$1/r^2$ *t-J* model in a magnetic field

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We study the one-dimensional supersymmetric t-J model with $1/r^2$ interaction threaded by magnetic flux. Because of the long-range interaction, the effect of this flux leads to a modification of the electron hopping term. We present an exact solution of this model for all values of the flux, concisely formulated as a set of Bethe-ansatz-like equations. Examination of the ground state shows that the persistent currents at zero temperature do not exhibit a parity effect despite the fact that the long-range t-J model falls in the Luttinger-liquid universality class. This exception to Leggett's conjecture arises because of the special nature of the long-range hopping. [S0163-1829(97)01730-X]

Exact solutions have provided us with an interesting way to deal nonperturbatively with systems of strongly correlated electrons. Notable examples are the electron systems with δ-function interaction,[1] the Hubbard model,[2] and the short-range t-J model.[3] These models are solvable by Bethe ansatz and have played an important role in understanding the physics of the one-dimensional electron gas. A particularly interesting class of lattice models that are exactly solvable despite long-range interactions are the Haldane-Shastry spin chain of $1/r^2$ exchange interaction[4,5] and its many variations. The latter include the supersymmetric t-J models of long-range hopping and exchange[6–11] as well as multicomponent generalizations. Since the closed spin chain admits a Bethe-ansatz-like solution, this indicates that the quasiparticle interactions are statistical in nature and arise from demanding periodicity of the chain.

While it is natural to close the chain by imposing periodic boundary conditions, we may also consider the case where the closed chain is threaded by magnetic flux. Due to the nontrivial topology, electrons transported around the chain then pick up an Aharonov-Bohm phase in the presence of a nonzero flux. In models with nearest-neighbor exchange it is straightforward to encode this phase (and hence the flux) by imposing twisted boundary conditions on the wave functions. However such a prescription needs to be modified in the presence of long-range interactions where particles may hop between any arbitrary pair of sites. We show how this may be done for the supersymmetric t-J model with inverse square interaction and investigate its ground state and full energy spectrum. Based on previous results in the absence of magnetic flux,[10] we derive a set of Bethe-ansatz-like equations appropriate to twisted boundary conditions.

We consider a system of electrons on a one-dimensional ring described by the SU(2) supersymmetric t-J model. For a uniform and closed chain of L lattice sites, the Hamiltonian takes the form

$$H_{tJ}=-\frac{1}{2}\sum_{\sigma=\uparrow,\downarrow}\sum_{1\le l\neq m\le L}[t(l-m)c^\dagger_{l\sigma}c_{m\sigma}+\text{H.c.}]+\frac{1}{2}\sum_{1\le l\neq m\le L}J(l-m)[P^\sigma_{lm}-(1-n_l)(1-n_m)], \tag{1}$$

where the hopping strength, $t(n)$, and exchange interaction, $J(n)$, are functions only of separation due to the rotational invariance of the ring. P^σ_{lm} is the spin exchange operator and n_l is the electron number operator. We have implicitly assumed a projection onto single occupancy at each site.

Without magnetic flux, the supersymmetric long-range model has an interaction strength given by $t(n)=J(n)=1/d^2(n)$ where $d(n)=(L/\pi)\sin(\pi n/L)$. While $d(n)$ is most readily interpreted as the chord length between sites n units apart, this interpretation is not easily generalized to encompass nonzero flux. In particular, an electron hopping along a chord must travel through the interior of the ring where it would be sensitive to the actual magnetic field and not just the flux. To avoid this difficulty, we use the alternate interpretation of $J(n)$ as the periodic version of $1/n^2$, namely[12,11,13]

$$J(n)=\sum_{k=-\infty}^{\infty}\frac{1}{(n+kL)^2}=\left(\frac{\pi}{L}\right)^2\frac{1}{\sin^2(\pi n/L)}, \tag{2}$$

which represents the sum of hopping over all multiples of the period L with periodic boundary conditions. It is now straightforward to generalize this to twisted boundary conditions appropriate to a ring threaded by flux. We introduce a dimensionless flux ϕ, represented by the vector potential $A=\phi(\phi_0/L)$ where $\phi_0=hc/e$, so that electrons pick up a phase $e^{2\pi i\phi}$ when transported once around the ring. In this case, the hopping interaction is twisted and becomes

$$t_\phi(n)=\sum_{k=-\infty}^{\infty}\frac{e^{2\pi i\phi(n+kL)/L}}{(n+kL)^2}. \quad (3)$$

Since the exchange interaction is insensitive to the flux, $J(n)$ is still given by Eq. (2). This interaction was introduced by Fukui and Kawakami for the twisted Haldane-Shastry model in Ref. 13 where the sum was carried out for rational twists, $\phi=p/q$. It turns out, however, that this infinite sum can actually be evaluated for arbitrary ϕ, yielding the result

$$t_\phi(n)=\left(\frac{\pi}{L}\right)^2\frac{z^{n\lfloor\phi\rfloor}\{[1-(\phi-\lfloor\phi\rfloor)]+(\phi-\lfloor\phi\rfloor)z^n\}}{\sin^2(\pi n/L)}, \quad (4)$$

where $z=e^{2\pi i/L}$ and $\lfloor\phi\rfloor$ is the greatest integer not exceeding ϕ. This expression is remarkable for the fact that it is piecewise *linear* and continuous in ϕ, even though the flux originally entered in the exponent of Eq. (3). We note that $t_\phi(n)$ satisfies the conditions $t_{-\phi}(n)=t_\phi(n)^*$ and $t_{-\phi}(-n)=t_\phi(n)$, which is readily apparent from Eq. (3), but hidden in the summed expression of Eq. (4). To simplify our subsequent discussion, we may restrict ϕ to lie in the range $0\leqslant\phi\leqslant1$, obviating the need for the greatest integer function. Other values of the flux may always be brought into this range by a gauge transformation with resultant shift in the lattice momentum. In this case, Eq. (4) becomes

$$t_\phi(n)=J(n)[(1-\phi)+\phi z^n], \quad (5)$$

demonstrating that the effect of an arbitrary flux is to simply give a linear interpolation between different systems, each with an integral value of ϕ.

Previous techniques for solving the Haldane-Shastry and t-J models without flux[4,6,10] are easily extended to the present case, given by the twisted hopping $t_\phi(n)$. In particular, the t-J model may be solved by introducing a basis of Jastrow wave functions describing the down-spin and hole excitations about a fully polarized up-spin background. While this background may appear unnatural in the presence of flux, it nevertheless allows an immediate generalization of the exact solution constructed in Ref. 10. In order to apply the techniques of Ref. 10, we need the sum formula

$$\sum_{n=1}^{L}z^{Jn}(1-z^n)^a t_\phi(n)=2\frac{\pi^2}{L^2}\{\delta_{a,0}[\tfrac{1}{6}(L^2-1)+\phi(1-\phi)-(J+\phi)(L-(J+\phi))]+\delta_{a,1}[L-2(J+\phi)-1]+2\delta_{a,2}\}, \quad (6)$$

which follows from Eq. (5) and the zero-flux sum formula.[4] This expression holds for non-negative J whenever $0\leqslant\phi\leqslant1$ and $0\leqslant a\leqslant L-1-J$ and generalizes the result for rational twists presented in Ref. 13, while reducing to the standard expression[4] when $\phi=0$. These results indicate that the restriction to rational twists is unnecessary, so that there is no distinction between rational and irrational twist angles in this strictly one-dimensional system. We note that the ϕ^2 terms cancel in the above sum formula as they must; it is the vanishing of this quadratic term (which persists in the ground-state energy) that is ultimately responsible for the disappearance of the ground-state parity effect in this model.

Based on the fact that the three terms in Eq. (6) correspond to constant, two-body, and three-body terms in the Hamiltonian,[4,6] we see that the flux has no effect on the cancellation of three-body terms. This immediately shows that the quasiparticles remain free, up to statistical interactions, even in the presence of flux. From the constant and two-body terms, it is apparent that ϕ acts to shift the quasiparticle momenta, leading to a modified dispersion relation. Since this is the extent of the modification to the solution caused by nonzero ϕ, the results of Ref. 10 are easily extended to the case of twisted hopping.

For a spin chain with $M_\downarrow$ down spins and Q holes, we introduce two sets of pseudomomenta: p_i ($i=1,2,\ldots,M_\downarrow+Q$) and q_α ($\alpha=1,2,\ldots,Q$). The solution to this supersymmetric t-J model may then be written in a Bethe-ansatz-like form

$$p_iL=2\pi J_i+\sum_{j\neq i}\theta(p_i-p_j)-\sum_\alpha\theta(p_i-q_\alpha),$$

$$\sum_i\theta(q_\alpha-p_i)=2\pi I_\alpha, \quad (7)$$

where the step function $\theta(x)=\pi\,\mathrm{sgn}\,(x)$ is the (statistical) scattering phase. The fermionic quantum numbers J_i and I_α are either integers or half-integers and are restricted to lie in the ranges $|J_i|\leqslant(L-M_\downarrow+1)/2$ and $-(M_\downarrow+Q)/2\leqslant I_\alpha\leqslant(M_\downarrow+Q)/2-1$, respectively. Since the q_α's label the hole degrees of freedom, it gives rise to a natural splitting of the pseudomomenta $\{p_i\}$ into $M_\downarrow$ spin and Q hole degrees of freedom. Namely we take $\mathcal{Q}$ to be the set of pseudomomenta p_i satisfying

$$\sum_\alpha[\theta(p_i-q_\alpha)-\theta(p_{i-1}-q_\alpha)]=2\pi. \quad (8)$$

There are exactly Q such pseudomomenta, corresponding to the hole excitations. The remaining $M_\downarrow$ pseudomomenta then correspond to spin excitations. Using this distinction, the energy spectrum and momentum of the system are given by

$$E(\phi)=\frac{\pi^2}{6}L(1-1/L^2)+\sum_{i\notin\mathcal{Q}}\epsilon_0(p_i)+\sum_{i\in\mathcal{Q}}\epsilon_\phi(p_i),$$

$$P(\phi)=(L-1)\pi+2\pi\phi(1-Q/L)-\sum_{i=1}^{M_\downarrow+Q}(p_i-\pi)\mathrm{mod}2\pi, \quad (9)$$

where the single-particle dispersion relation is

$$\epsilon_\phi(k)=\tfrac{1}{2}[(k+2\pi\phi/L)^2-\pi^2+4\pi^2\phi(1-\phi)/L^2]$$
$$=(1-\phi)\epsilon_0(k)+\phi\epsilon_0(k+2\pi/L), \quad (10)$$

for $0\leqslant\phi\leqslant 1$. This piecewise linear form of the dispersion relation follows directly from the behavior of the hopping term, Eq. (5). We wish to stress that, while this solution has the form of a Bethe ansatz, it was actually derived as an exact solution based on the construction of a complete basis of Jastrow wave functions and a proper ordering of the Hilbert space, as described in Refs. 12 and 10.

States in the excitation spectrum are labeled by individually nonoverlapping quantum numbers J_i and I_α. The J_i may be represented as a string of 0's and 1's of length $L-M_\downarrow$ with $M_\downarrow+Q$ occupied positions represented by 1's.[14] The I_i then in turn label which of these 1's correspond to hole excitations (and hence are sensitive to the flux). The interpretation of the Bethe-ansatz-like equations (7) is to separate the spin excitations by inserting a 0 before every spin excitation. The resulting string then specifies the pseudomomenta p_i, lying in the range $[-\pi,\pi)$. From Eq. (10), it is evident that states in the middle of the string have lowest energy. Therefore the ground state of the *t-J* model, in a sector of fixed $M_\downarrow$ and Q, has pseudomomenta of the general form $p_i\in(\ldots 00101011111 0101000\ldots)$, with the hole excitations centrally located (and underlined). In order to study the ground-state properties, we introduce uniform spin and hole momenta, J_s and J_h (integral or half-integral as appropriate), perturbing the string of 1's to the left or right. In this case, the corresponding eigenenergies are

$$\frac{L^2}{\pi^2}E(J_s,J_h)=E_0+2M_\downarrow J_s^2+2Q[(J_h+\phi)^2+(J_s+J_h+\phi)^2-2\phi^2], \quad (11)$$

where

$$E_0=\tfrac{1}{6}L(L^2-1)+\tfrac{2}{3}(Q+M_\downarrow)[(Q+M_\downarrow)^2-1]+4\phi Q-\tfrac{1}{2}(Q+M_\downarrow)L^2-\tfrac{1}{2}Q(Q+M_\downarrow)(Q+2M_\downarrow). \quad (12)$$

For fixed $M_\downarrow$ and Q, the ground state has both J_s and $J_h+\phi$ as close to zero as possible. These states correspond to exact Jastrow product wave functions describing the ground state as well as uniform excitations of the *t-J* model.

To further examine the ground state of this model, we work at a fixed hole fraction, $n_h\equiv Q/L$. Denoting the number of electrons by $N_e=M_\uparrow+M_\downarrow=L-Q$, the ground state is either a SU(2) singlet for even N_e or a doublet for odd N_e. Due to finite-size effects, the ground-state properties depend on the value of N_e mod 4. At zero flux, whenever $N_e\neq 2$ mod 4, the ground state carries momentum and is hence twofold degenerate. However this degeneracy is always lifted for nonzero ϕ which breaks time reversal symmetry. The exact ground-state energies and momenta are given in Table I, where the bulk quantities are

$$E_g=-\frac{\pi^2 L}{12}[n_h(3-n_h^2)+2(3+2n_h)/L^2],$$

$$P_g=2\pi\phi(1-n_h). \quad (13)$$

TABLE I. Exact ground-state energies $E(\phi)$ and momenta $P(\phi)$ of the *t-J* model with twisted boundary conditions for $0\leqslant\phi\leqslant\frac{1}{2}$. Taking into account a level crossing at $\phi=\frac{1}{2}$, the absolute ground state for $\frac{1}{2}\leqslant\phi\leqslant 1$ instead has energy $E(1-\phi)$ and momentum $-P(1-\phi)$.

N_e mod 4	$E(\phi)$	$P(\phi)$
0	$E_g+\pi^2 n_h/L$	$P_g+\pi(1+n_h)$
1	$E_g+\pi^2[2+n_h(1+8\phi)]/4L$	$P_g-\frac{\pi}{2}(1-n_h-1/L)$
2	$E_g+4\pi^2\phi n_h/L$	P_g
3	$E_g+\pi^2[2+n_h(1+8\phi)]/4L$	$P_g-\frac{\pi}{2}(1-n_h+1/L)$

We note that the linear spectral flow apparent from the cancellation of ϕ^2 terms in Eq. (10) indicates that the ground state at zero flux is always the absolute lowest energy state since any flow to lower energy has nowhere to terminate. Hence the ground state is always diamagnetic, regardless of the number of electrons.

It is well known that the ground state of a noninteracting one-dimensional electron gas is either diamagnetic or paramagnetic, depending on whether the number of electrons in the system is even or odd. Leggett has conjectured that this parity effect persists in the presence of interactions;[15] this has subsequently been proven by Loss for generic Luttinger liquids.[16] Thus it is somewhat of a surprise that this long range *t-J* model provides an exception to Leggett's conjecture despite the fact that its low-lying physics is still described by a Luttinger-liquid fixed-point Hamiltonian. The origin of this breakdown may be traced to the unusual quasiparticle dispersion relation, Eq. (10), which in turn is an artifact of the long-range nature of the electron hopping interaction. Therefore this model does not invalidate Leggett's conjecture, but rather emphasizes the peculiar features of long-range hopping in the presence of a magnetic field, as was already apparent in the subtleties in constructing $t_\phi(n)$ and its resulting linearity in ϕ. This shows that there are sharp differences between models with and without long-range hopping in the presence of a magnetic field.

We now turn to a generalization to the SU$(1|K)$ supersymmetric *t-J* model with long-range interactions. Since the two-body nature of the quasiparticle interactions is unaffected by the flux, we may approach the SU$(1|K)$ generalization via the asymptotic Bethe ansatz (ABA), which was constructed at zero flux in Refs. 7 and 17. Since the magnetic flux twists all K fermionic species identically, it is natural to write the ABA in terms of fermionic excitations above the purely bosonic vacuum (which we denote $F^K B$). For this choice of grading, only the first nesting is affected by the flux. To proceed, we introduce K sets of pseudomomenta

$$p_i^{(a)}:\quad i=1,2,\ldots,N_a, \quad (14)$$

where $a=1,2,\ldots,K$ and $N_a=\Sigma_{i=a}^K M_i$ (M_i is the number of electrons with spin component i). Note that N_1 gives the total number of SU(K) electrons. These quasimomenta satisfy the following ABA equations:

$$p_i^{(1)}L = 2\pi J_i + \sum_{\alpha} \theta(p_i^{(1)} - p_\alpha^{(2)}),$$

$$\sum_i \theta(p_\alpha^{(2)} - p_i^{(1)}) = 2\pi I_\alpha^{(2)} + \sum_\beta \theta(p_\alpha^{(2)} - p_\beta^{(2)}) - \sum_\gamma \theta(p_\alpha^{(2)} - p_\gamma^{(3)}),$$

$$\vdots$$

$$\sum_\gamma \theta(p_\alpha^{(K)} - p_\gamma^{(K-1)}) = 2\pi I_\alpha^{(K)} + \sum_\beta \theta(p_\alpha^{(K)} - P_\beta^{(K)}). \tag{15}$$

The quantum numbers $\{J_i\},\{I_\alpha^{(2)}\}, \ldots ,\{I_\alpha^{(K)}\}$ are integers or half-integers which are distinct within each set, respectively. This set of equations is unchanged from the case without flux;[7,17] the only place where ϕ enters is in the energy and momentum, given by

$$E(\phi) = -\frac{\pi^2}{6}L(1-1/L^2) - \sum_{i=1}^{N_1} \epsilon_\phi(p_i^{(1)}),$$

$$P(\phi) = \sum_{i=1}^{N_1} (p_i^{(1)} + 2\pi\phi/L - \pi) \bmod 2\pi. \tag{16}$$

This provides the exact energy spectrum of the SU(K) t-J model in the presence of flux ϕ, even in the nonasymptotic regime.

For the ordinary SU(2) t-J model threaded by flux, the ABA equations given in the F^2B grading provide a more symmetrical description than the microscopically derived equations, (9). Nevertheless, the BF^2 picture of Eq. (7) has an advantage in that the complete level degeneracies are understood[10] independent of the SU(1|2) supermultiplet structure, which is spontaneously broken for nonzero flux.

In summary, we have provided an exact solution to the long-range t-J model in the presence of arbitrary flux. This solution indicates that there is no meaningful difference between rational and irrational values of the flux. We have also given solutions for the full energy spectra of the general SU(K) t-J model based on an asymptotic Bethe ansatz. It should be feasible to prove these solutions exact by studying the system with a complete basis of Jastrow wave functions as was done for the SU(2) model. For this latter case, the form of the exact solutions, only slightly changed in the presence of flux, indicates that this model remains integrable, even though the manifest SU(1|2) supersymmetry has been lost. Thus it is apparent that part of the Yangian symmetry[18] remains. Work on finding an infinite set of commuting constants of motion in the presence of magnetic flux is currently in progress.

Note added: Recently, we became aware of Ref. 19, which independently addressed many of the same issues. B. Sutherland also recently informed us that the linear dependence of the ground-state energy on the flux can be derived for the twisted t-J model by imposing twisted boundary conditions on the continuum model,[20,21] and then by taking the strong interaction limit to decouple the lattice oscillation modes from the internal degrees of freedom at each site. This approach should enable one to handle the SU(1|K) t-J model with twisted boundary conditions in a straightforward manner.

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