Current-carrying states in superconducting multilayers with Josephson interlayer coupling for temperatures close to T_{c0}: A microscopic theory

Sergey V. Kuplevakhsky
Department of Physics, Kharkov State University, 310077 Kharkov, Ukraine

Sergey V. Naydenov
Institute for Single Crystals, 310164 Kharkov, Ukraine

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We present a complete, self-consistent, microscopic description of current-carrying states in all sorts of superconducting multilayers with interlayer Josephson coupling near the bulk critical temperature, T_{c0}: superconductor-insulator (SI) superlattices with or without intrabarrier exchange interactions and nonmagnetic impurities inside superconducting (S) layers, pure structures with point-contact-type interlayer coupling, superconductor–normal-metal (SN) superlattices with an arbitrary concentration of nonmagnetic impurities, and SN superlattices in the dirty limit with paramagnetic impurities inside N barriers. We have obtained closed analytical expressions for the Josephson current as a function of an S layer thickness, a. For all these systems drastic deviations from a single-junction case were found: a reduction of the critical Josephson current j_c for pure SI superlattices with $a \leqslant \xi_0$, nontrivial current-phase dependence for multilayers with point-contact-type coupling and $a \leqslant \xi_0$, and nontrivial temperature dependence of j_c for SN superlattices. Mathematically, our approach is based solely on the use of a microscopic free-energy functional. For $a \gg \xi_0$, we reduce this functional to a Ginzburg-Landau-type functional with an extra term accounting for the interface free energy. For SI superlattices, in an appropriate limit this latter reduces to a Lawrence-Doniach-type functional with microscopically defined coefficients. [S0163-1829(97)01929-2]

I. INTRODUCTION

Spatially periodic structures with alternating layers of a superconducting material and a nonsuperconducting material have long been subject to extensive experimental and theoretical studies in the physics of superconductivity.[1] Renewed interest in these systems has recently been stimulated, on the one hand, by the fabrication of artificial high-quality stacks of Josephson junctions[2] and, on the other hand, by the discovery of the intrinsic Josephson effect in high-T_c superconductors.[3]

With regard to theory, the main efforts were concentrated on the calculation of the superconducting transition temperature, quasiparticle excitations, and the effect of magnetic fields. By contrast, surprisingly little attention was paid to the problem of current-carrying states. Nevertheless, for multilayers with Josephson coupling, there are at least two serious reasons to expect deviations from single-junction behavior, resulting in strong dependence of the supercurrent on a superconducting (S) layer thickness a. First, the transition temperature of a multilayer T_c in the current-carrying state is always lower than the bulk transition temperature, T_{c0}, due to the pair-breaking effect of the supercurrent itself. As a consequence, combined with the influence of other pair-breaking factors [proximity effect in superlattices with normal-metal (N) barriers, intrabarrier exchange interactions in superlattices with insulating (I) barriers, etc] the second-order phase transition to the normal state can be induced at a certain critical S layer thickness a_c. Second, for pure systems with small a ($a \leqslant \xi_0$, where ξ_0 is the microscopic coherence length) nonlocal character of the supercurrent must come into play. These conjectures are supported by earlier theoretical results.

Thus, the existence of a_c was established on the basis of the macroscopic Ginzburg-Landau (GL) equation[4] with a phenomenological periodic δ-function potential.[5] For small a, nontrivial current-phase dependence was predicted in the framework of the transfer-Hamiltonian method.[6] Both these effects were also found in numerical studies of a microscopic Kronig-Penney model of an SN superlattice at $T \ll T_{c0}$.[7]

The primary objective of this paper is to investigate the influence of finite a on current-carrying states in superconducting multilayers with interlayer Josephson coupling in full detail. We restrict ourselves to temperatures close to the bulk transition temperature, T_{c0}, where complete self-consistency can be achieved and closed analytical expressions can be obtained. All kinds of systems are considered on an equal footing: SI superlattices with or without intrabarrier exchange interactions and nonmagnetic impurities inside S layers, pure structures with point-contact-type interlayer coupling, SN superlattices with an arbitrary concentration of nonmagnetic impurities, and SN superlattices in the dirty limit with paramagnetic impurities inside N barriers. To attain our goals, we develop a rigorous, self-contained, fully microscopic approach, based solely on the use of a microscopic free-energy functional, which is derived by means of field-theoretical methods from a second-quantized BCS-type Hamiltonian[8] in Appendix A.

In Sec. II, we describe the general mathematical formalism and derive the principal equations of our theory. We show that, concerning the S layer thickness, two major regimes can be discerned. For small $a \leqslant \xi_0$ (the mesoscopic regime, accessible only to SI superlattices and structures with point-contact-type coupling), the description can be achieved only on the basis of an exact Green's function, which we obtain in Appendix B. For $a \gg \xi_0$ (the Ginzburg-Landau regime), considerable simplifications arise due to local character of the theory. To treat this regime correctly, we derive (Appendix C), from the microscopic free-energy functional, a GL-type functional with an extra term accounting for the interface free energy. For SI superlattices, we also show (Appendix D) that in an appropriate limit this GL functional with the interface term reduces to a Lawrence-Doniach-type[9] (LD) free-energy functional.

In Sec. III, we obtain closed analytical expressions for the Josephson current in pure SI superlattices and multilayers with point-contact-type coupling valid in the whole region $p_0^{-1} \ll a \leqslant \infty$ (p_0 for the Fermi momentum). We show that in the mesoscopic regime the effect of nonlocality manifests itself in a drastic reduction of the critical Josephson current in SI superlattices, while multilayers with point-contact-type coupling are characterized by nontrivial current-phase dependence.

In Sec. IV, we discuss the GL regime for SN and SI superlattices in great detail. In the case of SN superlattices, we obtain a complete analytical solution for the critical Josephson current j_c, valid in the whole region $\pi\tilde{\zeta}(T) \equiv a_c \leqslant a \leqslant \infty$ [$\tilde{\zeta}(T)$ for the GL coherence length in the presence of impurities]. We show that temperature dependence of j_c in the immediate vicinity of T_c is characterized by nontrivial $(T_c - T)$ behavior, which changes over to $(T_c - T)^2$ at lower T. In the case of SI superlattices, we explicitly include the effect of intrabarrier exchange interactions and nonmagnetic impurities. For $\tilde{\zeta}(T) \ll a$, and in the LD limit, we perform calculations of the supercurrent to second order in tunneling probabilities. Some peculiar features of the LD limit are also discussed.

II. GENERAL FORMALISM

The following models and notations will be used throughout this paper. We consider superlattices composed of alternating superconducting (s-wave-type) and nonsuperconducting normal-metal and insulating barriers. The barrier interfaces are normal to the x axis of the xyz coordinate system. The barriers themselves are supposed to possess bilateral symmetry with a mirror plane normal to the x axis. The origin of the coordinate system is chosen in the plane of the symmetry of one of the barriers. The total length of the system is $2L = Nc \gg \zeta(T)$, where $\zeta(T)$ is the GL coherence length, $c = a + d$, with a and d being an S layer and barrier thicknesses, respectively. The diameter of the cross section of the system is taken to be much less than the GL penetration depth, so that the effect of the vector potential in the absence of externally applied magnetic fields can be neglected.[10] In SN superlattices, all the normal-state properties are supposed to be the same. (Except for one case in Sec. IV, where we consider paramagnetic impurities inside the N layers.) For the electron-electron coupling constant we assume the model

$$g(x) = \begin{cases} -|g|, & \text{inside the S layers,} \\ \epsilon \to -0, & \text{inside the barriers.} \end{cases} \tag{1}$$

(The negative infinitesimal quantity ϵ is introduced for mathematical convenience only and will be set equal to zero after transition to the mean-field approximation.)

Under these conditions, the system can be completely described by a microscopic free-energy functional in the form

$$\Omega[F,F^*] = S\int_{-L}^{+L} dx_1 \Bigg[-g(x_1)|F(x_1)|^2 + \frac{7\zeta(3)N(0)|g(x_1)|^4|F(x_1)|^4}{16\pi^2 T_{c0}^2} \Bigg] - S\int_{-L}^{+L} dx_1 \times \int_{-L}^{+L} dx_2 g(x_1) g(x_2) K(x_1,x_2) F(x_1) F^*(x_2). \tag{2}$$

(A field-theoretical derivation is sketched in Appendix A.) Here S is the area of the cross section of the system, T_{c0} is the bulk transition temperature, $\zeta(m)$ is the Riemann zeta function,[11] $N(0) = mp_0/2\pi^2$ is the one-spin density of states at the Fermi level (with p_0 being the Fermi momentum, and $\hbar = c = 1$). The integral kernel

$$K(x_1,x_2) = \frac{1}{2S}\left\langle \int d\boldsymbol{\rho}_1 \int d\boldsymbol{\rho}_2 \mathrm{tr}\Bigg[T\sum_{\omega} \hat{G}_{\omega}(x_1\boldsymbol{\rho}_1, x_2\boldsymbol{\rho}_2) \times \hat{\sigma}_2 \hat{G}^t_{-\omega}(x_2\boldsymbol{\rho}_2, x_1\boldsymbol{\rho}_1)\hat{\sigma}_2 \Bigg] \right\rangle, \tag{3}$$

where $\boldsymbol{\rho} = (y,z)$, and $\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli matrix in the spin space, is expressed via the matrix Matsubara Green's function in the normal state, obeying the equation

$$\left[i\omega + E_F + \frac{1}{2m}\frac{d^2}{d\mathbf{r}_1^2} - V_{\mathrm{imp}}(\mathbf{r}_1) - \hat{V}_b(\mathbf{r}_1) \right] \hat{G}_{\omega}(\mathbf{r}_1,\mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{4}$$

In Eq. (4), $\omega = \pi T(2n+1)$ (n is an integer), $\mathbf{r}_1 = (x_1, \rho_1)$, $V_{\mathrm{imp}}(\mathbf{r}_1)$ is the impurity potential in the S layers, and $\hat{V}_b(\mathbf{r}_1)$ is the barrier potential [the accent (ˆ) denotes a nontrivial matrix structure in the spin space]. The upper index (^{t}) in Eq. (3) means transposition in the spin space, the trace is taken over the spins, and the angle brackets stand for averaging over the impurity concentration and small-scale Friedel-type oscillations on the order p_0^{-1}. [In line with the general properties of the Green's function, $K(x_1,x_2) = K(x_2,x_1)$, $K^*(x_1,x_2) = K(x_1,x_2)$, and $K(x_1 + nc, x_2 + nc) = K(x_1,x_2)$. For symmetric barriers and our choice of the coordinate system, an additional symmetry $K(-x_1, -x_2) = K(x_1,x_2)$ appears.] The complex-valued function $F = |F|e^{i\varphi}$ is the amplitude-of-condensation field. [In equilibrium, $F = \langle \psi_\uparrow \psi_\downarrow \rangle$.] Both $|F|$ and φ are smooth functions of x, and $|F| \neq 0$ everywhere in the interval $[-L,L]$. The latter condition guarantees that the system as a whole is in a phase-coherent state. A complete definition of $\Omega[F,F^*]$ implies that certain boundary conditions are set at $x = \pm L$. Here, we adopt the cyclic boundary condition $F(+L) = F(-L)$. This means $|F(+L)| = |F(-L)|$ and $\varphi(+L) - \varphi(-L) = 2\pi k$, with k being an integer. (In a homogeneous system, $\pi k/L$ can be identified with the momentum of a Cooper pair.[12]) It is convenient to choose the gauge in which $\varphi(+L)=-\varphi(-L)$. Furthermore, as a result of implicit gauge invariance, $\Omega[F,F^*]=\Omega[|F|,\nabla\varphi]$. Thus, in our periodic systems, $|F|$ and $\nabla\varphi$ must obey the conditions imposed by translational invariance:

$$|F(x+nc)|=|F(x)|,\quad (n=0,\pm 1,\pm 2,\ldots,), \tag{5}$$

$$\nabla\varphi(x+nc)=\nabla\varphi(x). \tag{6}$$

From the latter, it follows

$$\varphi(x+nc)=\varphi(x)+n\phi, \tag{7}$$

where ϕ is a constant (for a given k). For symmetric barriers under consideration, $|F|$ and $\nabla\varphi$ reach their extreme values at the midpoint of each S layer (at points $x=c/2+nc$, for our particular choice of the coordinate system).

Minimization of Eq. (2) with respect to F^* yields the self-consistency equation

$$F(x)+\int_{-L}^{+L}dx_1 g(x_1)K(x,x_1)F(x_1) + \frac{7\zeta(3)N(0)|g(x)|^3|F(x)|^2F(x)}{8\pi^2T_{c0}^2}=0. \tag{8}$$

The density of the supercurrent is determined via the functional derivative of Eq. (2) with respect to $\nabla\varphi$:

$$j(x)=2e\left\{\frac{\delta\Omega[|F|,\nabla\varphi]}{\delta\nabla\varphi(x)}\right\}_0 = -2e\int_{-L}^{x}dx_1\int_{x}^{+L}dx_2 g(x_1)g(x_2)K(x_1,x_2) \times \mathrm{Im}[F(x_1)F^*(x_2)]_0, \tag{9}$$

where the subscripts ($_0$) denote that the equilibrium value of F given by Eq. (8) and its complex conjugate should be substituted. In Eqs. (8) and (9), one can take the limit $\epsilon=0$. Although in the following we will consider Eqs. (8) and (9) only inside the S layers, introducing the standard notation $\Delta(x)\equiv|g|F(x)$ for the pair potential, it is worth noting that these equations are equally valid inside the barrier regions as well. This point is of great theoretical significance because of the issue of charge conservation:[13] As can be easily seen, the imaginary part of Eq. (8) multiplied by F^* gives the conservation law $dj(x)/dx=0$.

Provided the kernel (3) is known, the evaluation of Eqs. (8) and (9), in principle, can supply the solution to the problem of current-carrying states for an arbitrary a in the interval $p_0^{-1}\ll a\leqslant\infty$. In the *mesoscopic* regime, for $p_0^{-1}\ll a\leqslant\xi_0$ ($\xi_0=v_0/2\pi T_{c0}$ is the BCS coherence length, $v_0=p_0/m$), only these equations are applicable. (See Sec. II for two particular examples.) But for $a\gg\xi_0$ (the *Ginzburg-Landau* regime), considerable mathematical simplifications arise due to the use of the local GL equations.[4]

As shown in Appendix C, in the GL regime Eq. (2) reduces to

$$\Omega_{\mathrm{GL}}[\Delta_n^*(r),\Delta_n(r);\Delta_n^*(+0),\Delta_n^*(a-0)]=SN(0)\sum_{n=-N/2}^{+N/2}\left\{\int_{+0}^{a-0}dr\left[-\tau|\Delta_n(r)|^2+\frac{7\zeta(3)}{12}\xi_0^2\chi(\xi_0/l)\left|\frac{d\Delta_n(r)}{dr}\right|^2 + \frac{7\zeta(3)}{16\pi^2T_{c0}^2}|\Delta_n(r)|^4\right]+\frac{7\zeta(3)}{24}\xi_0^2\chi(\xi_0/l)\{(\tilde\alpha+\tilde\beta)[|\Delta_n(+0)|^2+|\Delta_n(a-0)|^2] + (\tilde\beta-\tilde\alpha)[\Delta_n^*(+0)\Delta_{n-1}(a-0)+\Delta_n(+0)\Delta_{n-1}^*(a-0)]\}\right\}, \tag{10}$$

where $\tau=1-T/T_{c0}$, $\Delta_n(r)\equiv\Delta(d/2+nc+r)$ is the GL order parameter in the nth S layer whose value at the interfaces is given by $\Delta_n(+0)$ and $\Delta_n(a-0)$ [with ± 0 being the distances $x\simeq\pm\xi_0$, small on the GL scale $\zeta(T)$], $\tilde\alpha$ and $\tilde\beta$ are microscopic parameters, and $\chi(\xi_0/l)$ is the impurity factor (l is the electron mean free path) given by[14]

$$\chi(\xi_0/l)=\frac{8}{7\zeta(3)}\sum_{n=0}^{+\infty}(2n+1)^{-2}(2n+1+\xi_0/l)^{-1}. \tag{11}$$

The first square-bracketed term in Eq. (10) is the usual GL bulk free energy.[15] The second square-bracketed term arises due to the loss of the condensation energy at the barrier interfaces, as a result of a local depression of the pair potential. This form of the GL free-energy interface term covers all possible types of symmetric barriers (for asymmetric barriers, an additional microscopic constant, $\tilde\gamma$, would appear) and guarantees the conservation of supercurrent (see below). The constant $\tilde\beta$ accounts for pair breaking due to intrabarrier exchange interactions and the proximity effect (in SN superlattices). (As can be easily seen, in the absence of supercurrent, the constant $\tilde\alpha$ drops out.) Notice, that at a phenomenological level, with three undefined constants, the GL free-energy functional with an interface term was introduced by Andreev in his discussion of a single planar defect.[16] Beginning with De Gennes,[17,12] the constants $\tilde\alpha$ and $\tilde\beta$ were essentially calculated for all the situations of interest here by many authors,[18–22] without any reference to the GL free energy, in the context of establishing microscopic boundary conditions to the mean-field GL equations for a single-barrier Josephson junction. Actual values of $\tilde\alpha$ and $\tilde\beta$ for different types of nonsuperconducting symmetric barriers will be given in Sec. II. At this stage, the following two things are worth noting. First, as can be seen from Eq. (10), the dimensionless interlayer coupling parameter is $(\tilde{\alpha}-\tilde{\beta})\xi_0$. In SN superlattices with nonferromagnetic N barriers, the limit of weak coupling corresponds to $d \gg \xi_0$. In this limit, only the combination $(\tilde{\alpha}-\tilde{\beta})\xi_0$ is small, being proportional to $\exp(-d/\xi)$, with $\xi \leqslant \xi_0$. On the contrary, in SI superlattices in the weak-coupling limit, both $\tilde{\alpha}\xi_0$ and $\tilde{\beta}\xi_0$, are small, being proportional to the tunneling probabilities. (The latter, $\tilde{\beta}\xi_0$, is proportional to the exchange part of the tunneling probability.) Second, in SI superlattices one has

$$\tilde{\alpha} \equiv \chi^{-1}(\xi_0/l)\alpha, \quad \tilde{\beta} \equiv \chi^{-1}(\xi_0/l)\beta, \tag{12}$$

where α and β stand for the pure limit. An immediate consequence of these identities is a cancellation of the impurity factor $\chi(\xi_0/l)$ in the interface term in Eq. (10).

By minimizing Eq. (10) with respect to $\Delta_n^*(r)$, we arrive at the set of GL equations

$$\tau\Delta_n(r) + \frac{7\zeta(3)}{12}\,\xi_0^2\chi(\xi_0/l)\,\frac{d^2\Delta_n(r)}{dr^2} - \frac{7\zeta(3)}{8\pi^2T_{c0}^2}\,\Delta_n(r)|\Delta_n(r)|^2 = 0. \tag{13}$$

These equations are coupled through the boundary conditions

$$\frac{d\Delta_n}{dr}(+0) = \frac{1}{2}\,[(\tilde{\alpha}+\tilde{\beta})\Delta_n(+0) - (\tilde{\alpha}-\tilde{\beta})\Delta_{n-1}(a-0)], \tag{14}$$

$$\frac{d\Delta_{n-1}}{dr}(a-0) = -\frac{1}{2}\,[(\tilde{\alpha}+\tilde{\beta})\Delta_{n-1}(a-0) - (\tilde{\alpha}-\tilde{\beta})\Delta_n(+0)]$$

that are also obtained from Eq. (10) by means of minimization with respect to $\Delta_n^*(+0)$ and $\Delta_n^*(a-0)$. Accordingly, the supercurrent density can be derived by analogy with Eq. (9):

$$j_n(r) = i\,\frac{7\zeta(3)}{6}\,eN(0)\xi_0^2\chi(\xi_0/l) \times \left[\Delta_n(r)\,\frac{d\Delta_n^*(r)}{dr} - \Delta_n^*(r)\,\frac{d\Delta_n(r)}{dr}\right]. \tag{15}$$

The application of the boundary conditions (14) demonstrates the conservation of the supercurrent density at the interfaces: $j_{n-1}(a-0) = j_n(+0)$.

Following Ref. 8, it is reasonable to isolate the phase shift of the pair potential at the interfaces and, using the symmetry relations (5)–(7), introduce the representations

$$\Delta_n(r) = \Delta_\infty f(r)\exp[i2m\chi_n(r) + i\phi/2],$$
$$\Delta_{n-1}(r) = \Delta_\infty f(r)\exp[i2m\chi_{n-1}(r) - i\phi/2], \tag{16}$$

where $\Delta_\infty = \sqrt{8\pi^2T_{c0}^2/7\zeta(3)(1-T/T_{c0})}$ is the bulk value of the gap parameter, $f(r)$ is a real function $[0 \leqslant f(r) \leqslant 1]$, $\chi_n(r) \equiv \chi(d/2+nc+r)$, $d\chi_n/dr \equiv v_s$ is the "superfluid velocity" in the S layers. In terms of the quantities f and v_s, Eqs. (13)–(15) become

$$[1 - 4m^2\tilde{\zeta}^2(T)v_s^2(r)]f(r) + \tilde{\zeta}^2(T)\,\frac{d^2f(r)}{dr^2} - f^3(r) = 0, \tag{17}$$

$$j(r) = \frac{14\zeta(3)}{3}\,emN(0)\xi_0^2\chi(\xi_0/l)\Delta_\infty^2 f^2(r)v_s(r), \tag{18}$$

$$\frac{d}{dr}\,[f^2(r)v_s(r)] = 0, \tag{19}$$

$$f_- = f_+, \tag{20}$$

$$-f'_- = f'_+ = \frac{1}{2}\,[\tilde{\alpha}+\tilde{\beta} - (\tilde{\alpha}-\tilde{\beta})\cos\phi]f_+, \tag{21}$$

$$v_{s-} = v_{s+} = \frac{1}{4m}\,(\tilde{\alpha}-\tilde{\beta})\sin, \tag{22}$$

where

$$\tilde{\zeta}(T) \equiv \chi^{1/2}(\xi_0/l)\zeta(T) = \chi^{1/2}(\xi_0/l)\xi_0\sqrt{7\zeta(3)/12(1-T/T_{c0})^{-1}}$$

is the GL coherence length in the presence of impurities. In the boundary conditions, $f' \equiv df/dr$, the subindices ($_-$) and ($_+$) denote $r = a-0$ and $r = +0$, respectively. These boundary conditions look exactly the same as in the case of a single-barrier problem.[19,20] The effect of a finite thickness of the S layers enters only by virtue of the symmetry conditions described above:

$$\frac{df}{dr}(a/2) = 0, \quad \frac{d^2f}{dr^2}(a/2) < 0. \tag{23}$$

[A corollary of these and Eq. (19) are, of course, $v'_s(a/2) = 0$ and $v''_s(a/2) > 0$.] To finish with the discussion of the boundary conditions, we observe that Eq. (22) requires that $\tilde{\alpha} > \tilde{\beta}$, which is always the case for the systems under consideration here.[23]

At the second-order-phase-transition point, Eq. (17) can be linearized. Noticing that the effect of the superfluid velocity is of second order in the weak-coupling parameter $(\tilde{\alpha}-\tilde{\beta})\xi_0$ and can be neglected, making use of the boundary conditions, we arrive at the relation:

$$\left[\tilde{\zeta}^{-1}(T)\tan\frac{a}{2\tilde{\zeta}(T)}\right]_c = \frac{1}{2}\,[\tilde{\alpha}+\tilde{\beta} - (\tilde{\alpha}-\tilde{\beta})\cos\phi]. \tag{24}$$

For given values of the pair-breaking parameters $\tilde{\beta}$ and ϕ, this relation has a double-edged meaning. First, for a fixed a, it determines the transition temperature T_c. Second, for a fixed $T < T_c$, it can be regarded as a definition of the critical thickness, a_c, below which the system remains in the normal state.

For SN superlattices, $\tilde{\alpha}$ and $\tilde{\beta}$ are of order ξ_0^{-1} or larger, giving $\tilde{\alpha}\tilde{\zeta}(T) \gg 1$, $\tilde{\alpha}a \gg 1$. Thus, the transition temperature is

$$T_c = T_{c0} - \frac{7\zeta(3)\pi^2}{12} T_{c0}\chi(\xi_0/l)\xi_0^2 a^{-2}, \tag{25}$$

and the critical thickness is

$$a_c = \pi\tilde{\zeta}(T). \tag{26}$$

For SI superlattices, $\tilde{\alpha}$ and $\tilde{\beta}$ are proportional to ξ_0^{-1} times the tunneling probabilities which can be made arbitrarily small. [In the following, we shall assume at least $\tilde{\alpha}\tilde{\zeta}(T) \ll 1$.] Thus, two different limiting cases can be realized. For $a \gg \tilde{\alpha}^{-1}$, $\tilde{\alpha}\tilde{\zeta}(T_c) \gg 1$, the transition temperature being given by formula (25).

For $a \ll \tilde{\zeta}(T)$, $\tilde{\alpha}\tilde{\zeta}(T_c) \ll 1$, with the transition temperature

$$T_c = T_{c0} - \frac{7\zeta(3)}{12}[\alpha+\beta-(\alpha-\beta)\cos\phi]T_{c0}\xi_0^2 a^{-1}, \tag{27}$$

and the critical thickness

$$a_c = [\alpha+\beta-(\alpha-\beta)\cos\phi]\zeta^2(T). \tag{28}$$

The absence of the impurity factor is to be noticed in these equations. This is an obvious consequence of Anderson's theorem:[24] Nonmagnetic impurities do not affect thermodynamic properties of homogeneous superconductors. (In this *Lawrence-Doniach* limit,[9] the complex pair potential is constant in each S layer.) In terms of the free-energy functional, Ω_{GL} of Eq. (10) reduces to a LD-type functional in the absence of external magnetic fields:

$$\Omega_{\mathrm{LD}}[|\Delta|;\phi] = NaSN(0)\left\{-\tau|\Delta|^2 + \frac{7\zeta(3)}{16\pi^2T_{c0}^2}|\Delta|^4 + \frac{7\zeta(3)}{12}\frac{\xi_0^2}{a}\left|\Delta\right|^2[\alpha+\beta-(\alpha-\beta)\cos\phi]\right\}. \tag{29}$$

Phenomenological LD functionals are in wide use for the description of different aspects of the effect of magnetic fields in superconducting superlattices and layered superconductors.[25] The details of a rigorous mathematical derivation of Eq. (29) from Eq. (10) are shown in Appendix D. In Sec. IV, we obtain solutions to the principal equations of the theory in the GL regime and discuss the LD limit in more detail.

III. MESOSCOPIC REGIME

A. SI superlattices: A reduction of the critical Josephson current

Our aim here is to obtain a microscopic formula for the Josephson current in a superconducting SI superlattice in the pure limit, valid in the whole range of thicknesses of the superconducting layers $p_0^{-1} \ll a \leq \infty$. To simplify the analysis, we restrict ourselves to the evaluation of the supercurrent to first order in the tunneling probability and do not consider the effect of exchange interactions inside the barriers. The latter allows us to suppress in what follows inessential spin indices.

As a starting point, for the description of the insulating barriers we adopt a model pseudopotential of the form

$$V_b(x) = V\sum_{n=-\infty}^{+\infty}\delta(x-na), \quad V \equiv Ud > 0. \tag{30}$$

Accordingly, the electron-electron coupling constant is $g(x) = -|g| \equiv \mathrm{const}$ everywhere in the interval $[-L,L]$, and we introduce the notation $\Delta(x) \equiv |g|F(x)$. This model was previously employed by Kuplevakhsky Fal'ko in their consideration of Bloch-type subgap excitations in an SI superlattice induced by supercurrent.[26] To avoid possible misunderstanding, we emphasize that, as in the case of a single SIS junction,[27] the use of a δ function for the description of repulsive potentials in microscopic equations is physically fully justified as soon as the quasiclassical approximation can be employed and we are not concerned with distances on the order of p_0^{-1}.[28,29] Equation (3) now becomes

$$K(x_1,x_2) = \pi N(0)v_0 T\sum_\omega \int_0^1 dt\; t\langle G_\omega(x_1,x_2)G_{-\omega}(x_2,x_1)\rangle, \tag{31}$$

where the angle brackets denote averaging over Friedel oscillations on the order of p_0^{-1} only. The Fourier transform of the Green's function in the coordinates y, z, taken at the Fermi level, is governed by the equation

$$\left[i\omega + E_F t^2 + \frac{1}{2m}\frac{d^2}{dx_1^2} - V\sum_{n=-\infty}^{+\infty}\delta(x_1-na)\right]G_\omega(x_1,x_2) = \delta(x_1-x_2), \tag{32}$$

with $t \equiv \cos\theta$ being the angle of incidence at the interface. In Appendix B we have obtained an exact solution to Eq. (32). It has the form

$$G_\omega(x_1,x_2) = -\frac{m}{i\lambda}\exp[-i\lambda_1|n_1-n_2|a]\left\{\delta_{n_1,n_2}\exp[-i\lambda|r_1-r_2|] + (1-\delta_{n_1,n_2})\exp[-i\,\mathrm{sgn}(n_1-n_2)\lambda(r_1-r_2)] + \frac{(\sin\lambda_2 a - \sin\lambda_1 a)\cos[\lambda(r_1-r_2)] + (mV/\lambda)\cos[\lambda(a-r_1-r_2)]}{\sin\lambda_1 a}\right\}, \tag{33}$$

$$\begin{cases} x_{1,2} = n_{1,2}a + r_{1,2}, \\ 0 \leq r_{1,2} < a, \\ n_{1,2} = 0, \pm 1, \pm 2, \ldots, \end{cases} \tag{34}$$

where $\lambda=-\mathrm{sgn}\omega\sqrt{E_F t^2+i\omega}$, the quantities λ_1,λ_2 are defined by

$$\cos\lambda_1 a=\cos\lambda a+\frac{mV}{\lambda}\sin\lambda a,$$
$$\sin\lambda_2 a=\sin\lambda a-\frac{mV}{\lambda}\cos\lambda a, \tag{35}$$

and the prescription $\lambda_{1,2}\to\lambda$, for $V\to 0$. This function is explicitly translationally invariant with a period a. For $V=0$, it goes over into the free-electron Green's function (B2). In the case of impenetrable barriers ($V=\infty$), Eqs. (33)–(35) reduce to a set of Green's functions describing flat metallic films of thickness a in contact with vacuum.[30] On the other hand, choose consecutively: (i) $n_{1,2}=0$, $r_{1,2}<a/2$; (ii) $n_{1,2}=-1$, $a-r_{1,2}<a/2$; (iii) $n_1=0$, $r_1<a/2$, $n_2=-1$, $a-r_2<a/2$, or $n_2=0$, $r_2<a/2$, $n_1=-1$, $a-r_1<a/2$. After proceeding to the limit $a\to\infty$, one obtains the Green's function of a system with a single repulsive barrier at $x=0$.[27]

With Eqs. (33)–(35) now in hand, assuming $a\gg p_0^{-1}$, it is straightforward to compute the kernel (31) in the quasiclassical approximation. Expanding

$$\lambda\approx -p_0|t|\mathrm{sgn}\omega-i\frac{|\omega|}{v_0|t|},\qquad |t|\gg\sqrt{T_{c0}/E_F}, \tag{36}$$

and retaining only the leading terms, we get

$$K(x_1,x_2)=\frac{\pi N(0)}{v_0}T\sum_\omega\int_0^1\frac{dt}{t}\left[\frac{T(t)}{\cosh(2|\omega|a/v_0t)+\sqrt{\cosh^2(2|\omega|a/v_0t)-T^2(t)}}\right]^{|n_1-n_2|}$$
$$\times\left\{\delta_{n_1,n_2}\exp\left[-\frac{2|\omega|}{v_0t}|r_1-r_2|\right]+(1-\delta_{n_1,n_2})\exp\left[-\frac{2|\omega|}{v_0t}(r_1-r_2)\mathrm{sgn}(n_1-n_2)\right]\right.$$
$$\left.+[1-T(t)]\frac{\exp[-2|\omega|a/v_0t]\cosh[2|\omega|/v_0t(r_1-r_2)]+\cosh[2|\omega|/v_0t(a-r_1-r_2)]}{\sinh(2|\omega|a/v_0t)}\right\}, \tag{37}$$

where $T(t)=v_0^2t^2(v_0^2t^2+V^2)^{-1}$ is the tunneling probability. It is important to observe that the quasiclassical expression (37) does not depend on a concrete functional form of $T(t)$ and should also hold for any symmetric repulsive barrier with $d\ll\xi_0$. As expected, Eq. (37) assures correct transition to all major limiting cases. Setting $T(t)\equiv 1$, we arrive at the free-electron kernel[12]

$$K(x_1-x_2)=\int_0^1 dt\,K_t(x_1-x_2)\equiv\frac{\pi N(0)}{v_0}T\sum_\omega\int_0^1\frac{dt}{t}\exp\left[-\frac{2|\omega|}{v_0t}|x_1-x_2|\right]. \tag{38}$$

In the opposite limit $T(t)\equiv 0$ ($V=\infty$) and $n_1=n_2=n$, one obtains a set of kernels describing N isolated metallic films in contact with vacuum:

$$K^{(0)}(x_1,x_2)=T\sum_\omega K_\omega^{(0)}(x_1,x_2)\equiv\frac{\pi N(0)}{v_0}T\sum_\omega\int_0^1\frac{dt}{t}\left[\exp\left[-\frac{2|\omega|}{v_0t}|x_1-x_2|\right]\right.$$
$$\left.+\frac{\exp[-2|\omega|a/v_0t]\cosh[2|\omega|/v_0t(x_1-x_2)]+\cosh\{2|\omega|/v_0t[x_1+x_2-(2n+1)a]\}}{\sinh(2|\omega|a/v_0t)}\right]. \tag{39}$$

Now consider: (i) $n_{1,2}=0$, $r_{1,2}<a/2$; (ii) $n_{1,2}=-1$, $a-r_{1,2}<a/2$; (iii) $n_1=0$, $r_1<a/2$, $n_2=-1$, $a-r_2<a/2$, or $n_2=0$, $r_2<a/2$, $n_1=-1$, $a-r_1<a/2$. Rendering $a\gg\xi_0$, we obtain the single-barrier kernel[31]

$$K(x_1,x_2)=\int_0^1 dt[\{K_t(x_1-x_2)+[1-T(t)]K_t(x_1+x_2)\}[\Theta(x_1)\Theta(x_2)+\Theta(-x_1)\Theta(-x_2)]$$
$$+T(t)K_t(x_1-x_2)[\Theta(x_1)\Theta(-x_2)+\Theta(-x_1)\Theta(x_2)]], \tag{40}$$

$$\Theta(x)=\begin{cases}1, & x>0;\\ 0, & x<0.\end{cases} \tag{41}$$

Concerning the supercurrent density for $T(1)\ll 1$, it is advantageous to evaluate Eq. (9) at one of the barriers (say, at $x=0$):

$$j(0)=-2e\int_{-L}^0 dx_1\int_0^{+L}dx_2K(x_1,x_2)\mathrm{Im}[\Delta(x_1)\Delta^*(x_2)]_0, \tag{42}$$

As follows from Eq. (37), the integrand of the integral over t in this case contains as a factor powers of $T(t)$. In first-order calculations, one should include only contributions of

$|n_1 - n_2| = 0,1$ and take the pair potential in zero order in weak coupling $[T(t) \equiv 0]$. This leads us to investigate Eq. (8) with the kernel $K^{(0)}$ for $n_1 = n_2 = n$:

$$\Delta_n(r) = |g| \int_0^a dr_1 K^{(0)}(r,r_1)\Delta_n(r_1) - \frac{7\zeta(3)N(0)|g||\Delta_n(r)|^2\Delta_n(r)}{8\pi^2 T_{c0}^2}. \tag{43}$$

Dropping the nonlinear term in Eq. (43), we obtain the equation for the transition temperature

$$\Delta_n(r) = |g| T \sum_\omega \int_0^a dr_1 K_\omega^{(0)}(r,r_1)\Delta_n(r_1). \tag{44}$$

Setting $T = T_{c0}$ in Eq. (39), we easily establish the property

$$|g| T_{c0} \sum_\omega \int_0^a dr_1 K_\omega^{(0)}(r,r_1) = 1, \tag{45}$$

where the definition of the bulk transition temperature

$$\pi |g| N(0) T_{c0} \sum_\omega \frac{1}{|\omega|} = 1 \tag{46}$$

was used. [In Eq. (46) a cutoff at the Debye frequency ω_D is implied.] Hence, the largest eigenvalue of Eq. (44) is $T_c = T_{c0}$ with the eigenvector $\Delta_n^{(0)} = \text{const}$. Inserting $\Delta_n^{(0)} = \text{const}$ into Eq. (43) and making use of the expansion

$$\pi |g| N(0) T \sum_\omega \frac{1}{|\omega|} \approx 1 + |g| N(0)\left(1 - \frac{T}{T_{c0}}\right), \tag{47}$$

we get $|\Delta_n^{(0)}| = \Delta_\infty$, where Δ_∞ is the bulk value of the gap parameter [see Eq. (16)]. Thus the correct zero-order pair potential, satisfying the conditions of Sec. II, is

$$\Delta^{(0)}(x) = \Delta_\infty \sum_{n=-\infty}^{+\infty} e^{i(1/2+n)\phi}\Theta(x - na)\Theta[(n+1)a - x]. \tag{48}$$

Now let us return to Eq. (42). Substitution of the first-order approximation to Eq. (37) (with $T = T_{c0}$) together with Eq. (48) yields

$$j = \frac{emE_F\Delta_\infty^2 T_{c0}}{2\pi} \sum_\omega \frac{1}{\omega^2} \int_0^1 dt\ tT(t)\tanh^2 \frac{2|\omega|a}{v_0 t} \sin\phi. \tag{49}$$

This is the desired expression for the dc Josephson current as a function of the S layer thickness, valid in the whole region $p_0^{-1} \ll a \leq \infty$.

For $a \gg \xi_0$, Eq. (49) goes over into the well-known formula[27] for the Josephson current in a single tunnel junction

$$j = \frac{emE_F\Delta_\infty^2}{4\pi T_{c0}} \int_0^1 dt\ tT(t)\sin\phi. \tag{50}$$

By contrast, in the extreme mesoscopic regime, when $p_0^{-1} \ll a \ll \xi_0$, we discover the dramatic result

$$j = \frac{7\zeta(3)emE_F\Delta_\infty^2}{\pi^5 T_{c0}} \frac{a}{\xi_0} \int_0^1 dt\ T(t)\sin\phi. \tag{51}$$

We see that in addition to the averaged tunneling probability the Josephson current is proportional to another small parameter, a/ξ_0. The reduction of the Josephson current is a manifestation of the nonlocality of Eq. (42): In a pure infinite SIS junction the contributions to the current are ''collected'' over a spatial region $\sim \xi_0$, while in our case they are restricted to two adjacent S layers only. This conclusion becomes even more evident if we rewrite Eq. (51) as

$$j = \frac{14\zeta(3)emE_F\Delta_\infty^2}{\pi^4 T_{c0}} \int_0^1 dt\ tT(t)\left[\frac{a}{v_0 t T_{c0}^{-1}}\right]\sin\phi. \tag{52}$$

The factor $a/v_0 t T_{c0}^{-1}$ under the integral sign in Eq. (52) can be regarded as the quasiclassical probability of finding an unscattered electron with the x component of the velocity $v_0 t$ within one S layer during the characteristic time of the order of T_{c0}^{-1}.[34] Our next example below demonstrates another aspect of the nonlocality of Eq. (42) for small a.

B. Multilayers with point-contact-type interlayer coupling: Nontrivial current-phase dependence

The structures considered here is an obvious extension to multilayers of the single-point-contact model thoroughly studied in Ref. 32. Let R be the radius of the orifice at the interface between two adjacent S layers, where $p_0^{-1} \ll R \ll a, \xi_0$. As explained in Ref. 32, the expansion parameter now is R/ξ_0, and the tunneling probability may take arbitrary values $0 < T(t) \leq 1$.

As in the case of a single point contact, instead of the supercurrent density, it is convenient to compute the total current at one of the interfaces $(x=0)$: $I = S_0 j(0) \equiv \pi R^2 j(0)$. In leading order, one should set $T = T_{c0}$ in Eq. (37) and use the approximation (48), obtaining

$$I(\phi) = \sum_{k=1}^{+\infty} kI(k)\sin k\phi, \tag{53}$$

$$I(k) = \frac{8emS_0E_F\Delta_\infty^2 T_{c0}}{\pi} \sum_\omega \frac{1}{\omega^2} \int_0^1 dt\ tT(t)\left[\frac{T(t)}{\cosh(2|\omega|a/v_0 t) + \sqrt{\cosh^2(2|\omega|a/v_0 t) - T^2(t)}}\right]^k \times \sinh^2 \frac{|\omega|a}{v_0 t}\left[1 + \frac{1 - T(t)}{\sqrt{\cosh^2(2|\omega|a/v_0 t) - T^2(t)}}\right], \tag{54}$$

where $k \equiv |n_1 - n_2|$ [see Eq. (37)]. The expression of the form (53) was derived in Ref. 6 within the transfer-Hamiltonian method, but the explicit analytical expressions for the partial contributions (54) could only be calculated here on the basis of our true microscopic approach. The series in Eq. (53) can be summed up, yielding

$$I(\phi)=\frac{8emS_0E_F\Delta_\infty^2T_{c0}}{\pi}\sum_\omega \frac{1}{\omega^2}\int_0^1 dt\ t\,\frac{Z(t,\omega)[1-Z^2(t,\omega)]\sin\phi}{[1+Z^2(t,\omega)-2Z(t,\omega)\cos\phi]^2}\sinh^2\frac{|\omega|a}{v_0t}\left[1+\frac{1-T(t)}{\sqrt{\cosh^2(2|\omega|a/v_0t)-T^2(t)}}\right],$$

$$Z(t,\omega)=\frac{T(t)}{\cosh(2|\omega|a/v_0t)+\sqrt{\cosh^2(2|\omega|a/v_0t)-T^2(t)}}. \tag{55}$$

In the limit $T(1)\ll 1$, Eq. (55) reduces to Eq. (49) times S_0. For $a \gg \xi_0$, Eq. (55) formally coincides with Eq. (50) times S_0, but with an arbitrary $T(t)$.

The nontrivial current-phase dependence of Eq. (55) for large $T(t)$ and $a \leq \xi_0$ should be noted here. Previously, nontrivial current-phase dependence was derived for a single point contact at temperatures $T \ll T_{c0}$,[32] which occurred due to the current-carrying subgap state.[33] In our case, the nontrivial current-phase dependence arises as a result of already mentioned nonlocality of Eq. (42). (The contribution of current-carrying subgap states vanishes at temperatures close to T_{c0}.)

IV. GINZBURG-LANDAU REGIME

A. SN superlattices

In the case of weakly-coupled SN superlattices, according to Eq. (26), the GL regime is the only possible one near T_{c0}. For these structures, we study the dependence of the critical Josephson current j_c on a and T.

To start with, we write down expansions of $(\widetilde{\alpha}\xi_0)^{-1}$ and $(\widetilde{\beta}\xi_0)^{-1}$ to leading order in weak coupling:[18,19,21]

$$(\widetilde{\alpha}\xi_0)^{-1}\approx\widetilde{q}^{(0)}-\widetilde{q}^{(1)},\quad (\widetilde{\beta}\xi_0)^{-1}\approx\widetilde{q}^{(0)}+\widetilde{q}^{(1)}, \tag{56}$$

where $\widetilde{q}^{(0)}$ corresponds to the decoupling limit $d=\infty$, and $\widetilde{q}^{(1)} \ll \widetilde{q}^{(0)}$. In Ref. 19, expansions (56) were obtained for an arbitrary concentration of nonmagnetic impurities. As an illustration, we give here two limiting examples of major importance. In the pure limit $[l=\infty,\ \chi(\xi_0/l)=1]$,

$$(\alpha\xi_0)^{-1}\approx 0{,}64-3{,}5(\xi_0/d)\exp(-d/\xi_0),$$

$$(\beta\xi_0)^{-1}\approx 0{,}64+3{,}5(\xi_0/d)\exp(-d/\xi_0). \tag{57}$$

In the dirty limit $[l\ll\xi_0,\chi(\xi_0/l)\approx\pi^2 l/7\zeta(3)\xi_0]$,

$$(\widetilde{\alpha}\xi_0)^{-1}\approx 0{,}47\sqrt{l/\xi_0}-0{,}7\sqrt{l/\xi_0}\exp(-d\sqrt{3/\xi_0 l}),$$
$$(\widetilde{\beta}\xi_0)^{-1}\approx 0{,}47\sqrt{l/\xi_0}+0{,}7\sqrt{l/\xi_0}\exp(-d\sqrt{3/\xi_0 l}). \tag{58}$$

The calculations in the dirty limit were extended in Ref. 21 to allow for paramagnetic impurities inside the N barrier:

$$(\widetilde{\alpha}\xi_0)^{-1}\approx 1{,}73\sqrt{2\pi\tau_S T_{c0} l/\xi_0}-0{,}93\sqrt{2\pi\tau_S T_{c0} l/\xi_0}\times\exp(-d\sqrt{3/2\pi\tau_S T_{c0}\xi_0 l}),$$
$$(\widetilde{\beta}\xi_0)^{-1}\approx 1{,}73\sqrt{2\pi\tau_S T_{c0} l/\xi_0}+0{,}93\sqrt{2\pi\tau_S T_{c0} l/\xi_0}\times\exp(-d\sqrt{3/2\pi\tau_S T_{c0}\xi_0 l}), \tag{59}$$

where $\tau_S \ll T_{c0}^{-1}$ is the spin-flip scattering time.[35] [See Ref. 36 for the discussion of a basic difference between paramagnetic and ferromagnetic N barriers.]

To take full advantage of the boundary conditions (20)–(22), we evaluate Eq. (18) at $r=+0$:

$$j(+0)=\frac{7\zeta(3)}{6}\frac{\widetilde{q}^{(1)}}{[\widetilde{q}^{(0)}]^2}[f_+^{(0)}]^2 eN(0)\xi_0\chi(\xi_0/l)\Delta_\infty^2\sin\phi, \tag{60}$$

To find $f_+^{(0)}$, we discard the v_s^2 term in Eq. (17) and use the GL first integral in the form[15]

$$2\widetilde{\zeta}^2(T)\left[\frac{df(r)}{dr}\right]^2=[1-f^2(r)]^2-s^2, \tag{61}$$

where the constant s $(0\leq s\leq 1$, with $s=0$ for $a=\infty)$ is to be determined from the conditions (23). Applying Eqs. (21) and (23) in appropriate order, exploiting the smallness of the ratio $\widetilde{q}^{(0)}\xi_0/\widetilde{\zeta}(T)$, we arrive at the final expression for the critical Josephson current as a function of $a \geq a_c \equiv \pi\widetilde{\zeta}(T)$ [for $T<T_c$, with T_c given by Eq. (25)]:

$$j_c=\frac{6}{7\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)^2(1-s^2)\widetilde{q}^{(1)}, \tag{62}$$

where $n=(4/3)N(0)E_F$ $(E_F=p_0^2/2m$ being the Fermi energy) is the number of electrons per unit volume. The constant s is implicitly defined by the equation

$$(1+s)^{-1/2}K\left(\sqrt{\frac{1-s}{1+s}}\right)=\frac{a}{2\sqrt{2}\widetilde{\zeta}(T)}, \tag{63}$$

where $K(k)$ is the complete elliptic integral of the first kind.[11] Explicit analytical expressions can be obtained in limiting cases. For $a-a_c\ll a_c$, $1-s\ll 1$, and we get

$$j_c=\frac{96}{7\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)^2\left(\frac{a}{\pi\widetilde{\zeta}(T)}-1\right)\widetilde{q}^{(1)}. \tag{64}$$

In the opposite limit $a \gg a_c$, $s \ll 1$,

$$j_c=\frac{6}{7\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)^2\{1-32\exp[-\sqrt{2}a/\widetilde{\zeta}(T)]\}\widetilde{q}^{(1)}, \tag{65}$$

exhibiting smooth transition to a single SNS junction with $a=\infty$.[21]

To investigate the dependence of j_c on T for a fixed a, one should insert into Eq. (63) the definition of the transition temperature $a=\pi\widetilde{\zeta}(T_c)$ [see Eq. (25)]. For $T_c-T\ll T_{c0}-T_c$, in striking contrast to a single SNS junction, we observe

$$j_c=\frac{48}{7\zeta(3)}\frac{env_0}{p_0\xi_0T_{c0}^2}(T_{c0}-T_c)(T_c-T)\widetilde{q}^{(1)}. \tag{66}$$

At lower temperatures $T_c-T\gg T_{c0}-T_c$, the typical $(T_{c0}-T)^2$ dependence of a single SNS junction[12] is recovered:

$$j_c=\frac{6}{7\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)^2\widetilde{q}^{(1)}. \tag{67}$$

The ''crossover'' temperature (T^*) of transition from behavior (66) to behavior (67) is

$$T^*\simeq 2T_c-T_{c0}. \tag{68}$$

To conclude the discussion of SN superlattices, we point out an important role of nonmagnetic impurities [entering Eqs. (62), (63) by means of $\widetilde{q}^{(1)}$ and $\widetilde{\zeta}(T)$], as a result of strong spatial variations of the pair potential due to the proximity effect.

B. SI superlattices with magnetic tunnel barriers

For weakly coupled SI superlattices, the most general form of the microscopic coefficients $\widetilde{\alpha}$ and $\widetilde{\beta}$ is

$$\widetilde{\alpha}=\frac{3\pi^2}{7\zeta(3)}\chi^{-1}(\xi_0/l)\xi_0^{-1}\int_0^1 dt\; tT(t),$$
$$\widetilde{\beta}=\frac{6\pi^2}{7\zeta(3)}\chi^{-1}(\xi_0/l)\xi_0^{-1}\int_0^1 dt\; tT_S(t) \tag{69}$$

(see Appendix C). Here $T(t)$ is the total tunneling probability and $T_S(t)$ is the exchange part of the tunneling probability, with $T_S(1)\ll T(1)\ll 1$.

In this instance, the GL regime, besides straightforward inclusion of the effect of nonmagnetic impurities and intrabarrier exchange interactions, offers an opportunity to proceed beyond the first-order calculations of Sec. III. As should be clear from the results of Sec. II [compare Eqs. (25) and (27) for T_c], there is no universal (valid for arbitrary a) analytical high-order expansion of the supercurrent in powers of a weak-coupling parameter. So we again focus on two typical limiting cases.

For large $a\gg\widetilde{\zeta}(T)$, one can safely set $T_c\simeq T_{c0}$ and $f(a/2)\simeq 1$ [with $s\simeq 0$ in Eq. (61)]. We again apply Eqs. (18) and (61) at $r=+0$. Both $\widetilde{\alpha}\widetilde{\zeta}(T)$ and $\widetilde{\beta}\widetilde{\zeta}(T)$ being small, using the boundary condition (21), Eq. (61) can be expanded to first order. (In higher orders, the effect of v_s is to be included.) Thus, the supercurrent density to second order in tunneling probabilities reads

$$j=\frac{3\pi^2}{14\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)\int_0^1 dt\; t[T(t)-2T_S(t)]$$
$$\times\left[1-\frac{3\sqrt{2}\pi^2}{7\zeta(3)}\frac{\zeta(T)}{\chi^{1/2}(\xi_0/l)\xi_0}\int_0^1 dt\right.$$
$$\left.\times t\left\{2T_S(t)+[T(t)-2T_S(t)]\sin^2\frac{\phi}{2}\right\}\right]\sin\phi. \tag{70}$$

For sufficiently small $T(1)$, only the first-order term in Eq. (70) can be retained:

$$j=\frac{3\pi^2}{14\zeta(3)}\frac{env_0}{p_0\xi_0}\left(1-\frac{T}{T_{c0}}\right)\int_0^1 dt\; t[T(t)-2T_S(t)]\sin\phi. \tag{71}$$

[For $T_S(t)\equiv 0$, this is, of course, Eq. (50) of Sec. III.] The cancellation of the factor $\chi(\xi_0/l)$ in first-order formula (71) is a result of Anderson's theorem.[24] This is not the case for Eq. (70), where the second-order term allows for spatial variations of the pair potential near the interfaces due to depairing effect of intrabarrier exchange interactions and the supercurrent itself. Naturally, formulas (70) and (71) hold for a single SIS junction as well. [Compare the calculations in the pure limit of Ref. 20, where also the emergence of the factors $2T_S$ and $(T-2T_S)$ was elucidated.]

In the opposite LD limit [$a\ll\widetilde{\zeta}(T)\ll\widetilde{\alpha}^{-1}$], it is reasonable to start directly with the LD functional (29). Minimization of Eq. (29) with respect to $|\Delta|$ yields its equilibrium value, $|\Delta|_0$:

$$|\Delta|_0^2=\frac{8\pi^2T_{c0}^2}{7\zeta(3)}\left\{1-\frac{T}{T_{c0}}-\frac{7\zeta(3)}{12}\frac{\xi_0^2}{a}\right.$$
$$\left.\times[\alpha+\beta-(\alpha-\beta)\cos\phi]\right\}. \tag{72}$$

The variational derivative of Eq. (9) now becomes a partial derivative with respect to the phase shift ϕ:

$$j=\frac{2e}{NS}\left\{\frac{\partial\Omega_{\mathrm{LD}}[|\Delta|,\phi]}{\partial\phi}\right\}_0$$
$$=\frac{7\zeta(3)}{6}eN(0)\xi_0^2|\Delta|_0^2(\alpha-\beta)\sin\phi. \tag{73}$$

Inserting Eq. (72) and the explicit expressions for α and β [Eq. (69) with $\chi(\xi_0/l)\equiv 1$], we finally get

$$j=\frac{4\pi^4T_{c0}^2}{7\zeta(3)}eN(0)\xi_0\int_0^1 dt\; t[T(t)-2T_S(t)]$$
$$\times\left[1-\frac{T}{T_{c0}}-\frac{\pi^2}{2}\frac{\xi_0}{a}\int_0^1 dt\right.$$
$$\left.\times t\left\{2T_S(t)+[T(t)-2T_S(t)]\sin^2\frac{\phi}{2}\right\}\right]\sin\phi. \tag{74}$$

This expression is to be compared with Eq. (70). Here the second-order term in the supercurrent density appears rather as a consequence of a uniform depression of $|\Delta|_0$ owing to the pair breakers. As expected, for $\alpha^{-1} \gg \zeta^2(T)/a$, Eq. (74) goes over into Eq. (71).

V. DISCUSSION

Let us now briefly summarize what has been done. In this paper, we have achieved complete, self-consistent, microscopic description of current-carrying states in all major types of superconducting multilayers with interlayer Josephson coupling near T_{c0}. The effect of a finite S layer thickness has been fully elucidated. We have obtained closed analytical expressions for the Josephson current as a function of a for pure SI superlattices, multilayers with point-contact-type coupling, and SN superlattices with an arbitrary impurity concentration [Eqs. (49), (53)–(55), and Eqs. (62), (63)]. For all these systems drastic deviations from a single-junction case were found: the reduction of j_c for pure SI superlattices, nontrivial current-phase dependence for multilayers with point-contact-type coupling [Eq. (50) and Eqs. (53)–(55) in the mesoscopic regime], and nontrivial temperature dependence for SN superlattices [Eqs. (66)–(68)]. For SI superlattices in the GL regime, we explicitly included the effect of nonmagnetic impurities and intrabarrier exchange interactions and performed second-order calculations [Eqs. (70) and (74)].

We have accomplished our task using rigorous and powerful mathematical methods, which can also be applied to other problems of inhomogeneous superconductivity. For example, we have derived microscopically the general GL-type free-energy functional with the interface term (10). This functional not only yields correct microscopic boundary conditions to the mean-field GL equations (in a more general way than the standard procedure[12]), but it can also be used for studies of long-range thermal fluctuations. Here, we have employed this functional for a microscopic derivation of the LD-type functional (29). Currently, work is in progress to extend the consideration of this paper to take account of lower temperatures and unconventional pairing in S layers.

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APPENDIX A: DERIVATION OF THE MICROSCOPIC FREE-ENERGY FUNCTIONAL

A microscopic system satisfying the physical conditions of the beginning of Sec. II in the presence of an external magnetic field (with the vector potential **A**) can be described by a second-quantized Hamiltonian of the form[8]

$$H = \int_R d^3\mathbf{r}\,\psi_\alpha^+(\mathbf{r})\left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + V_{\rm imp}(\mathbf{r}) - E_F\right]\psi_\alpha(\mathbf{r}) - \frac{|g|}{2}\int_{R_s} d^3\mathbf{r}\,\psi_\alpha^+(\mathbf{r})\psi_{-\alpha}^+(\mathbf{r})\psi_{-\alpha}(\mathbf{r})\psi_\alpha(\mathbf{r}) + \int_{R_b} d^3\mathbf{r}\,\psi_\alpha^+(\mathbf{r})\hat{V}_{b\alpha\beta}(\mathbf{r})\psi_\beta(\mathbf{r}). \tag{A1}$$

Here R_s and R_b are the superconducting and barrier regions, respectively, $R = R_s \cup R_b$, a summation over spins is implied. Other notations are either standard or those of Sec. II.

The free energy of the system (A1) is given by

$$\Omega[\mathbf{A}] = -T\langle \ln Z[\mathbf{A}]\rangle, \tag{A2}$$

and the supercurrent density is[15]

$$j = -\frac{\delta\Omega[\mathbf{A}]}{\delta\mathbf{A}} = T\left\langle Z^{-1}[\mathbf{A}]\,\frac{\delta Z[\mathbf{A}]}{\delta\mathbf{A}}\right\rangle. \tag{A3}$$

The statistical sum $Z[\mathbf{A}]$ can be expressed[36,37] as the multiple path integral

$$Z[\mathbf{A}] = \lim_{\epsilon \to -0}\int D[F^*,F]\exp\{-\Omega[F^*,F;\mathbf{A}]/T\}, \tag{A4}$$

$$\Omega[F^*,F;\mathbf{A}]/T = -\int_0^{T^{-1}} d\tau \int_R d^3\mathbf{r}\, g(x)|F(\mathbf{r}\tau)|^2 - \frac{1}{2}\,{\rm Tr}\,\ln G^{-1}, \tag{A5}$$

$$G^{-1}(\mathbf{r}\tau,\mathbf{r}'\tau') = \left\{-\frac{\partial}{\partial\tau} + \frac{1}{2m}[\nabla - ie\mathbf{A}\gamma_3]^2\gamma_3 - V_{\rm imp}(\mathbf{r})\gamma_3 + E_F\gamma_3 + \frac{1}{2}(1-\gamma_3)\hat{V}_b^t(\mathbf{r}) - \frac{1}{2}(1+\gamma_3)\hat{V}_b(\mathbf{r}) + \frac{i}{2}g(x)F(\mathbf{r}\tau)\gamma_+\sigma_2 - \frac{i}{2}g(x)F^*(\mathbf{r}\tau)\gamma_-\sigma_2\right\}\delta(\mathbf{r}-\mathbf{r}')\,\delta(\tau-\tau'). \tag{A6}$$

Here γ_i and σ_i are the Pauli matrices in the Gor'kov-Nambu and spin spaces, respectively, $\gamma_\pm=(\gamma_1\pm i\gamma_2)/2$, $g(x)$ is given by Eq. (1). The trace in Eq. (A5) is taken over all continuous and discrete coordinates. The Bose fields $F=|F|\exp(i\varphi)$, $F^*=|F|\exp(-i\varphi)$ are periodic in the imaginary time τ with the period T^{-1}:

$$F(\mathbf{r}\tau)=T\sum_{m=-\infty}^{+\infty}F_m(\mathbf{r})e^{i2\pi Tm},$$

$$F^*(\mathbf{r}\tau)=T\sum_{m=-\infty}^{+\infty}F_m(\mathbf{r})e^{-i2\pi Tm}. \tag{A7}$$

With regard to spatial coordinates, they are supposed to obey the cyclic boundary conditions at $x=\pm L$, which are now fixed by the flux $\Phi=\int_{-L}^{+L}A_x dx$. One should bear in mind that in consequence of gauge invariance $\Omega[F^*,F;\mathbf{A}]=\Omega[|F|,\nabla\varphi-2e\mathbf{A}]$. Thus we can rewrite Eq. (A3) as

$$j=2e\left\langle Z^{-1}[\mathbf{A}]\lim_{\epsilon\to-0}\int D[F^*,F]\,\frac{\delta\Omega[F^*,F;\mathbf{A}]}{\delta\nabla\varphi}\right.$$
$$\left.\times\exp\{-\Omega[F^*,F;\mathbf{A}]/T\}\right\rangle \tag{A8}$$

and proceed to the limit $\mathbf{A}\to 0$. In equilibrium, only the Fourier components F_0 and F_0^* in the series (A7) should be retained. Furthermore, near T_{c0}, we can employ the expansion

$$\mathrm{Tr}\ \ln G^{-1}=\mathrm{Tr}\ \ln(G_0^{-1}-\check{\Delta})\approx\mathrm{Tr}\ \ln G_0^{-1}-\tfrac{1}{2}\ \mathrm{Tr}(G_0\check{\Delta})^2$$
$$-\tfrac{1}{4}\ \mathrm{Tr}(G_0\check{\Delta})^4, \tag{A9}$$

where

$$\check{\Delta}(\mathbf{r})\equiv-\frac{i}{2}\,g(x)F(\mathbf{r})\gamma_+\sigma_2+\frac{i}{2}\,g(x)F^*(\mathbf{r})\gamma_-\sigma_2, \tag{A10}$$

and take the fourth-order term in a local form. Evaluating the path integrals in the semiclassical approximation and assuming self-averaging of the fields F^*, F leads to Eqs. (2)–(4).

APPENDIX B: CALCULATION OF THE GREEN'S FUNCTION

Transform Eq. (32) into the equivalent equation

$$G_\omega(x_1,x_2)=G_\omega(x_1-x_2)+V\sum_{n=-\infty}^{+\infty}G_\omega(x_1-na)G_\omega(na,x_2), \tag{B1}$$

where the free-electron Green's function

$$G_\omega(x_1-x_2)=-\frac{m}{i\lambda}\exp[-i\lambda|x_1-x_2|], \tag{B2}$$

with $\lambda=-\mathrm{sgn}\omega\sqrt{E_F t^2+i\omega}$, obeys Eq. (32) with $V=0$. Using the representations

$$G_\omega(x_1-x_2)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq\ G(q)\exp[iq(x_1-x_2)], \tag{B3}$$

$$G(q)=-\frac{2m}{q^2-\lambda^2}, \tag{B4}$$

$$G_\omega(x_1,x_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}dq_1\int_{-\infty}^{+\infty}dq_2 G(q_1,q_2)$$
$$\times\exp(iq_1x_1-iq_2x_2), \tag{B5}$$

and

$$\sum_{n=-\infty}^{+\infty}\exp[-i(q_1-q_2)na]=\frac{2\pi}{a}\sum_{n=-\infty}^{+\infty}\delta\left(q_1-q_2+\frac{2\pi n}{a}\right), \tag{B6}$$

we make a Fourier transform of Eq. (B1):

$$G(q_1,q_2)=2\pi G(q_2)\delta(q_1-q_2)$$
$$+\frac{V}{a}G(q_1)\sum_{n=-\infty}^{+\infty}G\left(q_1+\frac{2\pi n}{a},q_2\right). \tag{B7}$$

From Eq. (B7), we find

$$\sum_{n=-\infty}^{+\infty}G\left(q_1+\frac{2\pi n}{a},q_2\right)$$

$$=\frac{2\pi G(q_2)\Sigma_{n=-\infty}^{+\infty}\delta(q_1-q_2+2\pi n/a)}{1-(V/a)\Sigma_{n=-\infty}^{+\infty}G(q_1+2\pi n/a)}. \tag{B8}$$

Substituting Eq. (B8) into Eq. (B7) and making an inverse Fourier transform, one obtains

$$G_\omega(x_1,x_2)=G_\omega(x_1-x_2)+\frac{2m^2V}{\pi a}\int_{-\infty}^{+\infty}dq\,\frac{\exp[iq(x_1-x_2)]}{q^2-\lambda^2}\sum_{n=-\infty}^{+\infty}\frac{\exp[-i(2\pi n/a)x_2]}{(q+2\pi n/a)^2-\lambda^2}$$
$$\times\left\{1+\frac{2mV}{a}\sum_{n=-\infty}^{+\infty}\left[\left(q_1+\frac{2\pi n}{a}\right)^2-\lambda^2\right]^{-1}\right\}^{-1}. \tag{B9}$$

The summation on the right-hand side of Eq. (B9) is made with the help of the formula

$$\sum_{n=-\infty}^{+\infty}\frac{\exp[-i(2\pi n/a)r]}{(q+2\pi n/a)^2-\lambda^2}=-\frac{a}{2\lambda}\,\frac{\exp[iq(a-r)]\sin\lambda r+\exp(iqr)\sin[\lambda(a-r)]}{\cos qa-\cos\lambda a}, \tag{B10}$$

where $0\leqslant r<a$. This formula can be easily proved by means of the representation

$$\sum_{n=-\infty}^{+\infty}\frac{\exp[-i(2\pi n/a)r]}{(q+2\pi n/a)^2-\lambda^2}=\frac{a^2}{8\pi^2 i}\oint_C\frac{dz\,\exp[-i(2\pi r/a)z]}{(aq/2\pi+z)^2-(a\lambda/2\pi)^2}\,\frac{\exp(-i\pi z)}{\sin\pi z}, \tag{B11}$$

where the closed contour C encircles the whole real axis in the counterclockwise direction in the complex z plane. The poles $z_{1,2}=(-aq\pm a\lambda)/2\pi$ of the integrand are supposed to lie outside the contour C. Thus we get

$$G_\omega(x_1,x_2)=G_\omega(x_1-x_2)-\frac{m^2V}{\pi a}\int_{-\infty}^{+\infty}dq\,\frac{\exp[iqa(n_1-n_2)]}{q^2-\lambda^2}\,\frac{\exp(iqa)\sin\lambda r_1+\sin[\lambda(a-r_1)]}{\cos qa-\cos\lambda a-(mV/\lambda)\sin\lambda a}\exp(-iqr_2), \tag{B12}$$

where $x_{1,2}=n_{1,2}a+r_{1,2}$, $0\leqslant r_{1,2}<a$, $n=0,\pm1,\pm2,\ldots$ The integral over q in Eq. (B12) is evaluated using the residue theorem. The integrand is a meromorphic function in the complex q plane with only simple poles. These poles are given by $q_{1,2}=\pm\lambda$ and the roots of the equation

$$\cos qa=\cos\lambda a+\frac{mV}{\lambda}\sin\lambda a. \tag{B13}$$

Hence, we can complete the contour of integration to a closed loop with a semicircle of an infinite radius either in the upper or the lower half-planes, depending on the analyticity of the integrand for different relations between n_1 and n_2. The resulting integrals along the contours are equal to $2\pi i$ times the sum over the residues. The contributions of the poles defined by Eq. (B13) amount to convergent infinite series that again can be summed up by means of Eq. (B10). Finally, we arrive at Eqs. (33)–(35).

APPENDIX C: DERIVATION OF Ω_{GL} WITH THE INTERFACE TERM

It is well known that the GL theory, being an expansion of rigorous microscopic theory in powers of $\xi_0/\zeta(T)$, breaks down at distances $\sim\xi_0$ from a sharp interface, where the pair potential may undergo strong spatial variations.[12] These spatial variations result in a loss of the condensation energy at the interfaces that are not accounted for in the original GL free-energy functional.[4] On the other hand, as shown in Sec. III, any spatial variations vanish in the case of impenetrable interfaces. This observation allows us to isolate the contribution of the interfaces in the microscopic free-energy functional (2), rewriting it identically as

$$\begin{aligned}\Omega[\Delta,\Delta^*]=S\int_{x_1\in S}dx_1\Bigg[\frac{1}{|g|}|\Delta(x_1)|^2-\int_{x_2\in S}dx_2\\ \times K^{(0)}(x_1,x_2)\Delta(x_1)\Delta^*(x_2)\\ +\frac{7\zeta(3)N(0)|\Delta(x_1)|^4}{16\pi^2T_{c0}^2}\Bigg]\\ -S\int_{x_1\in S}dx_1\int_{x_2\in S}dx_2[K(x_1,x_2)\\ -K^{(0)}(x_1,x_2)]\Delta(x_1)\Delta^*(x_2).\end{aligned} \tag{C1}$$

Here we have introduced the standard notation $\Delta(x)\equiv|g|F(x)$ for x being within an S layer, the spatial integrals are taken over S layers only. In the pure limit, the kernel $K^{(0)}(x_1,x_2)$ is that of Eq. (39), except for the change $x_1+x_2-(2n+1)a\to x_1+x_2-(2n+1)c$ $(c=a+d)$ in the last term. Both K and $K^{(0)}$ should be taken in asymptotic form for $a\gg\xi_0$ [see, e.g., Eq. (40)]. The first term on the right-hand side of Eq. (C1) can now be expanded in powers of $\xi_0/\zeta(T)$, yielding the usual GL bulk free energy. The second term should be identified with the desired interface free energy. In contrast to K and $K^{(0)}$, which contain long-range parts, the truncated kernel $K-K^{(0)}$ has an effective range of the order of ξ_0 and vanishes in the GL region. We can expand it to leading order in weak-coupling parameters, set $T=T_{c0}$ and extend the limits of spatial integration to infinity. One should also substitute for Δ,Δ^* solutions to the linearized version of microscopic Eq. (8) in the vicinity of each barrier. In view of the symmetry conditions of Sec. II, it is sufficient to consider only one particular barrier. Thus for $n=0$, we obtain

$$\Delta(x)=\int_{-\infty}^{-d/2}dx_1K(x,x_1)\Delta(x_1)+\int_{d/2}^{+\infty}dx_1K(x,x_1)\Delta(x_1), \tag{C2}$$

where we have extended the spatial integration to infinity. Within the same approximation, one should use the asymptotic form of K for $a \gg \xi_0$, expand it to required order in weak coupling, and set $T = T_{c0}$. Presently, there are well-developed mathematical methods of the treatment of such equations.[19,21] For example, it is convenient to decompose Δ into the symmetric (Δ_s) and antisymmetric parts (Δ_a). Because of the symmetry relations of Sec. II for K, Δ_s and Δ_a separately satisfy Eq. (C2). As a result, we obtain two independent linear uniform integral equations for Δ. Hence, the solution for Δ near the barrier $n=0$ depends on two arbitrary multiplicative constants that should be identified with the values of the GL order parameter at both interfaces, $\Delta(-d/2-0)$ and $\Delta(d/2+0)$. The solution near the nth barrier differs from this one by the factor $\exp(in\phi)$. By inserting these solutions into the second term on the right-hand side of Eq. (C1) we obtain the desired contribution from the interfaces to the GL free energy.

To illustrate this general mathematical scheme, we present calculations of the free-energy interface term for the case of SI superlattices with tunnel magnetic barriers. For simplicity, we focus on the pure limit and let $d \ll \xi_0$. As explained above, we need only the asymptotic form of K, which coincides with that of a single-junction problem. In both the ferromagnetic[20] and paramagnetic[22] limits, the general form of K for a single junction is

$$K(x_1,x_2)=\int_0^1 dt\{[K_t(x_1-x_2)+K_t(x_1+x_2)][\Theta(x_1)\Theta(x_2)+\Theta(-x_1)\Theta(-x_2)]\}-\int_0^1 dt\{[T(t)+2T_S(t)]K_t(x_1+x_2)$$
$$\times[\Theta(x_1)\Theta(x_2)+\Theta(-x_1)\Theta(-x_2)]+[T(t)-2T_S(t)]K_t(x_1-x_2)[\Theta(x_1)\Theta(-x_2)+\Theta(-x_1)\Theta(x_2)]\}, \quad \text{(C3)}$$

where $T(t)$ and $T_S(t)$ are the total and spin-flip tunneling probabilities, respectively, $[T_S(1) \ll T(1) \ll 1]$, other notations are those of Eqs. (40) and (41). In this expression for the kernel K, we have explicitly separated the parts $K^{(0)}$ [the first term on the right-hand side of Eq. (C3)] and $K-K^{(0)}$ (the second term). As we see, in this case the interface term in Eq. (C1) is of first order in tunneling probabilities. We therefore need the zero-order solution to Eq. (C2) with $K=K^{(0)}$. It is straightforward to find

$$\Delta^{(0)}(x)=\Delta(+0)\Theta(x)+\Delta(-0)\Theta(-x). \quad \text{(C4)}$$

Inserting Eq. (C4) into Eq. (C1), one immediately gets the interface free-energy term with

$$\alpha=\frac{3\pi^2}{7\zeta(3)}\,\xi_0^{-1}\int_0^1 dt\; tT(t), \qquad \beta=\frac{6\pi^2}{7\zeta(3)}\,\xi_0^{-1}\int_0^1 dt\; tT_S(t). \quad \text{(C5)}$$

These same values of α and β were previously obtained[20,22] in the framework of establishing the GL boundary conditions by means of the standard procedure[12] of identifying the asymptotics of the solution to Eq. (C2) at $x \to \pm\infty$ with the asymptotics of the solution to the GL equation at $x \to \pm 0$. A slight modification of the derivation[20,22] of K allows us to take account of the presence of nonmagnetic impurities in S layers, yielding $\tilde{\alpha}$ and $\tilde{\beta}$ of Eq. (69). Finally, we obtain Ω_{GL} in the form (10).

Analogous calculations can be carried out for all the systems considered in this paper. [In the case of SN superlattices, these calculations are more involved, because the solutions to Eq. (C2) are no longer constants.] In fact, there is no need to do this. As should be clear from the above-developed approach, one can simply capitalize on the results of the computation of $\tilde{\alpha}$ and $\tilde{\beta}$ for the GL boundary conditions in a single-junction problem (see references in Secs. II and IV) to obtain the desired form of Ω_{GL}.

APPENDIX D: LAWRENCE-DONIACH LIMIT

Consider Eq. (10) in the form

$$\Omega_{\mathrm{GL}}[\Delta_n^*(r),\Delta_n(r);\Delta_n^*(+0),\Delta_n^*(a-0)]=SN(0)\sum_{n=-N/2}^{+N/2}\left\{\int_{+0}^{a-0}dr\left[-\tau|\Delta_n(r)|^2+\frac{7\zeta(3)}{12}\,\xi_0^2\chi(l/\xi_0)\left|\frac{d\Delta_n(r)}{dr}\right|^2\right.\right.$$
$$\left.+\frac{7\zeta(3)}{16\pi^2T_{c0}^2}|\Delta_n(r)|^4\right]+\frac{7\zeta(3)}{24}\,\xi_0^2\{(\alpha+\beta)[|\Delta_n(+0)|^2+|\Delta_n(a-0)|^2]$$
$$\left.+(\beta-\alpha)[\Delta_n^*(+0)\Delta_{n-1}(a-0)\Delta_n(+0)\Delta_{n-1}^*(a-0)]\}\right\}, \quad \text{(D1)}$$

which is the general expression for the GL free-energy functional in SI superlattices. Let us introduce the representation

$$\Delta_n(r)=\Delta(r)\exp\left[i\left(n+\frac{1}{2}\right)\phi\right]. \quad \text{(D2)}$$

One can expand Δ into a Fourier series

$$\Delta(r)=\sum_{p}\Delta_{p}e^{ipr},\quad p=\frac{2\pi n}{a},\quad n=0,\pm1,\pm2,\ldots\ . \tag{D3}$$

Substitution of Eqs. (D2) and (D3) into Eq. (D1) yields

$$\Omega_{\rm GL}[\Delta_{p}^{*},\Delta_{p};\phi]=\Omega_{\rm LD}[\Delta_{0}^{*},\Delta_{0};\phi]+NaSN(0)\sum_{p\neq0}\left\{-\tau|\Delta_{p}|^{2}+\frac{7\zeta(3)}{12}\,\xi_{0}^{2}\chi(l/\xi_{0})p^{2}|\Delta_{p}|^{2}+\frac{7\zeta(3)}{16\pi^{2}T_{c0}^{2}}\left[|\Delta_{0}|^{2}\mathrm{Re}(\Delta_{p}^{*}\Delta_{-p})+\sum_{p1,p2}\mathrm{Re}(\Delta_{p}^{*}\Delta_{p1})\mathrm{Re}(\Delta_{p2}^{*}\Delta_{p1-p2-p})\right]+\frac{7\zeta(3)}{24}\,\frac{\xi_{0}^{2}}{a}\sum_{p1}\mathrm{Re}(\Delta_{p}^{*}\Delta_{p1})[\alpha+\beta-(\alpha-\beta)\cos\phi]\right\}. \tag{D4}$$

where $\Omega_{\rm LD}[\Delta_0^*,\Delta_0;\phi]$ is exactly the desired LD-type functional:

$$\Omega_{\rm LD}[\Delta_{0}^{*},\Delta_{0};\phi]=NaSN(0)\left\{-\tau|\Delta_{0}|^{2}+\frac{7\zeta(3)}{16\pi^{2}T_{c0}^{2}}\,|\Delta_{0}|^{4}+\frac{7\zeta(3)}{12}\,\frac{\xi_{0}^{2}}{a}\,|\Delta_{0}|^{2}\times[\alpha+\beta-(\alpha-\beta)\cos\phi]\right\}. \tag{D5}$$

[Note the absence in Eq. (D5) of the impurity factor $\chi(l/\xi_0)$, in line with the Anderson's theorem.[24]] If the thickness of superconducting layers satisfies the conditions

$$a\ll\tilde{\zeta}(T),\quad \alpha^{-1}\chi(l/\xi_{0}), \tag{D6}$$

the term $p^2|\Delta_p|^2$ gives a dominant contribution to the sum over $p\neq0$, and we can write

$$\Omega_{\rm GL}[\Delta_{p}^{*},\Delta_{p};\phi]\approx\Omega_{\rm LD}[\Delta_{0}^{*},\Delta_{0};\phi]+NaSN(0)\times\frac{7\zeta(3)}{12}\,\xi_{0}^{2}\chi(l/\xi_{0})\sum_{p\neq0}p^{2}|\Delta_{p}|^{2}, \tag{D7}$$

which means that in this limit the contributions of the Fourier components Δ_0 and $\Delta_{p\neq0}$ decouple. Minimization of Eq. (D1) with respect to $\Delta_{p\neq0}^*$ gives the obvious condition $\Delta_{p\neq0}=0$ [no spatial variations of $\Delta(r)$], and $\Omega_{\rm GL}$ actually reduces to $\Omega_{\rm LD}$. As could have been expected from the results of Sec. II, in view of the conditions (D6), a system in the LD limit can possess nontrivial minima with $|\Delta_0|\neq0$ only if

$$\tilde{\zeta}(T)\ll\alpha^{-1}\chi(l/\xi_{0}), \tag{D8}$$

with the transition temperature and the critical thickness given by Eqs. (27) and (28), respectively. Combining Eqs. (D6), (D8), and (28), we obtain the strict condition of the validity of the LD description:

$$a_{\rm min}\ll a\ll\tilde{\zeta}(T)\ll\alpha^{-1}\chi(l/\xi_{0}). \tag{D9}$$

Here, $a_{\rm min}$ is defined either by Eq. (28) $a_{\rm min}=a_c$, if $a_c\gg\xi_0$ or by the general condition of the validity of the GL regime $a_{\rm min}\gg\xi_0$, if $a_c\ll\xi_0$.

It is instructive to rewrite Eq. (D5), introducing the definition of the renormalized (due to pair breaking by exchange interactions inside the barriers) transition temperature in the absence of supercurrent, T_c':

$$\Omega_{\rm LD}[|\Delta|,\phi]=NaSN(0)\left[-\tau'|\Delta|^{2}+\frac{7\zeta(3)}{16\pi^{2}T_{c0}^{2}}\,|\Delta|^{4}+\frac{7\zeta(3)}{12}\,\frac{\xi_{0}^{2}}{a}\,|\Delta|^{2}(\alpha-\beta)(1-\cos\phi)\right], \tag{D10}$$

$$\tau'=\frac{T_c'-T}{T_{c0}},\qquad T_c'=T_{c0}-T_{c0}\,\frac{7\zeta(3)}{6}\,\frac{\xi_0^2}{a}\,\beta. \tag{D11}$$

(We have simplified the notation for the LD order parameter Δ_0, dropping the lower index.) This form of the LD functional clearly shows that interlayer coupling is determined by the typical difference $(\alpha-\beta)$.

A few concluding remarks would be appropriate here:

(1) The functional form of Eq. (D10) is the same both in the pure limit and in the presence of nonmagnetic impurities, but the effect of impurities enters implicitly via Eq. (D9).

(2) In phenomenological approaches, it is often demanded that the energy of interlayer coupling in $\Omega_{\rm LD}[|\Delta|,\phi]$ be much smaller than the intralayer condensation energy. Mathematically, this requirement is equivalent to $\zeta^2(T)a^{-1}\ll\alpha^{-1}$. This is an unnecessary constraint. As we have seen, the actual range of validity of the LD regime, set by Eq. (D9), is much larger.

(3) Owing to the implicit gauge invariance of our evaluation scheme, it is straightforward to include the effect of external magnetic fields, making use of the substitution $\phi\to\phi-2e\int_0^d dx\,A_x$ in Eq. (D10), where $\mathbf{A}$ is the vector potential. Further generalizations, allowing for spatial variations of Δ in the transverse yz plane, also pose no problem.

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