Synchronization in networks of superconducting wires

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We study the synchronization phenomena in networks of superconducting wires driven by quenched random external currents. Each wire interacts with all the other wires perpendicular to it; this system naturally presents an example of the well-known phase model. The equations of motion are obtained from current conservation conditions, which in turn lead to an effective Hamiltonian of the system. We then use the replica method to derive the self-consistency equations and obtain the phase diagram. Effects of alternating currents as well as the possibility of glassy behavior are also discussed. [S0163-1829(97)02225-X]

I. INTRODUCTION

A large population of mutually interacting self-oscillatory units often exhibits a coherent motion among their constituents over a relatively long distance. This remarkable feature, called *collective synchronization*, is prevalent in various oscillatory systems in physics, chemistry, biology, and even social sciences.[1–3] Theoretically, the case of the uniform infinite-range interaction is most tractable and serves as a starting point towards an understanding of the behavior of realistic systems. The globally coupled system was first analyzed by Kuramoto,[2] in which each oscillator has been modeled as a rotator with randomly distributed natural frequency. It was successfully shown that collective synchronization indeed sets in gradually as the coupling strength is increased beyond the critical value, which is reminiscent of the second-order equilibrium transition. In many physical systems, short-range interactions are more realistic, and the mean-field approach is regarded as a mathematically simple, albeit crude, approximation to the true underlying physics. On the other hand, there also exist a number of systems,[1–4] e.g., intracavity lasers, series arrays of Josephson junctions, and biological systems, where global coupling appears to be natural. In those systems, the mean-field approach is exact and should not be regarded as an approximation.

The network of superconducting wires, which has attracted recent interest,[5,6] provides another good example of the mean-field system. It consists of two orthogonal sets of N parallel superconducting wires which are coupled with each other by Josephson junctions at all points of crossing. The Hamiltonian of the system is given by the sum of individual Josephson energies, while the phases of the wires are regarded as classical thermodynamic variables. It should be noted here that the number of nearest neighbors scales with the system size, which naturally allows the mean-field approach. In view of the experimental situation, we consider the system *driven by external currents*, for which the Hamiltonian description is not adequate. Instead we deal with the equations of motion for the driven system, which can be derived from the current conservation conditions. We in particular consider the general case of nonuniform driving, and allow the current injected at one edge of a wire to be different from those injected at the edges of other wires. For sufficiently large N, the driving currents can be regarded as *quenched* random variables following a distribution with zero mean and appropriate variance. It should be noted that, unlike the Kuramoto model,[2] each wire interacts not with all the rest of the wires but with the half of the wires lying perpendicular to it. As a result of this *semiglobal* coupling, the equations of motion consist of two sets of equations; the two sets govern the dynamics of the horizontal and the vertical wires, respectively. Still the obtained equations of motion are closely related to those of the Kuramoto model, because each set of equations is of the mean-field nature. In fact, the system may be regarded as a generalization of the Josephson-junction series arrays, which can be mapped into the Kuramoto model in the proper limit.[4] Thus, this system may serve as a physical realization of the Kuramoto model, and many interesting features present in this system may be explained in terms of the synchronization. For example, the superconducting state with global phase coherence can be identified with the synchronized state and the superconducting-normal transition corresponds to the synchronization-desynchronization transition.

This paper consists of five sections: In Sec. II, we obtain the equations of motion and find the self-consistency equations at zero temperature, which yield results similar to those of the Kuramoto model. Section III is devoted to the finite-temperature analysis, which gives the phase diagram, while Sec. IV presents the effects of periodic driving currents. When the system is driven by alternating currents, it exhibits interesting phenomena of periodic synchronization. The transition into such a state with periodic synchronization may also be induced by changing the phase delay between the currents applied to horizontal wires and those to vertical wires. Finally, in Sec. V, a summary of the results is given, and the glassy behavior is shown not to appear in the absence of an external magnetic field, indicating the key role of the frustration.

II. EQUATIONS OF MOTION AT ZERO TEMPERATURE

We consider an array of two mutually perpendicular sets of N superconducting wires with Josephson junctions at each node. At one edge of each wire, uniform current I is injected and at the opposite edge the same current is extracted.

Neglecting capacitive effects and thermal noise currents, we can write the net current from the ith horizontal wire to the jth vertical wire as the sum of the Josephson current and the normal current:

$$I_{ij}=I_c \sin(\phi_i^{(1)}-\phi_j^{(2)})+\frac{V_{ij}}{R}, \tag{1}$$

where $\phi_i^{(1)}$ and $\phi_i^{(2)}$ are the phases of ith horizontal and vertical wires, respectively, I_c is the critical current of the junction, and R is the shunt resistance. V_{ij} $(\equiv V_i-V_j)$ is the potential difference across the junction, and related to the phase via the Josephson relation, $d(\phi_i^{(1)}-\phi_j^{(2)})/dt=2eV_{ij}/\hbar$.[7] The current conservation at each horizontal wire imposed on Eq. (1) yields a set of N coupled equations:

$$I_i^{\rm ext}=\sum_{j=1}^{N}\left[\frac{\hbar}{2eR}\frac{d}{dt}(\phi_i^{(1)}-\phi_j^{(2)})+I_c \sin(\phi_i^{(1)}-\phi_j^{(2)})\right],$$

where $I_i^{\rm ext}$ is the external current fed into the ith wire. It is convenient to write the above set of equations in the form

$$\dot{\phi}_i^{(1)}=\frac{1}{N}\sum_j \dot{\phi}_j^{(2)}+\omega_i^{(1)}-\frac{K}{N}\sum_j \sin(\phi_i^{(1)}-\phi_j^{(2)}), \tag{2}$$

where $\omega_i^{(1)}\equiv 2eRI_i^{\rm ext}/N\hbar$ corresponds to the natural frequency of the ith wire and $K\equiv 2eRI_c/\hbar$ measures the coupling strength between the horizontal and vertical wires. Similarly, the equations for vertical wires assume the form

$$\dot{\phi}_j^{(2)}=\frac{1}{N}\sum_i \dot{\phi}_i^{(1)}+\omega_j^{(2)}-\frac{K}{N}\sum_i \sin(\phi_j^{(2)}-\phi_i^{(1)}), \tag{3}$$

which, together with Eq. (2), constitute the equations of motion of the system. We consider the general case of nonuniform driving, and allow the current injected at one edge of a wire to be different from those injected at the edges of other wires. In the thermodynamic limit $(N\to\infty)$, the driving currents can be described by a distribution with appropriate variance. Here the current driving each wire does not change with time, and can be regarded as a *quenched* random variable. We thus assume that the corresponding natural frequencies $\omega^{(1)}$ and $\omega^{(2)}$ are distributed according to the Gaussian distributions $g_1(\omega)$ and $g_2(\omega)$ with variances σ_1 and σ_2, respectively. Without loss of generality, the mean values of the distributions can be taken to be zero.[8] (The detailed shape of the distribution is irrelevant while the simple case of uniform driving corresponds to the delta-function distribution with $\sigma_1=\sigma_2=0$.) It is convenient to write the above equations in the matrix form

$$M_{ij}\dot{\phi}_j=F_i(\omega_i,\boldsymbol{\phi}) \tag{4}$$

with F_i being an appropriate function of ω_i and $\boldsymbol{\phi}\equiv(\phi_1^{(1)},\ldots,\phi_N^{(1)},\phi_1^{(2)},\ldots,\phi_N^{(2)})$, where the components of the $2N\times 2N$ matrix M are given by

$$M_{ij}=\begin{cases}\delta_{ij} & \text{for } 1\leqslant i,j\leqslant N \text{ and for } N+1\leqslant i,j\leqslant 2N\\ -\dfrac{1}{N} & \text{otherwise.}\end{cases}$$

Here, the determinant of the matrix M vanishes, reflecting the U(1) symmetry of the whole system. We thus set the phase of one particular wire, say $\phi_1^{(1)}$, equal to zero all time, i.e., $M_{1i}=M_{i1}=0$ for all i. Then the inverse matrix of M can be easily obtained, and multiplication of both sides of Eq. (4) by M^{-1} yields more tractable equations of motion:

$$\dot{\phi}_i^{(1)}=\omega_i^{(1)}-K\Delta_2 \sin(\phi_i^{(1)}-\theta_2),$$

$$\dot{\phi}_j^{(2)}=\omega_j^{(2)}-K\Delta_1 \sin(\phi_j^{(2)}-\theta_1)+K\Delta_1\Delta_2 \sin(\theta_2-\theta_1), \tag{5}$$

where the order parameters $\Delta_\alpha \exp(i\theta_\alpha)\equiv(1/N)\Sigma_j \exp[i\phi_j^{(\alpha)}]$ measure the degrees of coherence of the horizontal $(\alpha=1)$ and vertical $(\alpha=2)$ wires. It is obvious that these equations of motion reduce precisely to those for the well-known Kuramoto model if one sets $\phi_i^{(1)}=\phi_i^{(2)}$ for all i. Thus this system can be considered to be a generalization of the Kuramoto model and to provide a physical realization of the latter.

The stationary state exhibited by the above equations of motion is described by the probability distributions of $\phi^{(1)}$ and $\phi^{(2)}$:

$$P_1(\phi^{(1)};\omega)=\begin{cases}\delta[\phi^{(1)}-\theta_2-\sin^{-1}(\omega/K\Delta_2)] & \text{for } |\omega|\leqslant K\Delta_2,\\ \sqrt{\omega^2-(K\Delta_2)^2}[2\pi|\omega-K\Delta_2 \sin(\phi^{(1)}-\theta_2)|]^{-1} & \text{otherwise}\end{cases}$$

and

$$P_2(\phi^{(2)};\omega)=\begin{cases}\delta\{\phi^{(2)}-\theta_1-\sin^{-1}[(\omega+\Delta)/K\Delta_1]\} & \text{for } |\omega+\Delta|\leqslant K\Delta_1,\\ \sqrt{(\omega+\Delta)^2-(K\Delta_1)^2}[2\pi|\omega+\Delta-K\Delta_1 \sin(\phi^{(2)}-\theta_1)|]^{-1} & \text{otherwise,}\end{cases}$$

with $\Delta \equiv K\Delta_1\Delta_2 \sin(\theta_2-\theta_1)$. The order parameters can be expressed in terms of the probability distribution

$$\Delta_\alpha \exp(i\theta_\alpha)=\int d\omega g_\alpha(\omega)\int_0^{2\pi} d\phi P_\alpha(\phi;\omega)e^{i\phi},$$

which in turn gives self-consistency equations

$$\Delta_1=K\Delta_2\int_{-1}^{1} dx g_1(K\Delta_2 x)\sqrt{1-x^2},$$

$$\Delta_2=K\Delta_1\int_{-1}^{1} dy g_2(K\Delta_1 y)\sqrt{1-y^2}, \tag{6}$$

$$\Delta=0$$

with $x\equiv\omega/(K\Delta_2)$ and $y\equiv\omega/(K\Delta_1)$. It is interesting to note that the relative phase difference of the two sets of wires, $\theta_1-\theta_2$, is always zero although the amplitudes Δ_1 and Δ_2 may have different values depending on the distribution functions of the applied currents. This implies that the whole system can be fully synchronized with each other even though the coupling is not wholly global but semiglobal. Thus, the obtained result is indeed a straightforward extension of that of the Kuramoto model despite the difference in the coupling. As K is increased from zero, the only possible solution of Eq. (6) is the trivial solution ($\Delta_1=\Delta_2=0$) until K reaches the critical value K_c. At this critical value, a nontrivial solution branch bifurcates from the zero branch; the latter becomes unstable beyond K_c.[9] Here, both the horizontal and vertical wires become coherent at a single K_c. Overall coherence across the system is thus attained for $K>K_c$, and the whole system becomes superconducting. Expansion of Eq. (6) near K_c straightforwardly yields $\Delta_\alpha \propto (K-K_c)^{1/2}$ for $\alpha=1$ and 2, with the critical value $K_c=2[\pi\sqrt{g_1(0)g_2(0)}]^{-1}$, where the critical behavior with the exponent 1/2 is a characteristic of mean-field systems. Here we again emphasize that the amplitudes of Δ_1 and Δ_2 may differ from each other even though they have the same exponent near K_c.

III. EFFECTIVE HAMILTONIAN AT FINITE TEMPERATURES

In this section, we consider the extension of the model to finite temperatures. We thus add thermal noises to the equations of motion and obtain

$$\dot{\phi}_i^{(1)}=\frac{1}{N}\sum_j \dot{\phi}_j^{(2)}+\omega_i^{(1)}-\frac{K}{N}\sum_j \sin(\phi_i^{(1)}-\phi_j^{(2)})+\gamma_i(t),$$

$$\dot{\phi}_j^{(2)}=\frac{1}{N}\sum_i \dot{\phi}_i^{(1)}+\omega_j^{(2)}-\frac{K}{N}\sum_i \sin(\phi_j^{(2)}-\phi_i^{(1)})+\gamma_j(t), \tag{7}$$

where the thermal noises are characterized by

$$\langle \gamma_i(t)\rangle=0,$$

$$\langle \gamma_i(t)\gamma_j(t')\rangle=2\Gamma\delta_{ij}\delta(t-t')$$

with Γ ($\geqslant 0$) measuring the strength of the noise. Equation (7) has the form of a set of Langevin equations, where two different sets of variables, representing horizontal and vertical wires, respectively, are coupled with each other. As in the previous section, one may introduce the order parameters Δ_1 and Δ_2, and write down two Fokker-Planck equations, for the two *one-wire* probability densities P_1 and P_2, the zero-temperature limits of which have been given in Sec. II. At finite temperatures, however, the two Fokker-Planck equations lead to self-consistency equations, which couple Δ_1 and Δ_2 quite complexly and cannot be solved in a transparent manner. Therefore we resort to the Fokker-Planck equation for the *2N-wire* probability density $P(\boldsymbol{\phi},t)$:

$$\frac{\partial P}{\partial t}=-\sum_i\left[\frac{\partial}{\partial\phi_i}h_i-\Gamma\frac{\partial^2}{\partial\phi_i^2}\right]P \tag{8}$$

with

$$h_i=\begin{cases} w_i^{(1)}-(K/N)\sum_j \sin(\phi_i^{(1)}-\phi_j^{(2)}) & \text{for horizontal wires,}\\ w_i^{(2)}-(K/N)\sum_j \sin(\phi_i^{(2)}-\phi_j^{(1)}) & \text{for vertical wires,}\end{cases}$$

which leads to a stationary solution $P_0(\boldsymbol{\phi})\propto\exp(-H[\boldsymbol{\phi}]/T)$ with the effective Hamiltonian

$$H=-\frac{1}{N}\sum_{i,j}\cos(\phi_i^{(1)}-\phi_j^{(2)})-\frac{1}{K}\sum_i(\omega_i^{(1)}\phi_i^{(1)}+\omega_i^{(2)}\phi_i^{(2)}) \tag{9}$$

at temperature $T\equiv\Gamma/K$. Note that the second term in Eq. (9), which has the form of a generalized washboard potential, is not periodic in ϕ. This apparent absence of the correct periodicity suggests that the above solution is not adequate for describing the dynamics of the system, which involves variations of ϕ larger than 2π. Similar problems arise in Josephson-junction arrays[10] and in networks of neuronal oscillators,[11] where the actions were regarded as periodic functions with period $2n\pi$ ($n\to\infty$). The standard Villain approximation, which gives an accurate description at low temperatures, was then shown to restore the correct periodicity and to yield results independent of n. We thus follow Ref. 10, and regard H in Eq. (9) as the effective Hamiltonian of the system, with the period $2n\pi$. Here, similarly to Ref. 10, it can also be seen that the corresponding free energy is independent of n, which allows us to take the integration interval of ϕ to be from $-\pi$ to π.

We then use the replica method, and obtain the free energy per wire:

$$f=\lim_{n\to 0}\frac{T}{nN}\left[1-\left(\frac{N}{2\pi T}\right)^{3n}\int\prod_a dA_a dB_a dC_a dD_a dE_a dF_a \exp(-N\Phi/T)\right] \tag{10}$$

with

$$\Phi=\frac{1}{2}\sum_a (A_a^2+B_a^2+C_a^2+D_a^2+E_a^2+F_a^2)-T\ln\left\langle\!\!\left\langle \sum_{[\phi_a]}\exp\left\{\frac{1}{T}\sum_a\left[A_a(\cos\phi_a^{(1)}+\cos\phi_a^{(2)})+iB_a\cos\phi_a^{(1)}+iC_a\cos\phi_a^{(2)}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+D_a(\sin\phi_a^{(1)}+\sin\phi_a^{(2)})+iE_a\sin\phi_a^{(1)}+iF_a\sin\phi_a^{(2)}+\frac{1}{K}(\omega^{(1)}\phi_a^{(1)}+\omega^{(2)}\phi_a^{(2)})\right]\right\}\right\rangle\!\!\right\rangle_\omega,$$

where $\langle\langle\cdots\rangle\rangle_\omega$ denotes the quenched average over the distributions of $\omega^{(1)}$ and $\omega^{(2)}$. In the thermodynamic limit $N\to\infty$, we perform the integral via the saddle-point method

$$\frac{\partial\Phi}{\partial A_a}=\frac{\partial\Phi}{\partial B_a}=\cdots=0,$$

which leads to

$$A_a=\langle\!\langle\ \langle\cos\phi_a^{(1)}+\cos\phi_a^{(2)}\rangle\ \rangle\!\rangle_\omega,\quad D_a=\langle\!\langle\ \langle\sin\phi_a^{(1)}+\sin\phi_a^{(2)}\rangle\ \rangle\!\rangle_\omega,$$
$$B_a=i\langle\!\langle\ \langle\cos\phi_a^{(1)}\rangle\ \rangle\!\rangle_\omega,\quad E_a=i\langle\!\langle\ \langle\sin\phi_a^{(1)}\rangle\ \rangle\!\rangle_\omega, \tag{11}$$
$$C_a=i\langle\!\langle\ \langle\cos\phi_a^{(2)}\rangle\ \rangle\!\rangle_\omega,\quad F_a=i\langle\!\langle\ \langle\sin\phi_a^{(2)}\rangle\ \rangle\!\rangle_\omega.$$

Here $\langle\mathcal{O}(\phi)\rangle$ stands for the average with respect to the action $\mathcal{L}$,

$$\langle\mathcal{O}(\phi)\rangle=\frac{\sum_{[\phi]}\mathcal{O}(\phi)e^{\mathcal{L}([\phi])}}{\sum_{[\phi]}e^{\mathcal{L}([\phi])}}$$

with the action given by

$$\mathcal{L}=\frac{1}{T}\sum_a\left[A_a(\cos\phi_a^{(1)}+\cos\phi_a^{(2)})+iB_a\cos\phi_a^{(1)}+iC_a\cos\phi_a^{(2)}+D_a(\sin\phi_a^{(1)}+\sin\phi_a^{(2)})+iE_a\sin\phi_a^{(1)}+iF_a\sin\phi_a^{(2)}\right.$$
$$\left.+(1/K)(\omega^{(1)}\phi_a^{(1)}+\omega^{(2)}\phi_a^{(2)})\right].$$

Since the order parameters B_a, C_a, E_a, and F_a in Eq. (11) are purely imaginary, we write $B_a\equiv iB_a'$, etc., for convenience, and omit the primes hereafter. After performing the quenched average, we obtain the free energy in the form

$$f=BC+EF-\frac{1}{K}\int Dz_1Dz_2\ln\left\{\mathrm{Tr}_{[\phi]}\exp\left[\frac{1}{T}\left(C\cos\phi^{(1)}+B\cos\phi^{(2)}+F\sin\phi^{(1)}+E\sin\phi^{(2)}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\sqrt{\sigma_1}}{KT}\phi^{(1)}z_1+\frac{\sqrt{\sigma_2}}{KT}\phi^{(2)}z_2\right)\right]\right\}=BC+EF-T\int Dz_1\ln\left[(\sqrt{\sigma_1}z_1/KT)^2\sum_n(-1)^n\frac{I_n(\sqrt{C^2+F^2}/T)}{(\sqrt{\sigma_1}z_1/KT)^2+n^2}\right]$$
$$-T\int Dz_2\ln\left[(\sqrt{\sigma_2}z_2/KT)^2\sum_n(-1)^n\frac{I_n(\sqrt{B^2+E^2}/T)}{(\sqrt{\sigma_2}z_2/KT)^2+n^2}\right], \tag{12}$$

where I_n's represent the modified Bessel functions of the first kind, and the replica-symmetric solution has been chosen, i.e., $B_a=B$ for all a, etc. Defining the order parameters Δ_1 and Δ_2 measuring the mutual coherence of wires in the same direction as $\Delta_1\equiv\sqrt{B^2+E^2}$ and $\Delta_2\equiv\sqrt{C^2+F^2}$, which are restricted in the range $[0,1]$, we finally get the self-consistency equations by differentiating the free energy with respect to the order parameters:

$$\Delta_1=\int Dz\sum_n(-1)^n\frac{I_n'(\Delta_2/T)}{(\sqrt{\sigma_1}z/KT)^2+n^2}\left[\sum_n(-1)^n\frac{I_n(\Delta_2/T)}{(\sqrt{\sigma_1}z/KT)^2+n^2}\right]^{-1},$$
$$\Delta_2=\int Dz\sum_n(-1)^n\frac{I_n'(\Delta_1/T)}{(\sqrt{\sigma_2}z/KT)^2+n^2}\left[\sum_n(-1)^n\frac{I_n(\Delta_1/T)}{(\sqrt{\sigma_2}z/KT)^2+n^2}\right]^{-1}, \tag{13}$$

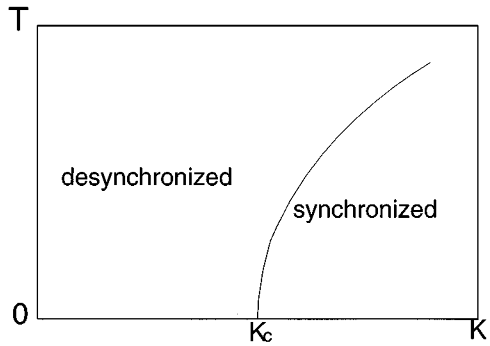

FIG. 1. Phase boundary on the (K,T) plane. The critical value K_c is given in the text.

where $I_n'(x) \equiv dI_n(x)/dx$, and $\int Dz$ denotes the average over the normalized Gaussian variable z.

We now check whether the obtained results are consistent with the zero-temperature results given in Sec. II: At zero temperature, the direct expansion of Eq. (13) fails because the arguments of the modified Bessel functions are divergent in all orders. Therefore, we adopt the spin-wave approximation which should be accurate at zero temperature, and expand the washboard-type potential in Eq. (12) about its minimum:

$$\exp(A\cos\phi + B\phi) \approx \exp[\tilde{A}\cos(\phi-\phi_0)] \approx \exp \tilde{A}\left[1-\frac{1}{2}(\phi-\phi_0)^2\right],$$

where $\phi_0 \equiv \sin^{-1}(B/A)$ and $\tilde{A} \equiv A\cos\phi_0$. After taking the trace over ϕ, one can easily confirm that Eq. (13) indeed reduces to Eq. (6), reproducing correct behavior in the zero-temperature limit. As the temperature is increased, on the other hand, vortex excitations begin to appear, making the simple spin-wave approximation invalid. It is thus necessary to investigate explicitly Eq. (13), which can be solved numerically. For a given value of the coupling strength K, the (numerical) solution of Eq. (13) gives the transition temperature T_c as a function of K: Above T_c, collective synchronization is not allowed whereas a coherent state emerges as the temperature becomes lower than T_c. Alternatively, at a given temperature the system may be considered to possess the critical coupling strength K_c, below which no synchronization appears. In particular, at zero temperature, the critical value K_c has been given in Sec. II: $K_c = 2[\pi\sqrt{g_1(0)g_2(0)}]^{-1}$, which depends on the distributions of external currents. At finite temperatures, the values of K_c may be obtained numerically to yield phase boundaries in the (K,T) plane. Here we have examined the numerical solutions of Eq. (13) for several different cases, and found that the overall phase boundaries remain qualitatively unchanged, regardless of the values of σ_1 and σ_2. Figure 1 displays a typical phase diagram in the (K,T) plane, showing the phase boundary below which collective synchronization sets in.

IV. PERIODICALLY DRIVEN SYSTEM

Periodically driven systems of globally coupled oscillators have been studied, and shown to display interesting behaviors.[12] We expect such periodic driving to play a similar role in the network of superconducting wires, and consider the network driven by combined direct and alternating applied currents, with particular attention to the phase delay between the driving currents in the horizontal wires and those in the vertical wires. At zero temperature the equations of motion take the form

$$\dot{\phi}_i^{(1)} = \omega_i^{(1)} - K\Delta_2 \sin(\phi_i^{(1)} - \theta_2) + I_i^{(1)}\cos\Omega t,$$

$$\dot{\phi}_j^{(2)} = \omega_j^{(2)} - K\Delta_1 \sin(\phi_j^{(2)} - \theta_1) + K\Delta_1\Delta_2 \sin(\theta_2-\theta_1) + I_j^{(2)}\cos\Omega(t+\tau), \tag{14}$$

where $\omega_i^{(1)} + I_i^{(1)}\cos\Omega t$ and $\omega_i^{(2)} + I_i^{(2)}\cos\Omega(t+\tau)$ represent the driving currents applied to the ith horizontal and vertical wires, respectively, with τ describing the phase delay between the two driving currents. With suitable redefinitions, each of Eq. (14) can be cast in the simple form

$$\dot{\phi} + \tilde{K}\sin\phi = \omega + I\cos\Omega t, \tag{15}$$

which describes a single resistively shunted Josephson junction with the coupling $\tilde{K} \equiv K\Delta_\alpha$, biased by an applied current $\omega + I\cos\Omega t$. It is well known that such a system can be locked to the external driving, which is characterized by the Shapiro steps[13]

$$\frac{\langle\dot{\phi}\rangle}{\Omega} = n$$

with n integer. On the nth step, the phase of the locked wire is thus given by

$$\phi \approx n\Omega t + \frac{I}{\Omega}\sin\Omega t + \phi_0,$$

where ϕ_0 is determined according to

$$\omega = n\Omega + (-1)^n \tilde{K} J_n(I/\Omega)\sin\phi_0$$

with J_n being the Bessel function of order n. Off the step, the wire is unlocked and its phase is given by

$$\phi \approx \omega t + \frac{I}{\Omega}\sin\Omega t + \phi^0,$$

where ϕ^0 is a constant independent of ω.

The equations for the order parameters are then obtained by imposing self-consistency. For the alternating current amplitude I distributed according to $f_\alpha(I)$, the self-consistency equations read

$$\Delta_\alpha e^{i\theta_\alpha} = \int dI f_\alpha(I) \int d\omega\, g_\alpha(\omega) \langle \exp[i\phi^{(\alpha)}]\rangle_{\omega,I}, \tag{16}$$

where $\langle\cdots\rangle_{\omega,I}$ denotes the average in the stationary state with given ω and I. Since only locked wires contributed to collective synchronization,[12] it is straightforward to compute the averages in Eq. (16) and obtain the self-consistency equations

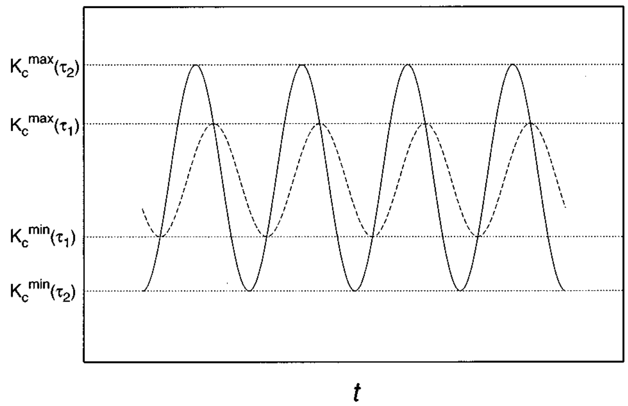

FIG. 2. K_c as a function of time for two different values of the phase delay, $\tau_1=\pi/2$ (represented by the solid line) and $\tau_2=\pi/3$ (represented by the dashed line). We have set $I_0=\Omega=1$ for convenience.

$$\Delta_1 e^{i(\theta_1-\theta_2)}=K\Delta_2\sum_n e^{in\Omega t}\int dI f_1(I)\exp\left(i\frac{I}{\Omega}\sin\Omega t\right)$$

$$\times J_n(I/\Omega)\int_{-1}^{1}dx g_1[n\Omega+K\Delta_2 J_n(I/\Omega)x]$$

$$\times[\sqrt{1-x^2}+i(-1)^n x],$$

$$\Delta_2 e^{i(\theta_2-\theta_1)}=K\Delta_1\sum_n e^{in\Omega t}\int dI f_2(I)\exp\left[i\frac{I}{\Omega}\sin\Omega(t+\tau)\right]$$

$$\times J_n(I/\Omega)\int_{-1}^{1}dy g_2[n\Omega-\Delta+K\Delta_1 J_n(I/\Omega)y]$$

$$\times[\sqrt{1-y^2}+i(-1)^n y], \tag{17}$$

where the integration variables x and y are defined to be

$$x\equiv\frac{\omega-n\Omega}{K\Delta_2 J_n(I/\Omega)},\quad y\equiv\frac{\omega-n\Omega+\Delta}{K\Delta_1 J_n(I/\Omega)}.$$

Note that *in the limit* $\tau\rightarrow 0$, Eq. (17) with $\Delta_1=\Delta_2$ and $\Delta=0$ reduces to the corresponding equation of Ref. 12, which was shown to display periodic synchronization depending on the coupling strength K. Such phenomena of periodic synchronization might be of relevance to the behaviors of many biological systems subject to periodic stimuli. Among those which have already been observed experimentally include oscillatory responses of neurons in the visual cortex of a cat.[14]

Here the behavior of the system depends on the additional parameter τ, the phase delay between driving currents applied to the horizontal and vertical wires. To examine the role of τ, we assume $K\Delta_{1,2}\ll 1$ near the transition to the coherent state, and expand g_1 and g_2 around $n\Omega$. We further assume that $g_1(\omega)$ and $g_2(\omega)$ are nonzero only for $|\omega|<\omega_c$ for a finite cutoff ω_c, and consider high-frequency driving such that $\Omega>\omega_c$. For simplicity, we consider the distribution of driving in the form

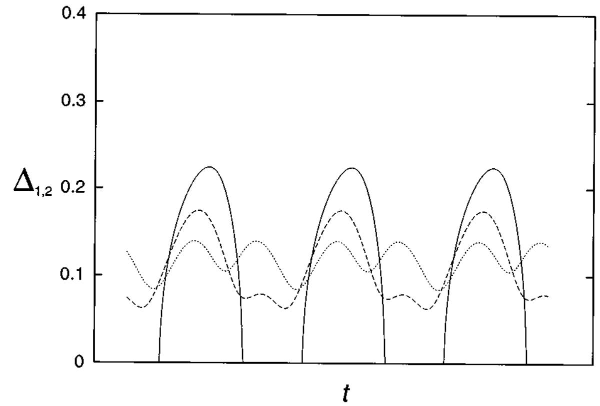

FIG. 3. Periodic behaviors of the order parameters with time for various values of the phase delay. The case $\tau=\pi/3$ represented by the solid line displays synchronization periodically, while the cases $\tau=\pi/2.2$ (dashed line) and $\tau=\pi/2$ (dotted line) exhibit synchronization all the time.

$$f_1(I)=f_2(I)=\frac{1}{2}[\delta(I-I_0)+\delta(I+I_0)],$$

which gives Eq. (17) in the form

$$\Delta_1=a(K\Delta_2)-b(K\Delta_2)^3+O(K\Delta)^5,$$

$$\Delta_2=c(K\Delta_1)-d(K\Delta_1)^3+O(K\Delta)^5 \tag{18}$$

with the coefficients given by

$$a=\frac{\pi}{2}g_1(0)J_0\left(\frac{I_0}{\Omega}\right)\cos\left(\frac{I_0}{\Omega}\sin\Omega t\right),$$

$$b=-\frac{\pi}{2}g_1''(0)J_0^3\left(\frac{I_0}{\Omega}\right)\cos\left(\frac{I_0}{\Omega}\sin\Omega t\right),$$

$$c=\frac{\pi}{2}g_2(0)J_0\left(\frac{I_0}{\Omega}\right)\cos\left[\frac{I_0}{\Omega}\sin\Omega(t+\tau)\right],$$

$$d=-\frac{\pi}{2}g_2''(0)J_0^3\left(\frac{I_0}{\Omega}\right)\cos\left[\frac{I_0}{\Omega}\sin\Omega(t+\tau)\right].$$

Equation (18) allows nontrivial solutions of Δ_1 and Δ_2 only if $K>K_c(\equiv 1/\sqrt{ac})$. Note that K_c is periodic in time and varies in the range $K_c^{\min}<K_c<K_c^{\max}$, where $K_c^{\max}$ and $K_c^{\min}$ are given by the maximum and the minimum values of $1/\sqrt{ac}$ for given τ. For $K<K_c^{\min}$, the system is always incoherent, while it displays synchronization all the time for $K>K_c^{\max}$. For intermediate couplings ($K_c^{\min}<K<K_c^{\max}$), we have periodically $K<K_c$ for some time intervals and $K>K_c$ for the rest of time. Thus the system oscillates between the synchronized and the desynchronized states. It is of interest here that $K_c^{\min}$ and $K_c^{\max}$ depend also on the phase delay τ, which suggests that we may induce transitions between different synchronization states by varying τ. For example, suppose that the value of the delay is varied from τ_1 to τ_2, where the coupling K lies between $K_c^{\max}(\tau_1)$ and $K_c^{\max}(\tau_2)$. This induces a transition from the (continuously) synchronized state into the periodically synchronized state; in the latter the system displays synchronization-desynchronization transitions periodically. (See Fig. 2.) In a similar way, we may observe the transition between the periodically synchronized state and the desynchronized one, when K lies in the interval $[K_c^{\min}(\tau_2),\ K_c^{\min}(\tau_1)]$. Figure 3 displays the numerical solutions of Eq. (18) in the simple case $g_1(\omega)=g_2(\omega)$, where the transition between the continuously synchronized state and the periodically synchronized state can be observed.

V. DISCUSSION

We have studied synchronization phenomena in networks of superconducting wires. In particular, we have considered the networks driven by external currents, which resemble more closely experimental situations. Since the Hamiltonian description is not adequate in this case, we have derived the equations of motion from the current conservation conditions, and investigated their behaviors both at zero temperature and at finite temperatures. The peculiar semiglobal coupling results in the equations of motion consisting of two sets of equations, which governs the dynamics of the horizontal and the vertical wires, respectively. With the constraint that the phases of the ith horizontal and vertical wires are locked to each other, the equations of motion have been found to reduce to those of the Kuramoto model. The network system thus includes the Kuramoto model as a special case, and serves as a physical realization of the (generalized) Kuramoto model, which exhibits collective synchronization in the proper regime. Here synchronization represents global phase coherence, which in turn corresponds to the superconducting state of the system. The synchronization-desynchronization transition can then be identified with the superconducting-normal transition. We have also considered the system driven by alternating currents, and examined the effects of such periodic driving, to reveal periodic synchronization. In particular, the role of the phase delay in the driving currents has been investigated, and it has been shown that transitions between states with different types of synchronization can be induced by adjusting the phase delay.

We finally consider the possibility of glassy behavior in this driven system. The system without driving currents has been shown to display glassiness in the presence of a transverse magnetic field.[6] In the presence of an applied magnetic field, the phase difference in the Josephson term is replaced by the gauge-invariant one: $\phi_i-\phi_j-A_{ij}$, where A_{ij} is the line integral of the vector potential from the ith wire to the jth one. In the strong-field limit, the bond angle A_{ij} takes rapidly fluctuating values depending on i and j; this suggests to treat A_{ij}'s essentially as quenched random variables, and to regard the system (without driving currents) as a realization of the XY gauge glass. On the other hand, in our system, driven by randomly distributed currents, there already exists quenched disorder associated with driving currents, which raises the interesting possibility of glassiness in the absence of a magnetic field. To examine such possibility, we introduce the glass order parameter q, which corresponds to the Edwards-Anderson order parameter in the spin system:

$$q\equiv\langle\langle|\langle e^{i\phi}\rangle|^2\rangle\rangle, \tag{19}$$

where $\langle\langle\cdots\rangle\rangle$ denotes the quenched average over the distribution of ω, and $\langle\cdots\rangle$ stands for the thermal average. For simplicity, we consider only the case that the phases of the ith horizontal and vertical wires are locked to each other, which amounts to the Kuramoto model. The corresponding Fokker-Planck equation can be written in terms of the single-wire probability density $P(\phi,t)$:

$$\frac{\partial P}{\partial t}=-\left[\frac{\partial}{\partial\phi}D_v-\Gamma\frac{\partial^2}{\partial\phi^2}\right]P, \tag{20}$$

where the drift coefficient D_v is given by

$$D_v=\omega-K\Delta\ \sin(\phi-\theta)$$

with the order parameter $\Delta\exp(i\theta)\equiv(1/N)\Sigma_j\exp(i\phi_j)$. The stationary solution $P^{(0)}$ of Eq. (20), which has been obtained in Ref. 15, allows us to compute Δ and q via the equations

$$\Delta e^{i\theta}=\int d\omega\ \ g(\omega)\int_0^{2\pi}d\phi\ \ P^{(0)}(\phi,\omega,\theta)e^{i\phi},$$

$$q=\int d\omega g(\omega)\left|\int_0^{2\pi}d\phi\ \ P^{(0)}(\phi,\omega,\theta)e^{i\phi}\right|^2.$$

These equations show that the glass order parameter q can have a nonzero value only if Δ is not zero. Accordingly, despite the quenched disorder associated with driving currents, there does not exist a glass phase. This is to be compared with the results of Ref. 6, which proposed glassiness induced by an applied magnetic field (without quenched disorder). The absence of glassiness in our system may be understood as follows: The effective Hamiltonian of the system can be written in the form

$$H=-\frac{K}{N}\sum_{i,j}\ \cos(\phi_i-\phi_j)-\sum_i\ \omega_i\phi_i \tag{21}$$

in a manner similar to that leading to Eq. (9). Provided that the driving is not too strong, the washboard-type potential in Eq. (21) can be transformed into the form $\cos(\phi_i-\phi_j-A_{ij})$, where the bond angle A_{ij} is related with the driving currents via $A_{ij}=(\omega_i-\omega_j)/K$. Although the bond angle here possesses quenched randomness arising from the randomly distributed driving currents, it can be gauged away via the simple transformation $\phi_i-\omega_i/K\to\phi_i$. In particular, it does not give rise to any frustration, which is believed to be essential in glassiness. It is thus concluded that an applied magnetic field is necessary for inducing frustration and consequently, for exhibiting glassy synchronization.

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