,

$$S_1[0]=\int d^2r\bigg\{-\frac{\sigma_{xx}^0}{32}\mathrm{tr}[\partial_\mu(zz^\dagger)\partial_\mu(zz^\dagger)+\partial_\mu(z^\dagger z)\partial_\mu(z^\dagger z)]$$
$$-\frac{\sigma_{xy}^0}{32}\mathrm{tr}[\epsilon_{\mu\nu}\partial_\mu(zz^\dagger z\partial_\nu z^\dagger)-\epsilon_{\mu\nu}\partial_\mu(z^\dagger zz^\dagger\partial_\nu z)]$$
$$-\frac{\sigma_{xx}^0}{64}\mathrm{tr}[\partial_\mu(zz^\dagger)\partial_\mu(zz^\dagger zz^\dagger)+\partial_\mu(z^\dagger z)\partial_\mu(z^\dagger zz^\dagger z)]$$
$$-\frac{\sigma_{xy}^0}{64}\mathrm{tr}[\epsilon_{\mu\nu}\partial_\mu(zz^\dagger zz^\dagger z\partial_\nu z^\dagger)$$
$$-\epsilon_{\mu\nu}\partial_\mu(z^\dagger zz^\dagger zz^\dagger\partial_\nu z)]+O[(zz^\dagger)^4]\bigg\}. \tag{3.24}$$

Notice that terms proportional to σ_{xy}^0 can all be written as total derivatives; therefore, they can be expressed as boundary terms. To calculate the ensemble average of any quantity $X[z,z^\dagger]$, we perform the following expansion:

$$\langle\langle X\rangle\rangle=\lim_{n\to0}\int D[z,z^\dagger]I[z,z^\dagger]X[z,z^\dagger]e^{S_0\{z,z^\dagger,0\}}$$
$$\times\sum_{m=0}^{\infty}\frac{1}{m!}S_1^m[z,z^\dagger]. \tag{3.25}$$

Here $I[z,z^\dagger]$ is the Jacobian needed to make the measure in the z, $z^\dagger$ space invariant under a $U(2n)$ rotation at each $\mathbf{r}$. The explicit form of this Jacobian will not be needed. Its only role is to cancel quadratically divergent diagrams that arise in perturbation theory, in a manner that is standard for all NLσM's (see, e.g., Ref. 57). The terms in the expansion can be written in terms of averages calculated using the quadratic action with $\mathbf{A}=0$, defined by

$$\langle\langle\cdots\rangle\rangle_0=\int D[z,z^\dagger]\cdots e^{S_0[\mathbf{A}=0]}/Z_0[0], \tag{3.26}$$

in which the functional integral in the denominator is the same as the numerator but with the insertion $\cdots$ omitted. The expressions can be evaluated by contracting pairs of z and $z^\dagger$, which gives the diffusion propagator

$$\frac{\sigma_{xx}^0}{4}\langle\langle z_{ij}^\dagger(\mathbf{r})z_{kl}(\mathbf{r}')\rangle\rangle_0=\delta_{il}\delta_{jk}d(\mathbf{r},\mathbf{r}'), \tag{3.27}$$

which obeys the same conditions (1.16) as in earlier sections. The basic perturbation expansion is now a series in powers of $1/\sigma_{xx}^0$, though it will also be convenient to expand in powers of $\gamma=\sigma_{xy}^0/\sigma_{xx}^0$ to obtain a double expansion.

We now return to the physical meaning of the Ward identities (3.12) and (3.16), that resulted from the gauge invariance of the action $S[\mathbf{A}]$. We wish to compare these with our physical expectation that the current is divergenceless inside the sample, and that no current flows in or out of the sample at the reflecting walls (as we have shown in Sec. II, and discussed in Sec. I, the current response obtained in the SCBA is not divergenceless, because of the bulk $\sigma_{xy}^{II,0}$ term, but that is a non-Fermi-energy effect that will not be considered in the present formalism until later in this subsection). The first difficulty that seems to arise [as with Eq. (3.18)] is that the Ward identity states $\mathbf{D}\cdot\mathbf{j}^\sigma=0$, not $\nabla\cdot\mathbf{j}=0$, as we might have expected. The vector potential $\mathbf{A}$ present in $\mathbf{D}$ will generate δ-function terms when further functional derivatives are taken to obtain Ward identities, as must be done for the bilocal conductivity tensor and analogous correlators of more than two currents. However, as we mentioned in connection with the conductance in Sec. III B, in practice, for the particular components of $\mathbf{A}$ that yield the physically relevant conductivities, this does not seem to occur. For example, in addition to the results cited in Sec. III B, we can show that the Ward identity implies

$$\partial_\mu\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle=0 \tag{3.28}$$

for the mean bilocal conductivity tensor calculated in the NLσM using the action $S[\mathbf{A}]$, and this is valid for all $\mathbf{r}$ and $\mathbf{r}'$ inside the sample, including $\mathbf{r}=\mathbf{r}'$, to all orders in perturbation theory.

The boundary condition (3.14) also involves the tangential covariant derivative of the edge current, not the usual partial derivative as one would want in the electronic system. In this case, we do find a clash between the theory as formulated and our intuition. Eq. (3.15) is more explicitly

$$j_{y,\mathrm{bulk}}^\sigma-\partial_x j_{x,\mathrm{edge}}^\sigma=-\tfrac{1}{2}\sigma_{xy}^{II,0}[A_x,Q]. \tag{3.29}$$

The left-hand side is the combination one might have expected to be zero. However, it is nonzero when A_x is nonzero, implying that conservation is violated by δ-function terms on the edge in the bilocal conductivity and its moments. Note that similar commutators $[A_x,Q]$ and $[A_y,Q]$ appear in $j_{y,\mathrm{bulk}}^\sigma$ but are not a problem.

To illuminate the point further, we can use the perturbation expansion and compute the mean bilocal conductivity within the NLσM as formulated so far. From $S_0[\mathbf{A}]$, the equation of motion for $z^\dagger$ (formulas for z are similar) is, in the bulk

$$\sigma_{xx}^0\nabla^2z^\dagger=-2i(\sigma_{xx}^0\partial_\mu A_\mu^{-+}+\sigma_{xy}^{I,0}\epsilon_{\mu\nu}\partial_\mu A_\nu^{-+}), \tag{3.30}$$

(as in Sec. I A, where, however, $\sigma_{xy}^0=\sigma_{xy}^{I,0}$ and $\sigma_{xy}^{II,0}=0$), and at the edge is the tilted boundary condition

$$\sigma_{xx}^{0}(\partial_y z^{\dagger}+2iA_y^{-+})-\sigma_{xy}^{0}(\partial_x z^{\dagger}+2iA_x^{-+})=0. \quad (3.31)$$

These equations give the generalization of the ''classical'' theory of Sec. I A to include the edge currents with coefficient $\sigma_{xy}^{II,0}$, on using the identifications given in Sec. I D. The currents are given, in the present approximation, by

$$\begin{aligned}\delta S_0/\delta A_\mu^{+-}&\equiv j_\mu^{-+}\\&=\tfrac{1}{2}i\{\sigma_{xx}^{0}(\partial_\mu z^{\dagger}+2iA_\mu^{-+})+\sigma_{xy}^{I,0}\epsilon_{\mu\nu}(\partial_\nu z^{\dagger}\\&\quad+2iA_\nu^{-+})+\sigma_{xy}^{II,0}[\delta(y-W)-\delta(y)]\delta_{\mu x}z^{\dagger}\}.\end{aligned} \quad (3.32)$$

The bulk equation of motion can therefore be written $\partial_\mu j_\mu^{-+}=0$, while at the edge we have

$$j_{y,\mathrm{bulk}}^{-+}-\partial_x j_{x,\mathrm{edge}}^{-+}=-\sigma_{xy}^{II,0}A_x^{-+}, \quad (3.33)$$

which can also be written

$$j_{y,\mathrm{bulk}}^{-+}-D_x j_{x,\mathrm{edge}}^{-+}=0, \quad (3.34)$$

where the covariant derivative is the linearized version of that in the full NLσM, namely $D_\mu z^{\dagger}\equiv\partial_\mu z^{\dagger}+2iA_\mu^{-+}$, while $D_\mu D_\nu z^{\dagger}\equiv\partial_\mu D_\nu z^{\dagger}$. Thus, for current j^{-+} as defined here, current conservation in the naive form is violated by δ-function terms at the edge. This can be rectified, but, before doing so, we calculate the bilocal conductivity of the present model in the same approximation.

The mean bilocal conductivity in the present approximation is obtained as [compare Eq. (3.4); we leave implicit the choice of all replica components equal to 1, and the $n\to 0$ limit]

$$\begin{aligned}\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_0&\equiv-\lim_{\mathbf{A}\to 0}\frac{\delta}{\delta A_\nu^{-+}(\mathbf{r}')}\langle\langle j_\mu^{-+}(\mathbf{r})\rangle\rangle_0\\&=(\sigma_{xx}^{0}\delta_{\mu\nu}+\sigma_{xy}^{I,0}\epsilon_{\mu\nu})\delta(\mathbf{r}-\mathbf{r}')\\&\quad-\langle\langle j_\mu^{-+}(\mathbf{r})j_\nu^{+-}(\mathbf{r}')\rangle\rangle_0.\end{aligned} \quad (3.35)$$

On evaluating this using Eq. (3.32) (with $\mathbf{A}=0$), the similar formula for j^{+-}, and the definition (3.27) of d, we obtain the same result as in Eq. (2.20), except that the bulk $\sigma_{xy}^{II,0}$ term is not present. This result obeys $\partial_\mu\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_0=0$ for all $\mathbf{r}$, $\mathbf{r}'$ in the bulk, and Eq. (3.33) implies that

$$\langle\sigma_{y\nu}(\mathbf{r},\mathbf{r}')\rangle_0-\partial_x\int_{W-0^+}^{W+0^+}dy\langle\sigma_{x\nu}(\mathbf{r},\mathbf{r}')\rangle_0=\sigma_{xy}^{II,0}\delta_{\nu x}\delta(\mathbf{r}'-\mathbf{r}) \quad (3.36)$$

for $\mathbf{r}$ at the upper edge $y=W$ (and similarly for the lower). In effect, in the edge channel, A_x^{-+} simply creates current, so the naive conservation law is violated.

In the SCBA, we did not directly address this issue, but derived $\partial_x j_{x,\mathrm{edge}}^{\mathrm{SCBA}}-j_{y,\mathrm{bulk}}^{\mathrm{SCBA}}=0$ for $\mathbf{r}\neq\mathbf{r}'$ only. On the other hand, in the SCBA we also found a non-Fermi-energy contribution $\sigma_{xy}^{II,0}\epsilon_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')$ in the bulk, which means that, in the presence of $\mathbf{E}$, there is an additional part $(e^2/h)\sigma_{xy}^{II,0}\epsilon_{\mu\nu}E_\nu$ in the bulk current

$$j_\mu^{\mathrm{SCBA}}=j_{\mu,E=E_F}^{\mathrm{SCBA}}+\frac{e^2}{h}\sigma_{xy}^{II,0}\epsilon_{\mu\nu}E_\nu \quad (3.37)$$

within the SCBA. If we introduce a corresponding change in the bulk current here, so that

$$j_{\mu,\mathrm{mod}}^{-+}=j_\mu^{-+}-\sigma_{xy}^{II,0}\epsilon_{\mu\nu}A_\nu^{-+}, \quad (3.38)$$

then the tilted boundary condition (3.33) becomes

$$j_{y,\mathrm{bulk,mod}}^{-+}-\partial_x j_{x,\mathrm{edge,mod}}^{-+}=0 \quad (3.39)$$

in the presence of $\mathbf{A}$. Thus the modified current is conserved (not covariantly) at the edge (and so is j_μ^{SCBA}); the current in the edge channel comes from the bulk. Physically, there are two modes of conduction response to an electric field in the system. One is the ''sliding'' of the total charge density, which gives the bulk Hall conductivity $\sigma_{xy}^{II,0}$. This is a non-Fermi-energy effect, and is a local (δ function) response to an electric field. The other is the Fermi-energy response, which is diffusive in the bulk (including the Hall effect with coefficient $\sigma_{xy}^{I,0}$) and is chiral along the edge. As discussed in Sec. I, the $\sigma_{xy}^{II,0}$ bulk effect implies $\nabla\cdot\mathbf{j}=0$, meaning that $\partial\rho/\partial t\neq 0$. At the edge, there is no charge accumulated. A tangential electric field at the edge can produce a bulk current normal to the edge, and also a Fermi-energy edge current that increases along the edge. These effects involve the same coefficent $\sigma_{xy}^{II,0}$, and the result is that no current is created, so no charge accumulates at the edge. This occurs because of a version of the Laughlin-Halperin gauge-invariance argument.[51,40] A change in the potential (which is essentially what z is) would accumulate a charge density of order the inverse velocity of the edge states, but the same velocity also appears in the edge current, which carries away the charge.

We now propose a modification of the NLσM action which incorporates this non-Fermi-energy effect so as to recover the SCBA bilocal conductivity tensor in full, and maintain current conservation at the edge (though not in the bulk), to all orders in perturbation theory. Our proposed action (in which we reinstate the full $2n\times 2n$ matrix $\mathbf{A}$) is

$$\begin{aligned}S_{\mathrm{mod}}[\mathbf{A}]=\int d^2r\{&-\tfrac{1}{8}\sigma_{xx}^{0}\mathrm{tr}(D_\mu QD_\mu Q)\\&-\tfrac{1}{8}\sigma_{xy}^{I,0}\mathrm{tr}(\epsilon_{\mu\nu}QD_\mu QD_\nu Q)\\&-\tfrac{1}{8}\sigma_{xy}^{II,0}\mathrm{tr}(\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q)\\&+\tfrac{1}{8}\sigma_{xy}^{II,0}\mathrm{tr}(\epsilon_{\mu\nu}Q[A_\mu,Q][A_\nu,Q])+\eta\ \mathrm{tr}\,(Q\Lambda)\}\\&+\frac{i}{2}\sigma_{xy}^{II,0}\int dx\{\mathrm{tr}[A_x(x,W)Q(x,W)]\\&-\mathrm{tr}[A_x(x,0)Q(x,0)]\}.\end{aligned} \quad (3.40)$$

The added $\sigma_{xy}^{II,0}$ term maintains SU$(2n)$ global, but not local gauge, symmetry, corresponding to the nonconservation of the corresponding modified current which contains an additional term in the bulk:

$$j_{\mu,\mathrm{mod}}^{\sigma}=\frac{\delta S_{\mathrm{mod}}}{\delta A_\mu}=j_\mu^{\sigma}-\frac{1}{2}\sigma_{xy}^{II,0}\epsilon_{\mu\nu}[A_\nu,Q]. \quad (3.41)$$

The modified equation of motion must be obtained from $\delta S_{\mathrm{mod}}\{Q',\mathbf{A}\}/\delta R=0$ (as in Sec. III B) without using gauge invariance. It can be written in terms of the modified current, to give the modified Ward identity in the bulk,

$$D_\mu j^{\sigma}_{\mu,\mathrm{mod}} = -\tfrac{1}{2}\sigma^{II,0}_{xy}\epsilon_{\mu\nu}\{\partial_\mu([A_\nu,Q]) + i[A_\mu,[A_\nu,Q]]\}. \tag{3.42}$$

The modified current is not covariantly conserved, because the modified action is not gauge invariant. However, the boundary condition Eq. (3.16) is unchanged, because the added term in S_{mod} contains no derivatives, so does not give rise to any boundary terms. Nonetheless, the *interpretation* of the boundary condition changes, because the current has been modified. In terms of the modified current, the boundary condition states that at the edge

$$j^{\sigma}_{y,\mathrm{mod}} - \partial_x j^{\sigma}_{x,\mathrm{edge,mod}} = 0, \tag{3.43}$$

which is ''current conservation.'' Strictly, our arguments imply that this modification of the action applies only for the $+-$ and $-+$ components of $\mathbf{A}$; for the other components, the correct form may depend on what is assumed in the underlying model the NLσM is supposed to represent.

For the quadratic part $S_{0,\mathrm{mod}}$ of the modified action S_{mod}, the corresponding formulas are the following: for the action,

$$S_{0,\mathrm{mod}} = S_0 - \sigma^{II,0}_{xy}\int d^2r\,\epsilon_{\mu\nu}\mathrm{tr}(A^{+-}_\mu A^{-+}_\nu); \tag{3.44}$$

for the current,

$$j^{-+}_{\mu,\mathrm{mod}} = j^{-+}_\mu - \sigma^{II,0}_{xy}\epsilon_{\mu\nu}A^{-+}_\nu, \tag{3.45}$$

as in Eq. (3.38) above; for the modified equation of motion,

$$\partial_\mu j^{-+}_{\mu,\mathrm{mod}} = -\sigma^{II,0}_{xy}\epsilon_{\mu\nu}\partial_\mu A^{-+}_\nu, \tag{3.46}$$

which corresponds to the earlier Eq. (1.27); and in the bilocal conductivity tensor, the bulk contact term $\sigma^{II,0}_{xy}\epsilon_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')$ appears as in the full SCBA result, Eq. (2.20). The boundary conditions on z and $z^\dagger$ are, however, unmodified.

When using the modified action for calculations of conductance, there is no change to the results, as long as one uses cross sections that are parallel to the y axis, and therefore the expressions contain the integrals of the x components of the currents, as we did earlier. For those calculations, we already showed at the end of Sec. III B that the conductance is independent of the positions of the sections. Since the vector potentials used there have x components only, the extra term in S_{mod} is zero. In addition to the usefulness of the general S_{mod} for reproducing the bilocal conductivity tensor and maintaining the current-conservation properties at the edge to all orders in perturbation theory, it is also crucial for the conductance if one calculates the flux of current through more general cross sections than those specified above. In general, a cross section could be any curve that intersects the edges of the sample just twice, once on each of the reflecting walls. Two such sections may intersect at isolated points (instead of along their whole length), and the intersections are then said to be *transversal*, that is, the normals to the curves at the point of intersection are not parallel (nor antiparallel). The question arises of whether the conductance and its moments calculated using such sections is independent of the position and shape of the sections, including the case in which they intersect. When they intersect transversally, the bulk $\sigma^{II,0}_{xy}$ contact term will contribute at the intersection point, even within the SCBA. Our preliminary investigations of this, which will not be included here, show that these contributions are needed to cancel effects of the $\sigma^{II,0}_{xy}$ terms at the edge, so as to maintain conservation of the total current, and that the conductance obtained is the same as for the straight sections, for any shape and position. We believe this to be true in general, to all orders in perturbation theory. This shows that the use of the modified action is obligatory for such general calculations. Note that, for more general geometries, such as four probes, such intersecting sections will be common.

D. Further details of perturbative calculations

1. Boundary perturbation expansion for the diffusion propagator

It is difficult to calculate the diffusion propagator as defined by Eq. (1.16) explicitly (however, see Ref. 39), although propagators for simpler geometry such as a half-plane,[48] an infinite strip[48,33] or an annulus[48] can be found (the results for the infinite strip can be obtained by conformal mapping from the half-plane[48]). We can perform a boundary perturbation expansion[56] in powers of γ using the propagator at $\gamma=0$, which we define as $d^0(\mathbf{r},\mathbf{r}')$. $d^0(\mathbf{r},\mathbf{r}')$ can be constructed out of the solutions for the following eigenvalue problem:

$$-\nabla^2\phi^0 = \Lambda\phi^0, \tag{3.47}$$

with the boundary conditions $\partial_y\phi^0(x,W) = \partial_y\phi^0(x,0) = 0$ and $\phi^0(0,y) = \phi^0(L,y) = 0$. The eigenfunctions are $\phi^0_{nm}(x,y) = (2/\sqrt{LW})\cos(n\pi y/W)\sin(m\pi x/L)$ with corresponding eigenvalues $\Lambda^0_{nm} = (n\pi/W)^2 + (m\pi/L)^2$, where $n=0,1,2,\ldots$, $m=1,2,\ldots$. We have

$$d^0(\mathbf{r},\mathbf{r}') = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\phi^0_{nm}(x,y)\phi^0_{nm}(x',y')}{\Lambda^0_{nm}}. \tag{3.48}$$

Using the bulk diffusion equations for $d^0(\mathbf{r},\mathbf{r}')$ and $d(\mathbf{r},\mathbf{r}')$, we obtain

$$d(\mathbf{r}_2,\mathbf{r}_1) = d^0(\mathbf{r}_1,\mathbf{r}_2) + \int_C d\mathbf{S}\cdot[d^0(\mathbf{r},\mathbf{r}_2)\nabla d(\mathbf{r},\mathbf{r}_1) - d(\mathbf{r},\mathbf{r}_1)\nabla d^0(\mathbf{r},\mathbf{r}_2)], \tag{3.49}$$

where C is a closed surface enclosing the disordered region including the edges (see Fig. 1). Let us divide the surface C into four parts C_1, C_2, C_3, and C_4. Applying the boundary conditions for the propagators for different sections, we have the following:

$$d(\mathbf{r}_2,\mathbf{r}_1) = d^0(\mathbf{r}_1,\mathbf{r}_2) + \int_{C_1+C_3} dS_n d^0(\mathbf{r},\mathbf{r}_2)\partial_n d(\mathbf{r},\mathbf{r}_1).$$

Plugging in the boundary condition $\partial_n d = \gamma\partial_t d$, we have

$$d(\mathbf{r}_2,\mathbf{r}_1) = d^0(\mathbf{r}_1,\mathbf{r}_2) + \gamma\int_{C_1+C_3} dS_n d^0(\mathbf{r},\mathbf{r}_2)\partial_t d(\mathbf{r},\mathbf{r}_1). \tag{3.50}$$

The above equation generates an expansion in terms of d^0 and in powers of γ. Using $\mathcal{B}$ to denote the operation $\int_{C_1+C_3} dS_n \gamma \partial_t$, we can write schematically

$$d = d^0 + d^0 \mathcal{B} d^0 + d^0 \mathcal{B} d^0 \mathcal{B} d^0 + \cdots. \tag{3.51}$$

2. 1D propagator

The above expansion is not valid for extremely narrow samples with $W \ll L$, where Eq. (3.51) needs to be summed exactly. However, the propagator itself approaches the 1D diffusion propagator with conductivity $\sigma_{xx}^0(1+\gamma^2)W$. Such limiting behavior can be demonstrated by rewriting the quadratic action in the following way:

$$S_0(z,z^\dagger) = -\frac{\sigma_{xx}^0}{4}\int d^2r\{(1+\gamma^2)\mathrm{tr}[\partial_x z \partial_x z^\dagger] + \mathrm{tr}[(\partial_y z + \gamma\partial_x z)(\partial_y z^\dagger - \gamma \partial_x z^\dagger)]\}. \tag{3.52}$$

In the limit $W \ll L$, the main contribution to $\langle\langle z^\dagger z\rangle\rangle_0$ comes from the low-lying eigenmodes of $-\nabla^2$ which satisfy $(\partial_y + \gamma\partial_x)\phi^L(\mathbf{r}) = 0$, $(\partial_y - \gamma\partial_x)\phi^R(\mathbf{r}) = 0$ in the entire strip. The eigenvalues of these modes are separated from those of the other modes by a gap of order L^2/W^2.[33] In other words, the term $\int d^2r\, \mathrm{tr}[(\partial_y z + \gamma\partial_x z)(\partial_y z^\dagger - \gamma\partial_x z^\dagger)]$ in the action can be ignored in the 1D limit:

$$\begin{aligned}\lim_{W/L\to 0}\int D[z,z^\dagger] z_{ij}^\dagger z_{kl} e^{S_0} &= \int D[z,z^\dagger] z_{ij}^\dagger z_{kl} e^{-(\sigma_{xx}^0/4)(1+\gamma^2)W\int dx\, \mathrm{tr}[\partial_x z \partial_x z^\dagger]} \\ &= 4\delta_{il}\delta_{jk} d^{1\mathrm{D}}/\sigma_{xx}^0 W,\end{aligned} \tag{3.53}$$

where

$$d^{1D}(x,x') = \frac{4}{(1+\gamma^2)L}\sum_{m=1}^{\infty}\frac{\sin(m\pi x/L)\sin(m\pi x'/L)}{(m\pi/L)^2}. \tag{3.54}$$

More generally, by symmetry, we obtain the NLσM in one dimension (where no topological term is possible) with $\sigma_{xx}^{0,1\mathrm{D}} = \sigma_{xx}^0(1+\gamma^2)W = W/\rho_{xx}^0$ as the coefficient in the action.

IV. TWO-PROBE CONDUCTANCE

A. Boundary contribution

From the nonlinear σ model, the average conductance is

$$\langle g\rangle = -\frac{1}{L^2}\lim_{\mathbf{A}\to 0, n\to 0}\int_0^L dx_1 \int_0^L dx_1' \left\langle\left\langle \frac{\delta S}{\delta A_{x,11}^{+-}(x_1)}\frac{\delta S}{\delta A_{x,11}^{-+}(x_1')} + \frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,11}^{-+}(x_1')}\right\rangle\right\rangle, \tag{4.1}$$

where

$$\left.\frac{\delta S}{\delta A_{x,11}^{+-}(x_1)}\right|_{\mathbf{A}=0} = \int_0^W dy_1\left[\frac{i\sigma_{xx}^0}{2}(\partial_x z_{11}^\dagger + \gamma\partial_y z_{11}^\dagger) + \frac{i\sigma_{xx}^0}{4}(z^\dagger\partial_x z z^\dagger)_{11} + O(z^5)\right],$$

$$\begin{aligned}\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,11}^{-+}(x_1')} = &-\int_0^W dy_1 \sigma_{xx}^0 \delta(x_1 - x_1') \\ &\times[1 - \tfrac{1}{4}(zz^\dagger)_{11} - \tfrac{1}{4}(z^\dagger z)_{11} \\ &- \tfrac{1}{16}(zz^\dagger zz^\dagger)_{11} - \tfrac{1}{16}(z^\dagger zz^\dagger z)_{11} \\ &+ \tfrac{1}{8}(zz^\dagger)_{11}(z^\dagger z)_{11} + O(z^6)].\end{aligned} \tag{4.2}$$

To leading order in $1/\sigma_{xx}^0$, we obtain

$$\begin{aligned} g^0 &= \frac{1}{L^2}\int_A d^2r\int_A d^2r'\,\sigma_{xx}^0\Big[\delta(\mathbf{r}-\mathbf{r}') - \frac{\sigma_{xx}^0}{4}(\partial_x + \gamma\partial_y) \\ &\quad\times(\partial_x' - \gamma\partial_y')\langle\langle z_{11}^\dagger(\mathbf{r}) z_{11}(\mathbf{r}')\rangle\rangle_0\Big] \\ &= \frac{1}{L^2}\int d^2r\int d^2r'\,\sigma_{xx}^0[\delta(\mathbf{r}-\mathbf{r}') \\ &\quad - (\partial_x + \gamma\partial_y)(\partial_x' - \gamma\partial_y')d(\mathbf{r},\mathbf{r}')].\end{aligned} \tag{4.3}$$

We have thus recovered the result from the diagrammatic expansion, and the other approaches described in Sec. I. Therefore the following results apply to any of these approaches.

The long-ranged term in the above expression comes from the ladder diagrams (i.e., diffusion) and in the absence of magnetic field, it does not contribute to the two-probe conductance.[38] In the presence of magnetic field, this is no longer the case. One can show that the local term gives the Ohmic conductance $\sigma_{xx}^0 W/L$. The long-ranged term, which involves volume integrals of total derivatives, gives the difference of boundary values of the diffusion propagator. Since the diffusion propagator goes to zero in the leads, the ∂_x and ∂_x' terms vanish upon volume integral. We are left with the boundary difference at the upper and lower edges,

$$\begin{aligned} g^0 = \sigma_{xx}^0\frac{W}{L} + \sigma_{xx}^0\gamma^2\frac{1}{L^2}\int_0^L dx\int_0^L dx'[d(x,W;x',W) \\ + d(x,0;x',0) - d(x,0;x',W) - d(x,W;x',0)].\end{aligned} \tag{4.4}$$

The σ_{xy}^0-dependent part is expressed as a boundary term and vanishes when the magnetic field is zero ($\gamma = 0$), or if the system is subject to periodic boundary condition in the transverse direction.

At the one-loop level, which is the next order in $1/\sigma_{xx}^0$, we have verified that the interference correction to $\langle g\rangle$ vanishes in the limit $n\to 0$. Therefore the presence of edges does not change the conclusion of the previous perturbative calculations[25–27] that there is no weak localization correction to σ_{xx} of relative order $1/\sigma_{xx}^0$. In general, we do not expect the presence of edges to have any effect on the renormalization flow of σ_{xx} in the perturbative regime, since it is dominated by short distance effects in the bulk. Whether σ_{xy} is ever renormalized perturbatively when the system has edges is less clear to us.

B. Small-γ correction for the two-probe conductance

For small γ, we can make use of the propagator d^0 at $\gamma=0$ to obtain the leading correction to g^0. Plugging $d^0(\mathbf{r},\mathbf{r}')$ in the boundary term of the two-probe conductance, we obtain

$$g^0(\gamma)=\sigma_{xx}^0 W/L[1+\gamma^2 f_1(L/W)]+O(\gamma^4), \tag{4.5}$$

where

$$f_1(L/W)=\frac{64}{\pi^4}\frac{L^2}{W^2}\sum_{m=1,odd}^{\infty}\sum_{n=1,odd}^{\infty}\frac{1}{m^2}\frac{1}{m^2+\frac{L^2}{W^2}n^2}$$

$$\to\begin{cases}14\zeta(3)L/\pi^3 W, & W\gg L\\ 1/2, & W=L\\ 1, & W\ll L.\end{cases}$$

[Here $\zeta(s)=\Sigma_{m=1}^{\infty}m^{-s}$ is the Riemann zeta function, and we note that $\zeta(3)\simeq 1.202$.] Thus, for an extremely wide sample, $W\gg L$, the effect of the edges can be ignored. In the 1D limit $(W/L\to 0)$, we obtain $g^0\to(1+\gamma^2)\sigma_{xx}^0 W/L$, which is consistent with our result that in the NLσM, $\sigma_{xx}^{0,1D}=(1+\gamma^2)\sigma_{xx}^0 W$. This is also consistent with other results, as noted in Sec. I A, valid for arbitrary γ, which show that the mean conductance in the 1D limit can be obtained from this one-dimensional conductivity $\sigma_{xx}^{0,1\mathrm{D}}=W/\rho_{xx}^0$.

V. VARIANCE OF THE CONDUCTANCE

In this section, we evaluate the variance of conductance to leading order in $1/\sigma_{xx}^0$. From the nonlinear σ model, we obtain

$$\begin{aligned}\langle g^2\rangle-\langle g\rangle^2=&\frac{1}{L^4}\lim_{\mathbf{A}\to 0,n\to 0}\int dx_1\int dx_1'\int dx_2\int dx_2'\left\langle\left\langle\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,11}^{-+}(x_1')}\,\frac{\delta^2 S}{\delta A_{x,22}^{+-}(x_2)\,\delta A_{x,22}^{-+}(x_2')}\right.\right.\\
&+\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{-+}(x_2')}\,\frac{\delta^2 S}{\delta A_{x,22}^{+-}(x_2)\,\delta A_{x,11}^{-+}(x_1')}+\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{+-}(x_2)}\,\frac{\delta^2 S}{\delta A_{x,11}^{-+}(x_1')\,\delta A_{x,22}^{-+}(x_2')}\\
&+\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,11}^{-+}(x_1')}\,\frac{\delta S}{\delta A_{x,22}^{+-}(x_2)}\,\frac{\delta S}{\delta A_{x,22}^{-+}(x_2')}+1\leftrightarrow 2+\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{-+}(x_2')}\,\frac{\delta S}{\delta A_{x,22}^{+-}(x_2)}\,\frac{\delta S}{\delta A_{x,11}^{-+}(x_1')}\\
&+1\leftrightarrow 2\\
&+\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{+-}(x_2)}\,\frac{\delta S}{\delta A_{x,22}^{-+}(x_2')}\,\frac{\delta S}{\delta A_{x,11}^{-+}(x_1')}+(+\leftrightarrow -)+\frac{\delta S}{\delta A_{x,11}^{+-}(x_1)}\,\frac{\delta S}{\delta A_{x,11}^{-+}(x_1')}\\
&\left.\left.\times\frac{\delta S}{\delta A_{x,22}^{+-}(x_2)}\,\frac{\delta S}{\delta A_{x,22}^{-+}(x_2')}\right\rangle\right\rangle_{\text{connected}},\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{-+}(x_2')}&=-\int dy_1\,\delta(x_1-x_2')\frac{\sigma_{xx}^0}{8}(z^\dagger z)_{12}(zz^\dagger)_{21}+O(z^6),\\
\frac{\delta^2 S}{\delta A_{x,11}^{+-}(x_1)\,\delta A_{x,22}^{+-}(x_2)}&=\int_0^W dy_1\,\delta(x_1-x_2)\frac{\sigma_{xx}^0}{2}z_{12}^\dagger z_{21}^\dagger,\end{aligned}\tag{5.2}$$

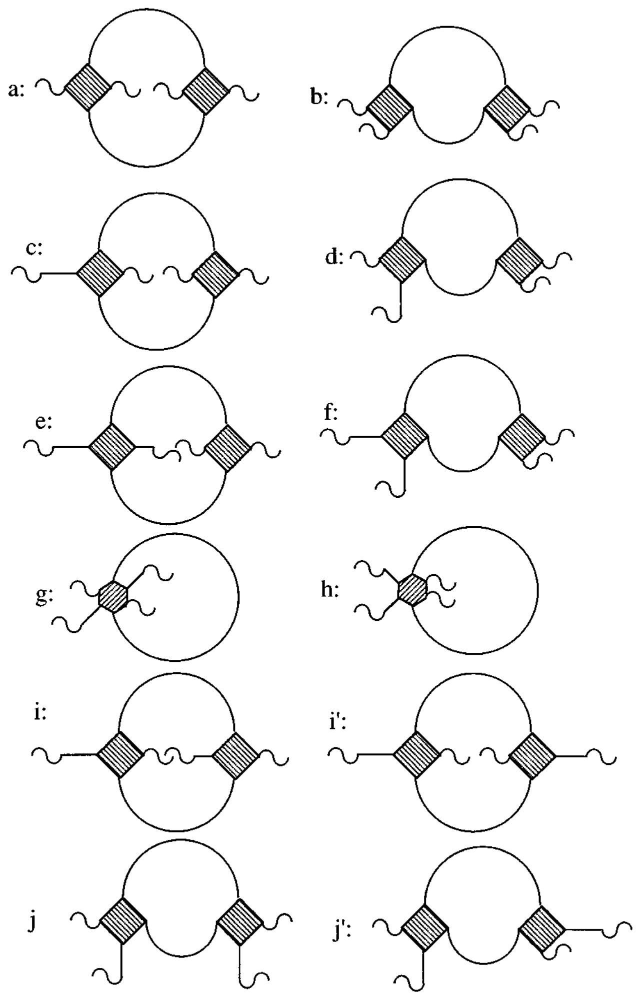

FIG. 4. The diagrams for the variance of conductance to leading order in $1/\sigma_{xx}^0$ and to order γ^2. The shaded polygons are vertices. The lines connecting the vertices are the diffusion propagators.

and the other functional derivatives are all evaluated at $\mathbf{A}=0$. The leading diagrams are shown in Fig. 4. Various vertices are denoted by polygons with the wavy tail indicating $\delta/\delta A$, while the lines linking the vertices are the diffusion propagators. The diamond-shaped vertex with no wavy tails comes from the four-point interaction term in S_1. We have also obtained the same set of diagrams using the diagrammatic approach. The diagrammatic approach is complicated because various vertices need to be evaluated separatedly. In the presence of magnetic field, the vertices are dressed with nonvanishing G^+G^+ and G^-G^- ladders, although in the end they can all be expressed in terms of σ_{xx}^0 and σ_{xy}^0. For the NLσM, the vertices can be obtained from the action. Figure 5 shows that one particular vertex from $\delta^2 S/\delta A_{x,11}^{+-}\delta A_{x,11}^{-+}$ is equal to the sum of four diagrams in the diagrammatic approach.

Figures 4(a) and 4(b) are the only diagrams considered in previous UCF theories. The rest of the diagrams were considered by Kane, Serota, and Lee,[38] and it is known that, for $\gamma=0$, they give rise to the long-ranged correlation in local current response but do not contribute to the variance of conductance in the absence of magnetic field.[38] One can show that these additional diagrams can all be written as boundary contributions, and that they vanish when $\gamma=0$ for

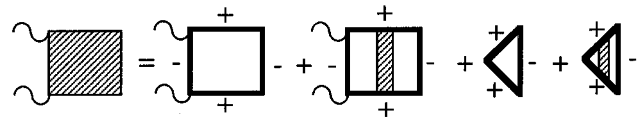

FIG. 5. The equivalence between the diagrammatic approach and the NLσM approach. One vertex from the NLσM, $\delta^2 S/\delta A_{x,11}^{+-}\delta A_{x,11}^{-+}$, is equal to the sum of four diagrams in the diagrammatic approach.

the same reason as the ladder series vanish in the case of the average conductance, when written in the area-averaged form. However, in the presence of the magnetic field the additional diagrams give rise to Hall-ratio-dependent contributions. The work of KY and ML discussed the effect of the tilted boundary condition on the diffusion propagator, but did not consider the additional diagrams.

From Figs. 4(a) and 4(b) alone, we obtain

$$\langle \delta g^2\rangle_{a,b}=\frac{1}{L^4}\{4\,\mathrm{Tr}(dd^T)+2\,\mathrm{Tr}(dd)\}. \tag{5.3}$$

Using the classical network model of Sec. I B, we calculated the diffusion propagator d for a range of values of γ and W/L. Using this propagator, we find that $\mathrm{Tr}(dd)$ and $\mathrm{Tr}(dd^T)$ are smooth functions of γ and L/W. The peaks reported by ML in the variance of the conductance [as given by Eq. (5.3)] are not observed in our exact numerical calculation. The argument advanced by ML for the existence of ''resonant'' peaks due to the tilted boundary condition effects is not supported by this calculation. We emphasize again that, in any case, Eq. (5.3) is not the full expression for the variance of the conductance, because there are other diagrams that were omitted by KY and ML.

A. Recovery of UCF result in the 1D limit

The importance of the additional diagrams can be best demonstrated in the quasi-1D limit ($W\ll L$), where σ_{xx}^0 and σ_{xy}^0 combine to form a single parameter, $\sigma_{xx}^{0,\mathrm{1D}}$. This limit is well described by the random-matrix theory of the unitary ensemble. For general reasons given earlier, we expect the variance of the conductance to approach the well-known 1D UCF result, independent of the value of γ.

Plugging in the 1D diffusion propagator of Eq. (3.54), we obtain

$$\langle \delta g^2\rangle_{a,b,\mathrm{1D}}=\frac{6}{\pi^4}\left(\sum_{m=1}^{\infty}\frac{1}{m^4}\right)\frac{1}{(1+\gamma^2)^2}=\tfrac{1}{15}(1-2\gamma^2)+O(\gamma^4), \tag{5.4}$$

where we used

$$\sum_{m=1}^{\infty}\frac{1}{m^4}=\frac{\pi^4}{90}.$$

This is essentially the argument used by KY and ML, except that they gave versions applicable at finite temperature. However, this result of these authors, that the variance of the conductance depends on γ, even in the 1D limit, is incorrect.

Among the additional diagrams, Figs. 4(c) and 4(d) and the sum of Figs. 4(i) and 4(i$'$) vanish to order γ^2 for all

W/L; Figs. 4(g), 4(h), 4(i), 4(i′), 4(j), and 4(j′) vanish as $W/L \rightarrow 0$. The dominant contributions come from Figs. 4(e) and 4(f):

$$\langle \delta g^2 \rangle_{e,f,1D} \simeq \frac{6}{\pi^4} \left(\sum_{m=1}^{\infty} \frac{1}{m^4} \right) 2\gamma^2 f_1(\infty) = \tfrac{1}{15}(2\gamma^2). \tag{5.5}$$

Figures 4(e) and 4(f) thus cancel the γ^2 correction from Fig. 4(a) and 4(b). We obtain in total, to order γ^2,

$$\langle \delta g^2 \rangle_{W/L \rightarrow 0} = \tfrac{1}{15}. \tag{5.6}$$

Thus, in 1D the UCF result of Refs. 4, 7, and 12 is recovered, at least to order γ^2. We remind the reader that the 1D UCF result holds only when the length L is less than the 1D localization length, ξ_{1D}, and that ξ_{1D} is of order $\sigma_{xx}^{0,1D} = W/\rho_{xx}^0$, which is much larger than the lower limit (W) on L in the diffusive regime $\rho_{xx}^0 \ll 1$.

B. Variance of the conductance in two dimensions

For a wide sample with W/L arbitrary, the variance of the conductance depends on the Hall ratio. We will calculate the correction to the usual result for the $\gamma=0$ unitary ensemble, to order γ^2. In the 2D limit, the individual diagrams, Figs. 4(e)–4(h), 4(j), and 4(j′), all have logarithmically-divergent parts, however, their logarithmic contributions cancel out. (The cancellation is guaranteed by the fact that S is not renormalized at one-loop level.) There can be even more divergent diagrams containing $\delta(0)$, but these diagrams are canceled by diagrams generated by the measure $I[z,z^\dagger]$. (Since they are at least of order γ^4, they are not explicitly calculated in this paper.) The total γ^2 correction is finite. Summing up the contributions from Figs. 4(a)–4(j′), for a square sample we obtain

$$(\delta g)^2_{L=W} = \left[9.06 \frac{1}{\pi^4} + 2.40 \gamma^2 \frac{256}{\pi^8} \right] + O(\gamma^4). \tag{5.7}$$

The expression for the variance for arbitrary W/L is given in Appendix D.

VI. CONCLUSION

In this paper, we considered the mesoscopic conductance and its fluctuations in the presence of a magnetic field for a realistic two-probe geometry. Our perturbation theory has a different structure from previous theories[4,7,32] because of the presence of two conductivity parameters σ_{xx}^0 and σ_{xy}^0. We found that σ_{xy}^0 not only enters the boundary condition for diffusion, as was noted in Refs. 33–35, but also appears in the current vertex and other vertices which govern the interference processes. As a result the two-probe conductance and its variance in the perturbative regime depend on the Hall ratio $\gamma = \sigma_{xy}^0/\sigma_{xx}^0$. Our calculations differ from the previous results,[33,34] since we have not only modified the boundary condition but also considered additional diagrams which vanish in the zero-field limit or in an edgeless system. Our main result is that UCF's are modified in the presence of edges; the variance of the two-probe conductance, although it is still of order 1, increases with the Hall ratio, as shown in Eq. (5.7). However, in the quasi-1D limit of a long sample, the usual universal result is recovered.

The reflecting boundary condition at the ''hard'' wall (or ''edge'') is crucial for the dependence of the conductance on the Hall ratio γ that we find. If this is replaced by a periodic transverse boundary condition (i.e., a system on the surface of a cylinder), the results of the usual unitary ensemble in two dimensions are obtained; the results of Xiong and Stone[32] are easily modified for this case, for which they are correct. While a cylinder may seem hard to realize experimentally, it can be mapped to an annulus by a conformal mapping. The annulus is sometimes known as the Corbino disk, in which there are no edges, and a radial voltage drop is applied to induce a current flow. Thus, for the disk, the conductance fluctuations should be a universal function of the ratio of the inner and outer radii, with no dependence on γ.

The experimental observation of the effects we find depends first on being in the regime $L_{in} \gg L, W$, so the system is phase coherent, and on having an elastic mean free path l due to impurities such that $L, W \gg l$. Our calculations only address the metallic regime of conductance fluctuations at large diagonal conductivity σ_{xx}^0, where perturbation theory is valid. In principle, this approach is valid for any value of the Hall ratio $\gamma = \sigma_{xy}^0/\sigma_{xx}^0$, or of the Hall angle $\theta_H = \tan^{-1}\gamma$. For simplicity, we expanded most of our results also to first nontrivial order in γ^2. The terms in $(1/\sigma_{xx}^0)^2$ that are left out cannot be neglected if the system size L or W exceeds the order of ξ_{pert}, the crossover scale at which the renormalized conductivity becomes of order 1 or less. If L, W are greater than ξ_{pert}, the system crosses over either to the localized regime where σ_{xy} becomes quantized, or, for Fermi energies near the critical values that lie near the centers of the Landau bands, to the critical transition region between the plateaus; our theory does not apply to either of these. Therefore, one must use mesoscopic systems that are not too large. Fortunately, since $\xi_{\text{pert}} \sim l e^{(\sigma_{xx}^0)^2}$, this is not difficult if $\sigma_{xx}^0 \gg 1$. According to the SCBA results reviewed in Sec. II B, σ_{xx}^0 will be large unless either the Landau-level index N of the highest partially occupied Landau level is of order 1, or the Fermi energy lies in the tail of the density of states of the disorder-broadened Landau bands, when $\omega_c \tau_0$ is large enough that these are well developed. Thus the magnetic field B must be large enough to suppress the Cooperons, so the system is in the unitary (broken time-reversal symmetry) regime, but not too large. [We do *not* generally require $\omega_c \tau_0 > 1$, though this would ensure that $\gamma \geqslant O(1)$.] In effect, for the observation of the effects found in our theory, ideal conditions would be that the system should exhibit Shubnikhov–de Haas (SdH) oscillations, but not well-developed quantized Hall plateaus, even for asymptotically low temperatures. As the Fermi energy or magnetic field varies through a Landau band, yielding such an oscillation in σ_{xx}^0 and σ_{xy}^0 varies monotonically, which implies that the ratio $\gamma = \sigma_{xy}^0/\sigma_{xx}^0$ varies. There is therefore a great deal of scope for varying γ by varying either the field B from low values ($\gamma \simeq 0$) to larger, or as the field sweeps through a single SdH oscillation. However, since the amplitude of the fluctuations depend on γ, it will be necessary to collect statistically independent values of the conductance without changing γ too much. Thus the simplest experimental method, which uses the magnetic field as the ergodic parameter, will not work, and some other technique must be used to vary the sample conductance at fixed B. Finally, as the quantized Hall plateas are reached, localization effects will suppress fluctuations strongly between the centers of the LLs, and our theory is not applicable (although such measurements would be interesting).

While the calculations in this paper have addressed only the weak-coupling regime at $\sigma_{xx} \gg 1$, it is interesting to speculate about the effects of σ_{xy} on the conductance and its fluctuations in the critical regime of the integer quantum Hall effect, when the system has edges. The critical regime can be defined by the conditions $L_{\rm in} \gg L$, W, and L and W between $\xi_{\rm pert}$ and ξ, where $\xi \gg \xi_{\rm pert}$ is the localization length, which diverges as E_F approaches any of the critical values E_{cN}, $N=0,1, \ldots$. We expect that the renormalized local conductivity parameters σ_{xx}, σ_{xy}^{I}, and σ_{xy}^{II} are still meaningful, and that σ_{xx} and $\sigma_{xy}=\sigma_{xy}^{I}+\sigma_{xy}^{II}$ take on universal values [$\equiv 1/2$ (mod 1), in the case of σ_{xy}] at the critical points. This raises the question of the renormalization of the two pieces σ_{xy}^{I} and σ_{xy}^{II}, and whether the values of these are universal separately at the critical point. We note that at the localized fixed point, the behavior may be described by saying $\sigma_{xy}^{I}=0$, $\sigma_{xy}^{II}\equiv 0$ (mod 1), so that these parameters do approach universal values in this regime. For σ_{xx}, there is a widespread belief that it takes the universal value $\frac{1}{2}$ at the critical fixed point, though it is not always clear if the calculations done to support this are describing the local conductivity parameter σ_{xx}, rather than a mean conductance in a particular geometry. The relation of these is not known in the critical regime at the present time, and, as we have seen, is not simple even in the perturbative regime, if the system has edges. We expect that for a two-probe system with a periodic transverse boundary condition, σ_{xy} should not contribute to the conductance in the critical regime, just as it does not in the perturbative theory in this paper. Even then, the mean conductance is not in general given by $\sigma_{xx}W/L$, since non-Ohmic behavior is expected at least for $L \gg W$ where the system approaches a localized quasi-1D limit. Thus, even in the case of a square sample with $W=L$ and periodic transverse boundary condition, it is not clear that $\langle g \rangle = \sigma_{xx}$. The effect of the edges in the critical region is nicely shown in a recent paper,[23] which examined the mean, variance, and distribution of the conductance in a two-probe geometry like ours, with $W=L$, and for both reflecting and periodic transverse boundary conditions, i.e. with and without edges. The results show that the boundary conditions do make a difference (however, finite-size effects are significant, as shown for the periodic transverse boundary condition case in Ref. 22). The authors tentatively attribute this to ''edge currents,'' but as we have seen in the perturbative regime, there are edge effects (described by σ_{xy}), that are not solely due to edge currents carried by edge states (which are described by σ_{xy}^{II}). The boundary effects make themselves felt throughout the system, due to long-range correlations in the critical regime. They are relatively unimportant only when $W \gg L$. In fact, dependence on the boundary conditions, say on whether they are periodic or reflecting, would occur even in the absence of σ_{xy}, as it does in the weak coupling regime (see, e.g., Ref. 32) A further implication, suggested by our results, is that the critical conductance properties, in a given geometry that possesses edges, may depend on the fixed-point value of σ_{xy}, i.e., on which transition is being studied. While the structure of the critical field theory [including σ_{xy} (mod 1), σ_{xx}, and the critical exponents] should be universal, this may not be true for the conductance, because the edge brings in a dependence on the integer part of σ_{xy}. A first example of this is the simple fact that the mean of the Hall conductance, that can be defined in a multiprobe geometry (with edges), depends on which transition is being studied, thus violating universality to this extent. The same may be true of the critical conductance fluctuations in geometries with edges. On the other hand, for a Corbino disk, which has no edges, there should be full universality among the integer quantum Hall transitions. Clearly, it would be of interest to study this numerically or experimentally. One way to do so numerically would be using the Chalker-Coddington model,[43] with additional comoving edge channels coupling to the edges by hopping terms to obtain $|\sigma_{xy}^{0}|>1$.

Returning to the perturbative, metallic regime, $\sigma_{xx} \gg 1$, we expect that similar phenomena to those studied here in two dimensions should occur also in higher dimensions, for example in three dimensions. No isotropic topological term containing only two gradients is possible in higher dimensions. However, the Hall conductivity should make an appearance in the NLσM effective action, since it is a part of the measurable conductivity. It appears in a generalization of the 2D action to three dimensions,[58]

$$S=-\tfrac{1}{8}\sigma^{0}\int d^{3}r\ \mathrm{tr}[\partial_{\mu}Q\partial_{\mu}Q] \\ -\tfrac{1}{8}\sigma_{H}^{0}\int d^{3}r\,\epsilon_{\lambda\mu\nu}n_{\lambda}\mathrm{tr}[Q\partial_{\mu}Q\partial_{\nu}Q]. \qquad (6.1)$$

Here $\mathbf{n}$ is a unit vector in the direction of the magnetic field $\mathbf{B}$, and σ^{0} and σ_{H}^{0} are the diagonal (dissipative) and Hall conductivities, respectively. Thus the action is anisotropic, because the $\mathbf{B}$ field specifies a direction. (However, for simplicity we neglected the possible anisotropy in the diagonal conductivity σ^{0}.) The action can be viewed as resulting directly from considering layers stacked perpendicular to the magnetic field, each of which has a Hall conductivity and is descibed by the 2D NLσM action, plus a transition amplitude for electrons hopping between the layers. Such models have recently been studied numerically.[59] For systems with boundaries, the σ_{H}^{0} term in the action now leads in perturbation theory to phenomena similar to those in two dimensions, such as a tilted boundary condition, a dependence of $\langle g \rangle$ and the conductance fluctuations on $\sigma_{H}^{0}/\sigma^{0}$, and so on. Thus, in three dimensions, and also in still higher dimensions, the conductance fluctuations in general depend on the Hall ratio (or angle). However, for the localization transition in three dimensions, which would be expected to be in the unitary class since time-reversal symmetry is broken by the magnetic field, we suspect that the σ_{H}^{0} term is irrelevant at the critical fixed point, so that the properties of the transition are universal, independent of the bare Hall ratio, at least to the same extent as in two dimensions, as discussed above. Similarly to the 2D case, the σ_{H}^{0} term contributes to the action of configurations in which each layer has a nonzero instanton number (insofar as this number is well defined, if the system has boundaries). In three dimensions, there also exist topologically stable point-singular configurations of the Q field (known as ''hedgehogs'' in the literature), which may be viewed as points at which the instanton number changes from one layer to the next. The σ_H^0 term counts the number of layers with each value of the instanton number, and thus is sensitive to the presence and location of the hedgehogs. However, in the case of NLσM's studied in connection with antiferromagnets, it appears that the hedgehogs are irrelevant as far as the critical properties are concerned, even though they may affect the behavior in the phases on either side of the transition (see, e.g., Ref. 60). Therefore, we suspect that, while the σ_H^0 term plays a role in the metallic phase, and also (after renormalization) in the 3D quantized Hall phase of layered systems,[59] it may have no effect on the critical properties, except perhaps for the conductance when edges are present. Clearly, these are questions that may repay further study.

ACKNOWLEDGMENTS

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APPENDIX A: BOUNDARY CONDITION AT THE HARD WALLS FOR THE WHITE-NOISE MODEL

In this appendix we briefly derive the boundary condition on the diffusion propagator within the SCBA, by using current conservation at the reflecting walls. Using the SCBA equation $[E-H_0-\Sigma^{\pm}(\mathbf{r})]\overline{G^{\pm}}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$, we can show that

$$\nabla\cdot\mathbf{J}^{+-}(\mathbf{r},\mathbf{r}')=\nabla\cdot\left[\overline{G^{-}}(\mathbf{r}',\mathbf{r})\left(\frac{-i\hbar}{2m_e}\overleftrightarrow{D}\right)\overline{G^{+}}(\mathbf{r},\mathbf{r}')\right]$$
$$=\frac{i}{\hbar}(\Sigma^{+}-\Sigma^{-})[-\delta(\mathbf{r}-\mathbf{r}')/u+\overline{G^{+}}(\mathbf{r},\mathbf{r}')\overline{G^{-}}(\mathbf{r}',\mathbf{r})]. \tag{A1}$$

Let us define the G^+G^- ladder diagram with n impurity lines as $S^{+,-,(n)}$ and the ladder diagram with one current vertex attached to the left as

$$\mathbf{v}^{L,(n)}(\mathbf{r},\mathbf{r}')=u\int d^2r_1\mathbf{J}^{+-}(\mathbf{r},\mathbf{r}_1)S^{+,-,(n)}(\mathbf{r}_1,\mathbf{r}').$$

Using the above property of $\mathbf{J}^{+-}$, we obtain the following recursive relation:

$$\nabla\cdot\mathbf{v}^{L,(n)}=\tau[S^{+,-,(n)}-S^{+,-,(n+1)}]. \tag{A2}$$

Summing up all ladder diagrams, we obtain

$$\nabla\cdot\mathbf{v}^{L}(\mathbf{r},\mathbf{r}')=-\tau\delta(\mathbf{r}-\mathbf{r}'), \tag{A3}$$

where $\mathbf{v}^L(\mathbf{r},\mathbf{r}')$ represents the $\mathbf{J}^{+-}S^{+,-}(\mathbf{r},\mathbf{r}')$. This shows that, on the finest length-scale resolution, $\langle\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')\rangle_{\rm SCBA}$ obeys $\nabla\cdot\sigma(\mathbf{r},\mathbf{r}')=0$, for $\mathbf{r}$ sufficently far from $\mathbf{r}'$.

For $\mathbf{r}$ at the reflecting boundary, the normal component of $\mathbf{j}$ *outside* the sample is zero. In the presence of a boundary current, which, from a coarse-grained, large-scale point of view, can be treated as δ functions $\delta(y-W)$, $\delta(y)$ in the components tangential to the edge, the surface integral of the current emerging from a small box centered on the top edge reduces to

$$\int_{W-0^+}^{W+0+}dy\,\partial_x\langle\sigma_{x\nu}(\mathbf{r},\mathbf{r}')\rangle_{\rm SCBA}-\langle\sigma_{y\nu}(\mathbf{r},\mathbf{r}')\rangle_{\rm SCBA}|_{y=W}=0 \tag{A4}$$

for $\mathbf{r}\neq\mathbf{r}'$. Thus any normal current (just inside the edge) must be converted to a δ-function tangential current at the edge. This condition was discussed for the current $\mathbf{j}'$ in Sec. I A. Within the SCBA, it leads [using Eq. (2.20)] to the conclusion that it is $\gamma=\sigma_{xy}^0/\sigma_{xx}^0$, not $\sigma_{xy}^{I,0}/\sigma_{xx}^0$, which appears in the boundary condition (1.16) on the diffusion propagator d. A similar argument holds for the $\mathbf{r}'$ dependence. The extension of this discussion to include the situation $\mathbf{r}=\mathbf{r}'$ is given in Sec. III C.

APPENDIX B: PROOF OF THE CURRENT CONSERVATION IDENTITIES WITHIN SCBA

We will show that, within the SCBA, $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}')$ satisfies the constraints imposed by current conservation. We start with the σ^{+-} term. We can write the ladder diagrams in the following fashion:

$$\sigma_{\rm ladder}^{+-}(\mathbf{r},\mathbf{r}')=\frac{\hbar^4}{4m_e^2}u\int d^2r_2v^L(\mathbf{r},\mathbf{r}_2)J^{-,+}(\mathbf{r}',\mathbf{r}_2). \tag{B1}$$

Using the recursive relation (A2) for $\mathbf{v}^{L,(n)}$, and denoting $S^{+-,(n)}J^{-+}$ as $\mathbf{v}^{R,(n)}$, we obtain another recursive relation:

$$\nabla\cdot\boldsymbol{\sigma}^{+-,(n)}(\mathbf{r},\mathbf{r}')=\frac{2m_e}{\hbar^2}(\Sigma^{+}-\Sigma^{-})[\mathbf{v}^{R,(n-1)}(\mathbf{r},\mathbf{r}')-\mathbf{v}^{R,(n)}(\mathbf{r},\mathbf{r}')].$$

Summing up all the ladder diagrams, we obtain

$$\nabla\cdot\boldsymbol{\sigma}^{+-}(\mathbf{r},\mathbf{r}')=\frac{2m}{\hbar^2}\sum_N\sum_{N'}\{\overline{G_N^{+}}(\mathbf{r},\mathbf{r}')\overleftrightarrow{D}_j'P_{N'}(\mathbf{r}',\mathbf{r})-P_N(\mathbf{r},\mathbf{r}')\overleftrightarrow{D}'\overline{G_{N'}^{-}}(\mathbf{r}',\mathbf{r})\},$$

where $P_N(\mathbf{r},\mathbf{r}')$ is the projection operator onto the Nth Landau level. The right-hand side is a short-ranged function of $|\mathbf{r}-\mathbf{r}'|$, which we can treat as a δ function. We can write

$$\nabla\cdot\boldsymbol{\sigma}^{+-}(\mathbf{r},\mathbf{r}')=\mathbf{c}^{+-}\delta(\mathbf{r}-\mathbf{r}'), \tag{B2}$$

where

$$c_j^{+-}=-i\hbar\sum_n\sum_{n'}(\overline{G_n^{+}}v_{nn'}^j-v_{nn'}^j\overline{G_{n'}^{-}}),$$

where $v_{nn'}$ is the matrix elements of the velocity operator, and

$$c_j^{+-}\begin{cases} =0 & \text{for} \quad B=0 \\ \neq 0 & \text{for} \quad B\neq 0. \end{cases}$$

To evaluate $\sigma^{++}(\mathbf{r},\mathbf{r}')$ and $\sigma^{--}(\mathbf{r},\mathbf{r}')$, we use the following trick:

$$\sigma^{++}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{E} dE' f(E') \lim_{E_1=E_2\to E'} \frac{\partial}{\partial E_1} \times \sigma^{++}(\mathbf{r},\mathbf{r}';E_1,E_2),$$

$$\sigma^{--}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{E} dE' f(E') \lim_{E_1=E_2\to E'} \frac{\partial}{\partial E_2} \times \sigma^{--}(\mathbf{r},\mathbf{r}';E_1,E_2), \tag{B3}$$

where $\sigma^{aa}(E_1,E_2)$ involves the ladder sum S^{aa} $(a=+,-)$. Define $\mathbf{v}^{R,aa,(n)}=S^{aa,(n)}\mathbf{J}^{aa}$, we can show that for the nth ladder diagram,

$$\begin{aligned}\nabla\cdot\sigma^{++,(n)}(\mathbf{r},\mathbf{r}';E_1,E_2) &= \frac{2m_e}{\hbar^2}[\Sigma^+(E_1)-\Sigma^+(E_2)]\mathbf{v}^{R,++,(n-1)}(\mathbf{r},\mathbf{r}') \\ &\quad -\frac{2m_e}{\hbar^2}[\Sigma^+(E_1)-\Sigma^+(E_2)]\mathbf{v}^{R,++,(n)}(\mathbf{r},\mathbf{r}') \\ &\quad +\frac{2m_e}{\hbar^2}[E_1-E_2]\mathbf{v}^{R,++,(n)}(\mathbf{r},\mathbf{r}').\end{aligned} \tag{B4}$$

One can see that there is cancellation between the nth ladder diagram and the $(n+1)$th ladder diagram. Similar relations can be derived for $\nabla\cdot\sigma^{--,(n)}(\mathbf{r},\mathbf{r}';E_1,E_2)$. Summing up all the ladder diagrams, taking the derivative over energy and then the limit $E_1=E_2\to E'$, we can show that

$$\begin{aligned}\nabla\cdot[\sigma^{++}(\mathbf{r},\mathbf{r}')+\sigma^{--}(\mathbf{r},\mathbf{r}')] &= -c^{+-}\delta(\mathbf{r}-\mathbf{r}')-\frac{\hbar^2}{2m_e}\int dE' f(E') \\ &\quad \times[\mathbf{v}^{++,R}(\mathbf{r},\mathbf{r}',E')-\mathbf{v}^{--,R}(\mathbf{r},\mathbf{r}',E')].\end{aligned} \tag{B5}$$

We can see that $\nabla\cdot\sigma^{+-}(\mathbf{r},\mathbf{r}')$ is canceled by contributions from $\nabla\cdot\sigma^{++}(\mathbf{r},\mathbf{r}')$ and $\nabla\cdot\sigma^{--}(\mathbf{r},\mathbf{r}')$,

$$\begin{aligned}\nabla\cdot\sigma(\mathbf{r},\mathbf{r}') &= -\frac{\hbar^2}{2m_e}\int_{-\infty}^{E} dE' f(E')[\mathbf{v}^{++,R}(\mathbf{r},\mathbf{r}',E') \\ &\quad -\mathbf{v}^{--,R}(\mathbf{r},\mathbf{r}',E')].\end{aligned} \tag{B6}$$

Since

$$\nabla'\cdot\mathbf{v}^{R,++,(n)}(\mathbf{r},\mathbf{r}') = \frac{2m_e}{\hbar^2}[S^{++,(n-1)}(\mathbf{r},\mathbf{r}')-S^{++,(n-1)}(\mathbf{r},\mathbf{r}')]=0$$

and

$$\nabla'\cdot\mathbf{v}^{R,--,(n)}(\mathbf{r},\mathbf{r}') = \frac{2m_e}{\hbar^2}[d^{--,(n-1)}(\mathbf{r},\mathbf{r}')-d^{--,(n-1)}(\mathbf{r},\mathbf{r}')]=0,$$

we obtain

$$\vec{\nabla}\cdot\langle\sigma(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}\cdot\overleftarrow{\nabla}'=0. \tag{B7}$$

Using Eqs. (B6) and (B7) and the asymptotic property of the Green's function[42]

$$\int dS'|_{\mathbf{r}'=\infty} G^{\pm}(\mathbf{r},\mathbf{r}')\overleftrightarrow{D}' G^{\pm}(\mathbf{r}',\mathbf{r})=0,$$

we can show finally that

$$\nabla\cdot\int \langle\sigma(\mathbf{r},\mathbf{r}')\rangle_{\mathrm{SCBA}}\cdot d\mathbf{S}'=0. \tag{B8}$$

APPENDIX C: REMARKS ON EDGE STATES AND QUANTIZATION

Here we return to the topological considerations of Sec. III B, and relate them to edge states and the quantization of the Hall conductance in the localized regime. The topological considerations of Sec. III B are closely related to the problem of setting up a path integral for a quantum spin, by which we mean an irreducible representation of the symmetry group, which is SU$(2n)$ here (for a review, see e.g., Ref. 60), and this connection is also utilized in the mapping from the Chalker-Coddington model (representing a network of edge states) to a quantum spin chain or the NLσM (Ref. 49) (the connection between the latter two problems, and the relation to the quantum Hall effect, was discussed earlier[61]). In the quantum spin problem, we would take imaginary time, with a periodic boundary condition in the time direction, and the action would contain only the $\sigma_{xy}^{II,0}$ terms from $S[\mathbf{A}]$, Eq. (3.8); the system would be taken to be a disk, with the single edge corresponding to the world line of the quantum spin with its periodic boundary condition. For the two-probe geometry, this corresponds to regarding x as imaginary time, and the two edge channels are then a pair of quantum spins, with the spins fixed at $Q=\Lambda$ at the initial and final ''times'' $x=0$, L. In the absence of the rest of the action, quantum-mechanical consistency requires in either geometry that the coefficient $\sigma_{xy}^{II,0}$ be quantized to integer values, for reasons closely related to the properties of ''large'' (topologically nontrivial) gauge transformations; for the case of SU(2), this corresponds to $2S=$ integer, as usual. Essentially, the argument says that, since the only degree of freedom in the problem is the value of Q on the edge, then its continuation into the interior, needed to write the topological term, is arbitrary, and the path integral should be invariant under a change in Q in the interior that does not affect the edge; such changes are the ''large'' gauge transformations. Since the change in the action under such a change is $2\pi i\sigma_{xy}^{II,0}q$ for some integer q, this implies that $\sigma_{xy}^{II,0}$ is an integer. This is related to arguments for quantization of the Hall conductivity, once localization sets in Refs. 29 and 30. In this case, we may imagine that the localized system is described by the NLσM but with σ_{xx}^0 replaced by a renormalized value σ_{xx}, equal to zero because of localization. Then a similar argument requires that the renormalized σ_{xy}^{II} is quantized to integer values. Thus quantization of the Hall conductance and quantization of spin are closely connected.[62] This argument is also connected[30] with the gauge-invariance argument for quantization.[40] The edge states, that are the only degrees of freedom able to transport current over large distances in the localized regime, correspond to the quantum spin (in the $n\to 0$ limit). We note that from this point of view of the edge states, in which x plays the role of imaginary time, A_x plays the role of an external magnetic field, in the sense of the familiar Zeeman coupling in the SU(2) case. It is coupled to Q, which corresponds to the spin operator, or the current operator for the edge state. Q corresponds for SU$(2n)$ to the three-component unit vector $\mathbf{\Omega}$ that describes an SU(2) spin, which can be obtained explicitly by writing, for $n=1$, $Q=\mathbf{\Omega}\cdot\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is the vector of Pauli matrices. For SU(2), the corresponding operators in the quantum theory, after rescaling to absorb the coefficient analogous to our $\sigma_{xy}^{II,0}$, are the familiar operators $\mathbf{S}$, which generate SU(2) rotations and are conserved when the Hamiltonian is SU(2) invariant. In the presence of the vector potential $\mathbf{A}$, which enters multiplied by the magnitude of the spin S, to give the Zeeman coupling, just as in our action, the equation of motion (or Ward identity) for a single quantum spin describes the familiar precessional dynamics, which can therefore also be viewed as the covariant conservation of the spin.

By contrast, for the full action in the weak coupling regime $\sigma_{xx}\gg 1$, where the rest of the action depends on the form of Q in the interior, there is no reason why either σ_{xy}^{0} or $\sigma_{xy}^{II,0}$ should be quantized, in accordance with our physical expectations. The same applies to derivations of $S[\mathbf{A}]$ starting from the network model,[49] which also represents only the Fermi energy response, except that in this case the splitting of σ_{xy}^{0} into $\sigma_{xy}^{I,0}$ and $\sigma_{xy}^{II,0}$ is a matter of an arbitrary definition, as we discussed for the linearized model in Sec. I B. In the network model, the links of the lattice can be viewed as quantized edge channels, and these are the only degrees of freedom, so the coupling of $\mathbf{A}$ to these links is of the edge form discussed above, but with $\sigma_{xy}^{II,0}$ replaced by the quantized value 1. Of course, the coarse-grained values of σ_{xx}^{0}, $\sigma_{xy}^{I,0}$, and $\sigma_{xy}^{II,0}$ are determined by the parameters of the vertices in the network model, and by the definition of the coarse-grained currents,[47] so they are not quantized.

APPENDIX D: COMPUTATION OF THE VARIANCE DIAGRAMS

The conventional diagrams Figs. 4(a) and 4(b) depend on γ through the diffusion propagator. Using the boundary perturbation expansion, we obtain

$$\begin{aligned}\mathrm{Tr}(dd)=&\mathrm{Tr}\,(d^0d^0)+\mathrm{Tr}(d^0d^0\mathcal{B}d^0)+\mathrm{Tr}(d^0\mathcal{B}d^0d^0)\\&+2\,\mathrm{Tr}(d^0d^0\mathcal{B}d^0\mathcal{B}d^0)+\mathrm{Tr}(d^0\mathcal{B}d^0d^0\mathcal{B}d^0)+O(\gamma^3),\end{aligned}\tag{D1}$$

$$\begin{aligned}\mathrm{Tr}(dd^T)=&\mathrm{Tr}(d^0d^0)+\mathrm{Tr}(d^0d^0\mathcal{B}^Td^0)+\mathrm{Tr}(d^0\mathcal{B}^Td^0d^0)\\&+2\,\mathrm{Tr}(d^0d^0\mathcal{B}^Td^0\mathcal{B}^Td^0)+\mathrm{Tr}(d^0\mathcal{B}^Td^0d^0\mathcal{B}^Td^0)\\&+O(\gamma^3),\end{aligned}\tag{D2}$$

where the matrix $\mathcal{B}$ has the following elements in the basis of ϕ_{nm}^0:

$$\langle n'm'|\mathcal{B}|nm\rangle=-\frac{\gamma}{WL}\frac{8mm'}{(m')^2-m^2}\delta_{m+m',\mathrm{odd}}\delta_{n+n',\mathrm{even}}.$$

The linear term in γ is zero, because the matrix $\mathcal{B}$ is antisymmetric. We obtain

$$(\delta g)^2_{a,b}=\frac{1}{\pi^4}\left[6f_2-\gamma^2\frac{64}{\pi^4}(12f_3-2f_4)\right],\tag{D3}$$

where

$$f_2\left(\frac{L}{W}\right)=\sum_{n=0}\sum_{m=1}\frac{1}{(m^2+n^2L^2/W^2)^2},\tag{D4}$$

$$f_3\left(\frac{L}{W}\right)=\sum_{n=0,m=1}\sum_{n'=0,m'=1}\frac{1}{\left(m^2+n^2\frac{L^2}{W^2}\right)^3}\frac{1}{m'^2+n'^2\frac{L^2}{W^2}}\times\frac{(mm')^2}{[(m')^2-m^2]^2}\delta_{m+m',\mathrm{odd}}\delta_{n+n',\mathrm{even}},\tag{D5}$$

$$f_4\left(\frac{L}{W}\right)=\sum_{n=0,m=1}\sum_{n'=0,m'=1}\frac{1}{\left(m^2+n^2\frac{L^2}{W^2}\right)^2}\times\frac{1}{\left(m'^2+n'^2\frac{L^2}{W^2}\right)^2}\frac{(mm')^2}{[(m')^2-m^2]^2}\times\delta_{m+m',\mathrm{odd}}\delta_{n+n',\mathrm{even}}.\tag{D6}$$

The upper bounds for n and m in all sums are $n_{\max}\sim W/l$ and $m_{\max}\sim L/l$. For a square sample with $L=W$, we have $f_2\simeq 1.51$,

$$(\delta g)^2_{a,b,L=W}=\frac{1}{\pi^4}[9.06-2.35\gamma^2]+O(\gamma^3).\tag{D7}$$

Figures 4(e) and 4(f) give

$$\begin{aligned}(\delta g)^2_{e,f}=&\frac{256}{\pi^8}\gamma^2\frac{L^2}{W^2}\sum_{n_1,m_1=\mathrm{odd}}\sum_{n_2,m_2=\mathrm{odd}}\sum_{n_3=0,m_3=1}\frac{1}{m_1}\frac{1}{m_2}\frac{1}{m_1^2+n_1^2L^2/W^2}\frac{1}{m_2^2+n_2^2L^2/W^2}\frac{1}{(m_3^2+n_3^2L^2/W^2)^2}\{3(m_3^2D_3D_2\\&+n_3^2L^2/W^2D_1D_4)+3(m_1m_2D_5D_2+n_1n_2L^2/W^2D_1D_6)+10(m_1m_3D_7D_2+n_1n_3L^2/W^2D_1D_8)\},\end{aligned}\tag{D8}$$

where

$$D_1(m_1,m_2,m_3,m_3)=\delta_{m_1,m_2}-\tfrac{1}{2}\delta_{m_1,2m_3+m_2}-\tfrac{1}{2}\delta_{m_1,-2m_3+m_2}+\tfrac{1}{2}\delta_{m_1,2m_3-m_2},$$
$$D_2(n_1,n_2,n_3,n_3)=\delta_{n_1,n_2}+\tfrac{1}{2}\delta_{n_1,2n_3+n_2}+\tfrac{1}{2}\delta_{n_1,-2n_3+n_2}+\tfrac{1}{2}\delta_{n_1,2n_3-n_2},$$
$$D_3(m_1,m_2,m_3,m_3)=\delta_{m_1,m_2}+\tfrac{1}{2}\delta_{m_1,2m_3+m_2}-\tfrac{1}{2}\delta_{m_1,-2m_3+m_2}-\tfrac{1}{2}\delta_{m_1,2m_3-m_2},$$
$$D_4(n_1,n_2,n_3,n_3)=\delta_{n_1,n_2}-\tfrac{1}{2}\delta_{n_1,2n_3+n_2}+\tfrac{1}{2}\delta_{n_1,-2n_3+n_2}-\tfrac{1}{2}\delta_{n_1,2n_3-n_2},$$
$$D_5(m_1,m_2,m_3,m_3)=\delta_{m_1,m_2}-\tfrac{1}{2}\delta_{m_1,2m_3+m_2}-\tfrac{1}{2}\delta_{m_1,-2m_3+m_2}-\tfrac{1}{2}\delta_{m_1,2m_3-m_2},$$
$$D_6(n_1,n_2,n_3,n_3)=\delta_{n_1,n_2}+\tfrac{1}{2}\delta_{n_1,2n_3+n_2}+\tfrac{1}{2}\delta_{n_1,-2n_3+n_2}-\tfrac{1}{2}\delta_{n_1,2n_3-n_2},$$
$$D_7(m_1,m_2,m_3,m_3)=-\tfrac{1}{2}\delta_{m_1,2m_3+m_2}+\tfrac{1}{2}\delta_{m_1,-2m_3+m_2}+\tfrac{1}{2}\delta_{m_1,2m_3-m_2},$$
$$D_8(n_1,n_2,n_3,n_3)=\tfrac{1}{2}\delta_{n_1,2n_3+n_2}-\tfrac{1}{2}\delta_{n_1,-2n_3+n_2}+\tfrac{1}{2}\delta_{n_1,2n_3-n_2}. \tag{D9}$$

This term has a logarithmic part [it diverges with system size as $\ln(L/l)$]. It comes from the first term in the curly brackets, when the two derivatives of the four-point interaction are applied to the closed loop of two diffusion propagators. The second term results from applying the two derivatives to the two external propagators. The third term arises when one of the derivative is applied to the closed loop, one is applied to the external propagator.

Figures 4(g) and 4(h) are both logarithmic. They are of opposite signs, but the amplitude of Fig. 4(g), which is positive, is twice that of Fig. 4(h). We obtain

$$(\delta g)^2_{g,h}=\frac{256}{\pi^8}\gamma^2\frac{L^2}{W^2}\sum_{n_1,m_1=\text{odd}}\sum_{n_2,m_2=\text{odd}}\sum_{n_3=0,m_3=1}\frac{1}{m_1}\frac{1}{m_2}\frac{1}{m_1^2+n_1^2L^2/W^2}\frac{1}{m_2^2+n_2^2L^2/W^2}\frac{1}{m_3^2+n_3^2L^2/W^2}D_1D_2. \tag{D10}$$

Figures 4(j) and 4(j$'$) also have logarithmic divergence. We obtain from Fig. 4(j), we obtain

$$(\delta g)^2_j=\frac{256}{\pi^8}\gamma^2\frac{L^2}{W^2}\sum_{n_1,m_1=\text{odd}}\sum_{n_2,m_2=\text{odd}}\sum_{n_3=0,m_3=1}\sum_{n_4=0,m_4=1}\frac{1}{m_1}\frac{1}{m_2}\frac{1}{m_1^2+n_1^2L^2/W^2}$$
$$\times\frac{1}{m_2^2+n_2^2L^2/W^2}\frac{m_3}{m_3^2+n_3^2L^2/W^2}\frac{m_4}{m_4^2+n_4^2L^2/W^2}[\delta_{m_1,m_4\pm m_3}-\delta_{m_1,m_3-m_4}][\delta_{m_2,m_3\pm m_4}-\delta_{m_2,m_4-m_3}]$$
$$\times[\delta_{n_1,n_4\pm n_3}+\delta_{n_1,n_3-n_4}][\delta_{n_2,n_4\pm n_3}+\delta_{n_2,n_3-n_4}]. \tag{D11}$$

From Fig. 4(j$'$)

$$(\delta g)^2_{j'}=-\frac{256}{\pi^8}\gamma^2\frac{L^2}{W^2}\sum_{n_1,m_1=\text{odd}}\sum_{n_2,m_2=\text{odd}}\sum_{n_3=0,m_3=1}\sum_{n_4=0,m_4=1}\frac{1}{m_1}\frac{1}{m_2}\frac{1}{m_1^2+n_1^2L^2/W^2}$$
$$\times\frac{1}{m_2^2+n_2^2L^2/W^2}\frac{m_3^2}{m_3^2+n_3^2L^2/W^2}\frac{1}{m_4^2+n_4^2L^2/W^2}[\delta_{m_1,m_4\pm m_3}-\delta_{m_1,m_3-m_4}][\delta_{m_2,m_4\pm m_3}-\delta_{m_2,m_3-m_4}]$$
$$\times[\delta_{n_1,n_4\pm n_3}+\delta_{n_1,n_3-n_4}][\delta_{n_2,n_4\pm n_3}+\delta_{n_2,n_3-n_4}]. \tag{D12}$$

Both Figs. 4(j) and 4(j$'$) are negative. Their logarithmic parts combine to cancel those from Figs. 4(e), 4(f), 4(g), and 4(h). The variance, which is the sum of Figs. 4(a)–4(j$'$), is finite.

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