Hall conductance of Bloch electrons in a magnetic field

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We study the energy spectrum and the quantized Hall conductance of electrons in a two-dimensional periodic potential with perpendicular magnetic field *without* neglecting the coupling of the Landau bands. Remarkably, even for weak Landau band coupling significant changes in the Hall conductance compared to the one-band approximation of Hofstadter's butterfly are found. The principal deviations are the rearrangement of subbands and unexpected subband contributions to the Hall conductance. [S0163-1829(97)07728-X]

I. INTRODUCTION

Since many decades the problem of electrons under the influence of a two-dimensional periodic potential (Bloch electrons) and a perpendicular magnetic field is of great interest.[1] Each of the limiting cases, just a periodic potential and just a magnetic field, was solved in the early days of quantum mechanics.[2,3] Their solutions are translation invariant Bloch waves with energy *bands* and rotation invariant oscillator functions with discrete Landau *levels*, respectively. Away from the limiting cases the system must combine these adverse properties. For very weak and for very strong magnetic fields, compared with the potential strength, this combination gives rise to a fractal energy spectrum—the famous Hofstadter butterfly[4] (see Fig. 2). It is based on a one-band approximation that leads to the tight-binding Harper equation.[5,6] In the intermediate regime, where the magnetic field is of comparable strength to the potential, one has to take into account the coupling between the Landau levels or between the Bloch bands. In doing so, one obtains a vectorial tight-binding equation,[7] which has the correct chaotic classical limit,[7,8] whereas the one-band approximation of the Harper equation has an integrable classical limit. The coupling causes considerable changes in the Hofstadter butterfly and is of importance for experimental observations.[7,9,10]

Currently, one tries to find signatures of Hofstadter's butterfly in lateral superlattices with periods of about 100 nm on GaAs-$Al_xGa_{1-x}As$ heterojunctions. Straightforward spectroscopic measurements are not yet feasible[11] and, instead, the efforts are concentrated on magnetotransport measurements.[12,13] In fact, substructure in the Shubnikov-de Haas oscillations (oscillations in the longitudinal resistance due to the Landau levels) was found, which demonstrates the splitting of Landau levels due to the periodic potential.[13]

Beyond these qualitative findings in the longitudinal resistance the study of the Hall conductance would provide a quantitative demonstration of the Landau level substructure: Von Klitzing, Dorda, and Pepper discovered in 1980 that the Hall conductance σ is quantized between Landau levels in integer multiples of e^2/h (Ref. 14) [Fig. 1(a)]. Under the influence of a weak periodic potential each Landau level broadens into a Landau band with so-called minigaps, and one might have thought that the Hall conductances in these minigaps were rational multiples of e^2/h. Thouless *et al.*,[15] however, showed, with an argument by Laughlin,[16] that the Hall conductance is quantized even in these minigaps in *integer* multiples of e^2/h. For Hofstadter's butterfly they found that these integer values vary irregularly from gap to gap according to a diophantic equation [see Figs. 1(b) and 2].[17] Whereas the longitudinal resistance is zero in every gap, the Hall conductance differs from gap to gap, and thus contains quantitative information about the Landau band substructure.

In order to make the minigaps observable in the presence of finite disorder broadening,[18] one has to increase the potential strength sufficiently. This increases the width of the Landau bands and the minigaps, but at the same time increases the coupling between the Landau bands. This coupling, however, changes the structure of the energy spectrum

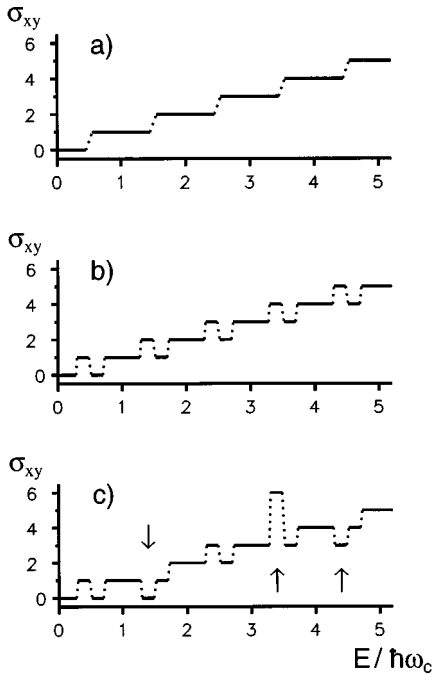

FIG. 1. The Hall conductance (solid lines) in the energy gaps in units of (e^2/h) is plotted schematically vs energy in units of cyclotron energy $\hbar\omega_c$ in the cases (a) without a periodic potential (quantum Hall effect), (b) with a periodic potential neglecting coupling of Landau bands, and (c) including coupling of Landau bands [see also Fig. 3(b)]. The magnetic flux per unit cell is 3/2. The dotted lines serve as a guide to the eye. One can see that the coupling can dramatically change the Hall conductances (arrows).

considerably,[7] and the results for the Hall conductance of Hofstadter's butterfly no longer apply [Fig. 1(c)]. The integer quantization of the Hall conductance in any gap, on the other hand, is ensured by Laughlin's argument even in this general case. Therefore the question arises: How will the Landau band coupling influence the integer values of the Hall conductance?

We will answer this question by studying the energy spectrum and the Hall conductance for different strengths of the Landau band coupling. The Hall conductance in a gap can change only if the gap closes and reopens as a function of the coupling strength. Surprisingly, this happens even for weak Landau band coupling, and we find the following principal deviations from the Hall conductance in Hofstadter's butterfly: (i) opening of previously closed gaps, (ii) rearrangement of subbands, including their contributions to the Hall conductance, and (iii) unexpected subband contributions to the Hall conductance. We finally make some remarks on the observability in lateral superlattices on semiconductor heterojunctions.

In the last years, electrons in two dimensions have also been studied under the influence of a magnetic modulation,[19] instead of an electric modulation. It was shown that there exists a connection between these two cases, e.g., the Hall in systems with magnetic modulation conductance is also quantized in an energy gap. Consequently, the phenomena discussed in this paper may also apply to these systems.

In Sec. II the model is introduced. The Hall conductances, when neglecting the Landau band coupling, are presented in Sec. III. In Sec. IV we study the influence of Landau band coupling on the energy spectrum and on the quantized values of the Hall conductance. The experimental observability of these findings is discussed in Sec. V, and Sec. VI gives concluding remarks.

II. MODEL

The one-particle Hamiltonian for an electron with charge $-e$ and effective mass m^* in a magnetic field and in a two-dimensional potential has the form

$$H=\frac{1}{2m^*}(\mathbf{p}+e\mathbf{A})^2+V(x,y), \tag{1}$$

where we neglect spin and electron-electron interaction. For a homogeneous magnetic field $\mathbf{B}$ in z direction the vector potential in the Landau gauge is given by $\mathbf{A}=B(0,x,0)$, and the periodic potential can be written in its Fourier decomposition

$$V(x,y)=V_0\sum_{r,s} v_{r,s}e^{2\pi i(rx/a+sy/b)}, \tag{2}$$

where a and b are periods in the x and y directions, respectively, and V_0 is the difference between the maximum and the minimum of the potential.

As a basis we choose the eigenfunctions of the case without potential in coordinate representation, namely,

$$\langle x,y|\nu,\theta\rangle=Ne^{i(y/b)\theta}\,\Psi_\nu\left(\frac{x}{l}+\frac{\theta l}{b}\right), \tag{3}$$

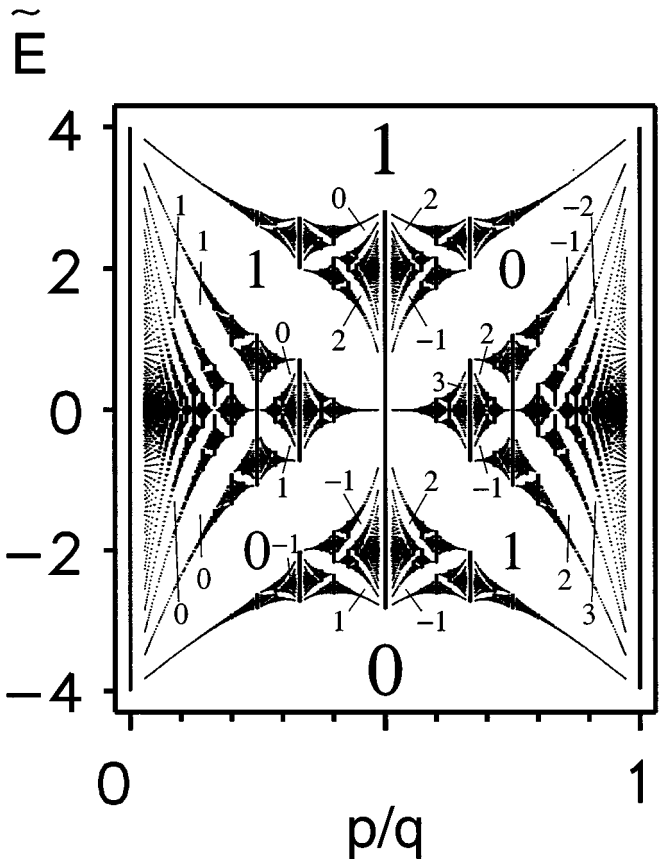

FIG. 2. Neglecting the Landau band coupling the scaled energy $\widetilde{E}$ [Eq. (21)] vs the inverse magnetic flux p/q yields for every Landau band the same spectrum, namely, the Hofstadter butterfly. The numbers in the energy gaps are the quantized Hall conductances in units of (e^2/h) to which one has to add the Landau band index ν.

where $l=\sqrt{\hbar/(eB)}$ is the magnetic length, $\Psi_\nu(z)=\exp(-z^2/2)\,H_\nu(z)$, where H_ν is the νth Hermite polynomial, normalization N,[20] $\nu=0,1,2,\ldots$, and $\theta\in\mathbb{R}$. The matrix elements of the first part of the Hamiltonian read

$$\langle\mu,\varphi|\frac{1}{2m^*}(\mathbf{p}+e\mathbf{A})^2|\nu,\theta\rangle=\hbar\omega_c(\nu+1/2)\,\delta_{\mu,\nu}\delta_{\varphi,\theta}, \tag{4}$$

which are the Landau levels with cyclotron frequency $\omega_c=eB/m^*$, and those of the potential

$$\langle\mu,\varphi|v_{r,s}e^{2\pi i(rx/a+sy/b)}|\nu,\theta\rangle = P_{\mu\nu}(r,s)e^{-i(\Phi_0/\Phi)r\varphi}\delta_{\varphi,\theta+2\pi s}, \tag{5}$$

with

$$P_{\mu\nu}(r,s)=v_{r,s}e^{irs\pi(\Phi_0/\Phi)}e^{-u/2}\left(\frac{\nu!}{\mu!}\right)^{1/2}\left(\pi\frac{\Phi_0}{\Phi}\right)^{(\mu-\nu)/2} \times(s\alpha^{-1}+ir\alpha)^{\mu-\nu}L_\nu^{\mu-\nu}(u),\quad \mu\geqslant\nu \tag{6}$$

and

$$P_{\mu\nu}(r,s)=v_{r,s}e^{irs\pi(\Phi_0/\Phi)}e^{-u/2}\left(\frac{\mu!}{\nu!}\right)^{1/2}\left(\pi\frac{\Phi_0}{\Phi}\right)^{(\nu-\mu)/2}$$
$$\times(-s\alpha^{-1}+ir\alpha)^{\nu-\mu}L_\mu^{\nu-\mu}(u),\quad \nu\geqslant\mu \tag{7}$$

$L_\mu^\nu(u)$ are the Laguerre polynomials, $\alpha=\sqrt{b/a}$, and $u=[\pi(r^2\alpha^2+s^2\alpha^{-2})\Phi_0/\Phi]$. Here Φ_0/Φ is the magnetic flux quantum $\Phi_0=h/e$ divided by the flux Φ through a unit cell of the periodic potential

$$\frac{\Phi_0}{\Phi}=\frac{h/e}{abB}. \tag{8}$$

We divide the parameter θ into $\theta=2\pi n+\vartheta$ with integer $n\in\mathbb{Z}$ and phase $\vartheta\in[0,2\pi)$, as the Hamiltonian is diagonal in ϑ. The eigenstates $|j,\vartheta\rangle$ of the Hamilton operator are decomposed into the basis states by

$$|j,\vartheta\rangle=\sum_{\nu,n}a_n^\nu(j,\vartheta)|\nu,n,\vartheta\rangle. \tag{9}$$

Inserting this into the Schrödinger equation, one obtains for every ϑ the eigenvalue equation

$$\mathbf{A}_n a_n+\sum_{s\neq0}\mathbf{T}_{n,s}a_{n+s}=\frac{E}{\hbar\omega_c}a_n, \tag{10}$$

with the vector

$$a_n=(a_n^0,a_n^1,\ldots,a_n^\nu,\ldots) \tag{11}$$

and the matrices $\mathbf{A}_n$ and $\mathbf{T}_{n,s}$,

$$\mathrm{A}_n^{\mu\nu}=(\nu+1/2)\delta_{\nu,\mu}+K\frac{\Phi_0}{\Phi}\sum_r P_{\mu\nu}(r,0)e^{-ir(2\pi n+\vartheta)(\Phi_0/\Phi)}, \tag{12}$$

$$\mathrm{T}_{n,s}^{\mu\nu}=K\frac{\Phi_0}{\Phi}\sum_r P_{\mu\nu}(r,-s)e^{-ir(2\pi n+\vartheta)(\Phi_0/\Phi)}. \tag{13}$$

Here the important parameter

$$K=2\pi m^*abV_0/h^2 \tag{14}$$

is a measure for the strength of the coupling of the Landau bands.

Equation (10) is an infinite dimensional matrix equation, which cannot be diagonalized numerically. Only if Φ_0/Φ is a rational number

$$\frac{\Phi_0}{\Phi}=\frac{h}{eBab}=\frac{p}{q}\quad\text{with}\quad p,q\in\mathbb{N}, \tag{15}$$

which means that q flux quanta penetrate p unit cells, one can make use of the magnetic translation operators.[21,22] For the vector potential in the Landau gauge they are defined by

$$M_a=e^{iya/l^2}e^{a\partial_x}\quad\text{and}\quad M_b=e^{b\partial_y}, \tag{16}$$

and displace a wave function by one unit cell in the x or y direction. The magnetic translation operators commute with the Hamilton operator, but in general not with each other. Only in the case when Φ_0/Φ is a rational number [Eq. (15)], can one enlarge the unit cell of the periodic potential by a factor of p to a new magnetic unit cell, and find

$$[M_{pa},M_b]=0. \tag{17}$$

Then the eigenfunctions of the Hamilton operator are also eigenfunctions of M_{pa} and M_b with eigenvalues $e^{i\kappa}$ and $e^{i\theta}$, respectively, and $\kappa\in[0,2\pi)$ and $\theta\in[0,2\pi q)$. It follows that

$$a_{n-q}^\mu(j,\vartheta,\kappa)=e^{i\kappa}a_n^\mu(j,\vartheta,\kappa), \tag{18}$$

so that n can be restricted to $0,1,\ldots,q-1$, producing q subbands. Furthermore, if we take only N consecutive Landau bands into account, Eq. (10) reduces to a finite $(Nq\times Nq)$ eigenvalue equation for every κ and every ϑ. As one can see from Eqs. (12) and (13), the energy spectrum in units of $\hbar\omega_c$ for a given potential shape depends only on the number of flux quanta per unit cell Φ_0/Φ and the Landau band coupling K.

III. HALL CONDUCTANCE WITHOUT LANDAU BAND COUPLING

If one neglects the coupling of the Landau bands, i.e., neglects the terms with $\nu\neq\mu$ in Eqs. (12) and (13), and takes into account only the lowest Fourier components [Eq. (2)], i.e., the $\cos x+\cos y$ potential

$$V(x,y)=\frac{V_0}{4}\left[\cos\left(\frac{2\pi x}{a}\right)+\cos\left(\frac{2\pi y}{a}\right)\right], \tag{19}$$

one obtains the Harper equation[5]

$$a_{n+1}+a_{n-1}+2\cos\left[(2\pi n+\vartheta)\frac{\Phi_0}{\Phi}\right]a_n=\widetilde{E}a_n, \tag{20}$$

with the scaled energy

$$\widetilde{E}=[E-\hbar\omega_c(\nu+1/2)]/[P_{\nu,\nu}(1,0)V_0]. \tag{21}$$

For every Landau band ν the resulting energy subbands plotted against the inverse flux p/q show the well-known Hofstadter butterfly[4] (see Fig. 2).

The Hall conductance for this case was derived by Thouless *et al.*[15] as the solution of a diophantic equation. In units of e^2/h the Hall conductance σ in the gth gap of Hofstadter's butterfly is given by

$$g=wp+\sigma q\quad\text{with}\quad|w|\leqslant q/2, \tag{22}$$

$g,p,q\in\mathbb{N}$, and $w,\sigma\in\mathbb{Z}$.[23] Figure 2 shows the Hofstadter butterfly with the Hall conductance of the lowest Landau band written in the large gaps. For the Hall conductance in higher Landau bands one has to add the Landau band index ν to these values. Figure 2 shows that the Hall conductance is not always a monotonous function of energy, as in the case without potential [Fig. 1(a)], a fact that could be the first experimentally obtainable hint for the internal band structure. But will this figure remain valid if the coupling of the Landau bands is taken into account?

IV. HALL CONDUCTANCE WITH LANDAU BAND COUPLING

The influence of the coupling between the Landau bands on the spectrum is determined by the coupling strength K

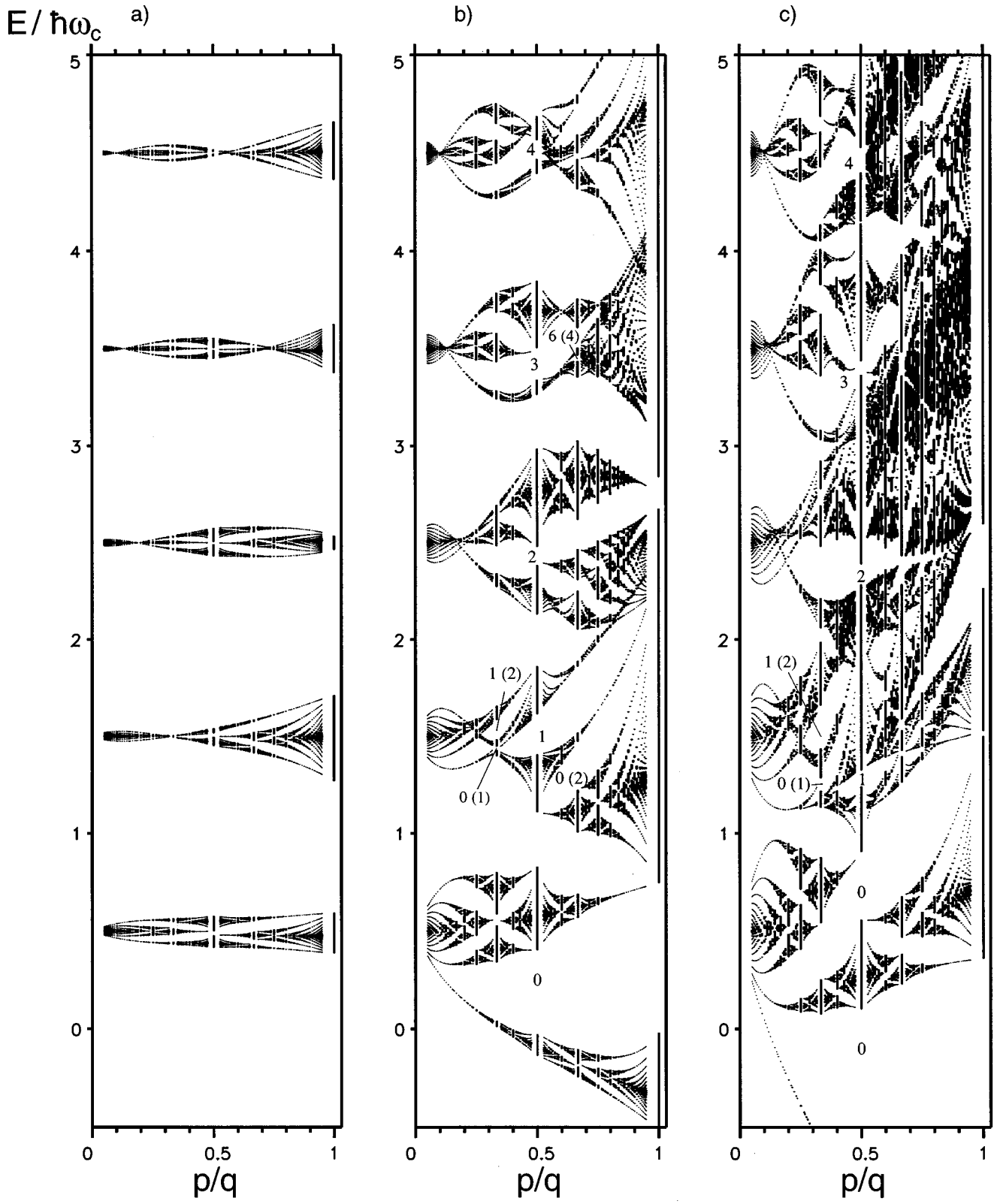

FIG. 3. The five lowest Landau bands for the cos$x$+cos$y$ potential [Eq. (19)] are plotted for increasing coupling strength $K=1$, 6, and 12. For $p/q=1/2$ the Hall conductances in the minigaps are shown, and also for $p/q=1/3$ and 2/3, if they deviate from the corresponding value of Hofstadter's butterfly (given in brackets). The number of Landau bands considered numerically is 6, 9, and 13, respectively.

[Eq. (14)]. How does an increase of the coupling strength affect the spectrum?

Figure 3 shows the energy spectrum[24] as a function of p/q for three different values of the coupling strength K for the cos$x$+cos$y$ potential (19). For small K [Fig. 3(a)] each Landau band resembles the Hofstatder butterfly multiplied with $P_{\nu,\nu}$ [Eq. (21)]. The Laguerre polynomials in this expression become zero for certain Φ_0/Φ, so that the width of

the corresponding Landau band vanishes, the so-called flatband positions. Their number increases with the Landau band index. With increasing K each Landau band becomes wider, even at the original ($K \ll 1$) flatband positions, and more distorted [Fig. 3(b)]. For p/q with even q the $q/2$th minigap, which is closed without coupling, opens due to the coupling. If the coupling is strong enough, Landau bands may even overlap. In this case the classification into Landau bands becomes meaningless [Fig. 3(c)]. Similar effects also occur for other periodic potentials, e.g., for potentials of the form

$$V(x,y) = V_0 \left[\cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right) \right]^{\beta}, \tag{23}$$

which are used as model potentials for antidots[25] [see Fig. 5(b)]. Such spectra for Bloch electrons in a magnetic field with Landau band coupling were previously studied in Refs. 7 and 9, where slight mistakes in the matrix elements of the potential, however, led to different results (in Ref. 9 only for antidot potentials [Eq. (23)] with $\beta > 2$).

The diophantic equation (22) for the Hall conductance is valid for the general case with coupling and arbitrary periodic potential only without the constraint $|w| \leqslant q/2$;[26,27] however, it then allows many solutions for the Hall conductance in a given gap. Instead, one may take advantage of a formula derived by Středa:[28] For any energy gap the Hall conductance (in units of e^2/h) is given by

$$\sigma = \frac{\partial N(E)}{\partial B} \frac{h}{e}, \tag{24}$$

with $N(E)$ the number of states per unit area having energy lower than the gap energy. As the Hall conductance in the gap is quantized, it is possible to replace $\partial N/\partial B$ by $\triangle N/\triangle B$ for adjacent rationals p_1/q_1 and p_2/q_2 which share this gap. Using the fact that the number of states per unit area in a subband is eB_i/hq_i ($i=1$, and 2), and Eq. (15), one finds

$$\sigma = \left(\frac{n_1}{p_1} - \frac{n_2}{p_2}\right) \Big/ \left(\frac{q_1}{p_1} - \frac{q_2}{p_2}\right) = \frac{n_1 p_2 - n_2 p_1}{q_1 p_2 - q_2 p_1}, \tag{25}$$

where n_i is the number of subbands below the gap for the corresponding magnetic field B_i.

The Hall conductance in a gap can only change if the gap closes and opens again as a function of the Landau band coupling.[29,19] We find three types of deviations from the Hall conductance of Hofstadter's butterfly: (i) opening of previously closed gaps, (ii) rearrangement of subbands, including their contributions to the Hall conductance, and (iii) unexpected subband contributions to the Hall conductance. We will now discuss these deviations in more detail.

(i) The gaps in the middle of a Landau band for p/q with q even are closed in Hofstadter's butterfly (Fig. 2), but, due to the coupling of Landau bands they may open (Fig. 3). Consequently, there appears an additional plateau in the Hall conductance. As an aside, we note that near flatband positions, where a Landau band is small, the minigap for $p/q = 1/2$ is often surprisingly large [see the 11th and 14th Landau band in Fig. 5(a)].

(ii) The second effect occurs, e.g., at $p/q = 1/3$ in Fig. 3(b) (see also Fig. 4). Without coupling one would expect the Hall conductance in the second minigap of the second Landau band to be 2, namely, 1 from Hofstadter's butterfly (Fig. 2) plus 1 from the one Landau band below. With coupling, instead, we find Hall conductance 1. One can understand this effect by looking at the entire spectrum for a given K (Fig. 3): With increasing p/q the uppermost (for small p/q) branch of the second Landau band is bent downward in such a way that at $p/q = 1/3$ the middle branch lies at the top of the Landau band and so does its contribution $(+1)$ to the Hall conductance. This rearrangement gives rise to the change of the Hall conductance from 2 to 1 in the second minigap. It has consequences also for $p/q = 2/3$ in the second Landau band. There the sequence of the Hall conductances in the minigaps reads, starting below the Landau band, $\sigma = \{1,0,1,2\}$ instead of $\{1,2,1,2\}$. The subband carrying the Hall conductance $(+1)$ is exchanged with the one carrying (-1).

(iii) A surprising effect is found in Fig. 3(b) [see also Fig. 1(c)]. Applying the diophantic equation (22) to the case $p/q = 2/3$, each subband carries a Hall conductance $(+1)$ or (-1). But in the fourth Landau band the sequence of the plateaus is found to be $\sigma = \{3,6,3,4\}$, which means that two of the subbands contribute instead with $(+3)$ and (-3), respectively. Something similar happens in Figs. 3(b) and 3(c) for $p/q = 1/3$: Without the coupling of the Landau bands, one expects for the second Landau band a monotonous sequence $\sigma = \{1,1,2,2\}$; with coupling it reads $\{1,0,1,2\}$, and now the lowest subband carries a negative conductance. These two examples are in full agreement with a formula derived by Dana, Avron, and Zak,[26] which gives all possible contributions of subbands to the Hall conductance for any periodic potential, namely,

$$1 = mp + \triangle\sigma q, \tag{26}$$

where $\triangle\sigma$ is the contribution of a subband and $m \in \mathbb{Z}$.

But this still leaves the question: How do such unexpected contributions arise? In Fig. 2 one sees that there exists a minigap with a Hall conductance $(+3)$ to the left of $p/q = 2/3$, which ends at the second subband of $p/q = 2/3$. In fact, as a function of the coupling strength the first minigap for $p/q = 2/3$ in the fourth Landau band closes and opens in such a way that now it includes the minigap with Hall conductance $(+3)$, giving rise to the sequence $\sigma = \{3,6,3,4\}$. Similarly, one can explain the above example at $p/q = 1/3$, where one finds a minigap to the left of the first subband of $p/q = 1/3$ with the Hall conductance (-1) (Fig. 2).

Remarkably, even for very weak Landau band coupling we found examples for subband rearrangement (ii). For vanishing coupling and for the $\cos x + \cos y$ potential (19), the width of the Landau band at flatband positions is zero. For example, in the second Landau band near $p/q = 1/3$ the bottom, middle, and top branches of the band cross in one point [Fig. 4(a)]. This degeneracy is lifted as soon as the Landau band coupling is turned on, and subbands are rearranged, including their contribution to the Hall conductance. The Hall conductance in the second Landau band at $p/q = 1/3$ thus changes from $\sigma = \{1,1,2,2\}$ to $\{1,1,1,2\}$ [Fig. 4(b)]. With increasing coupling strength the p/q range of this rearrange-

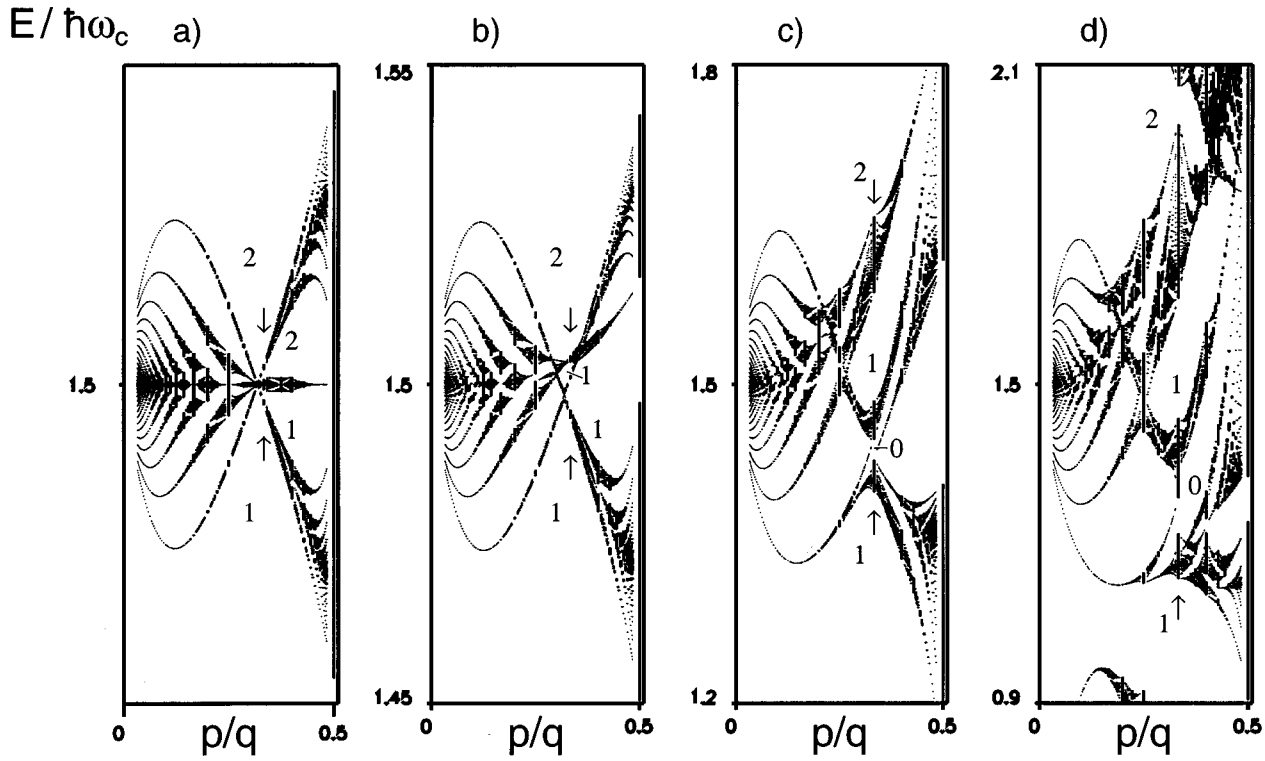

FIG. 4. The second lowest Landau band of the cos$x$+cos$y$ potential [Eq. (19)] is shown (a) without and (b) with weak ($K=1$), (c) moderate ($K=6$), and (d) strong coupling of Landau bands ($K=12$). One can see that even for a very weak periodic potential there can exist deviations from the diophantic equation (22) near flatband positions, here at $p/q=1/3$ (arrows), as discussed in the text.

ment expands [Fig. 4(c)]. Furthermore, one finds in Figs. 4(c) and 4(d) the Hall conductance 0 (instead of 1) in the first minigap of $p/q=1/3$, discussed above as an example of deviation (iii). It has its origin in the correspoding crossing of branches in Fig. 4(b).

Another example of a deviation from the diophantic equation (22) even for weak Landau band coupling can be found in the fourth Landau band near $p/q=1$ for ratios of the form $(q-1)/q$. Each subband adds one unit of conductance, except for the middle one that carries the large negative conductance $2-q$, so that the sum of the contributions for a Landau band equals one. Specifically, for $p/q=6/7$ the sequence of the values of the Hall conductance in the gaps between the subbands for the fourth Landau band reads, without coupling with increasing energy,

$$\sigma=\{3,4,5,6,1,2,3,4\}, \tag{27}$$

as can be seen in Fig. 2 with $(+3)$ added for the lower three Landau bands. Even for weak coupling [Fig. 3(a)], one instead finds

$$\sigma=\{3,4,5,0,1,2,3,4\}, \tag{28}$$

where the step to lower values occurs earlier in the sequence. One can interpret this effect by the exchange of the subband carrying (-5) with one carrying $(+1)$. Again, it is due to the sensitivity of flatband positions to the Landau band coupling.

Figure 5 compares a sequence of 15 Landau bands for the cos$x$+cos$y$ potential (19) and the antidot potential (23), and shows the deviations from the Hall conductances of Hofstadter's butterfly for $p/q=1/2$, $1/3$, and $2/3$. For the antidot potential one finds even without Landau band coupling several deviations from the diophantic equation (22) due to the different potential shape. Taking the coupling into account, we find the three principal types (i)–(iii) of deviations from the case without coupling as discussed above. Again, even weak Landau band coupling gives rise to these deviations near crossings (as a function of p/q) of branches of the spectrum.

To summarize, we find that crossings of branches of the spectrum, e.g., at flatband positions, may lead to deviations of the Hall conductance from the diophantic equation (22) even for weak coupling (Fig. 4). With increasing coupling the p/q range of these deviations expands.

V. REMARKS ON OBSERVABILITY

Bloch electrons in a magnetic field may be experimentally studied in lateral superlattices on semiconductor heterojunctions. The main obstacle for observing the subband structure is that most minigaps are small compared to the disorder broadening. A crude estimation of the disorder broadening can be given in the self-consistent Born approximation:[30] A single Landau level (or one of the q subbands) will be broadened by disorder to a sharp half-ellipse with a total width of 2Γ (or $2\Gamma/\sqrt{q}$), with Γ given by

$$\Gamma^2=\frac{2}{\pi}\frac{\hbar}{\mu m^*/e}\hbar\omega_c, \tag{29}$$

with the mobility μ. For typical values $\mu=50\ \mathrm{m}^2/(\mathrm{V\,s})$, $a=100$ nm, and $m^*=0.067m_e$ this equation gives

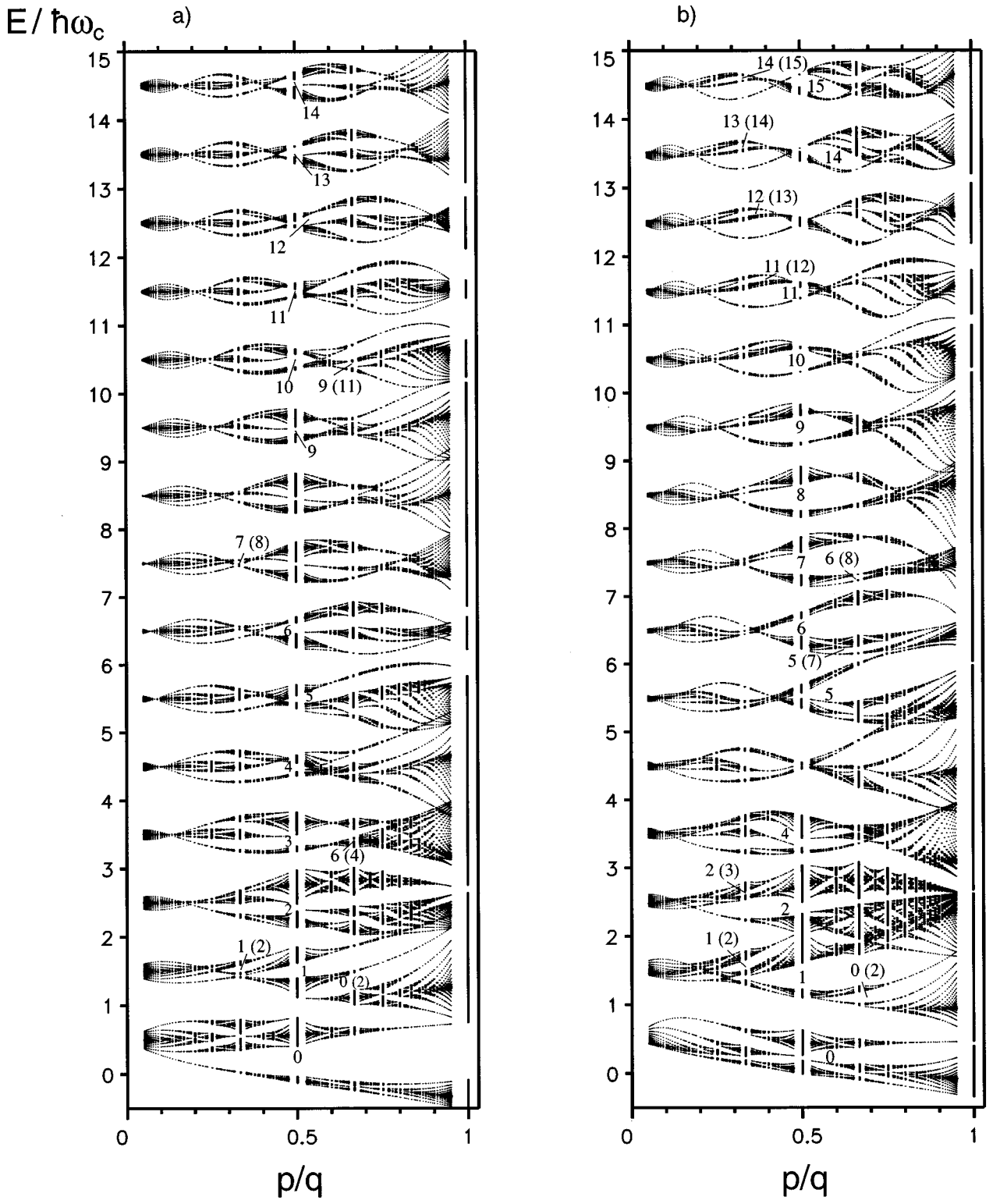

FIG. 5. The lowest 15 Landau bands are plotted for $K=6$ for the $\cos x+\cos y$ potential [Eq. (19)], and the antidot potential [Eq. (23)] with $\beta=2$. For $p/q=1/2$ the Hall conductances in the minigaps are shown, and also for $p/q=1/3$ and 2/3, if they deviate from the corresponding value of Hofstadter's butterfly (given in brackets). 21 Landau bands were numerically taken into account.

$\Gamma=0.18\sqrt{p/q}\hbar\omega_c$. All the minigaps in Fig. 3(a) and all, except for the largest ones, in Fig. 3(b) are closed by such a disorder broadening.

Thus, for a given disorder broadening one has to increase the strength of the potential in order to enlarge the internal gaps, using the fact that the width of a Landau band increases proportionally to V_0. Increasing the potential strength, however, also increases the coupling strength, which influences the spectrum. If the coupling is too strong, the Landau bands merge, many gaps are closed, and it is difficult to interpret the spectra in terms of Landau bands [Fig. 3(c)]. One has to choose the coupling strength K in

such a way that the Landau bands are as wide as possible in order to obtain observable gaps, while not overlapping with adjacent bands in the desired range of Φ_0/Φ. A quantitative estimation beyond $V_0=\hbar\omega_c$, which is equivalent to $K=\Phi/\Phi_0$, would have to consider that the shape of the Landau bands differs from band to band, and that it depends strongly on strength and Fourier components of the potential, as can be seen in Fig. 5.

Restricting ourself to the case of nonoverlapping Landau bands and $p/q<1$, we find for different potential shapes and strengths, that the largest minigaps usually occur at $p/q\approx 1/2$, 1/3, and 2/3. The corresponding Hall conductances will thus be the first to be found experimentally. They will differ from the Hall conductances of Hofstadter's butterfly, as discussed in Sec. IV and shown in Fig. 5.

VI. CONCLUSION

We have studied the energy spectrum of electrons in a two-dimensional periodic potential with perpendicular magnetic field *without* neglecting the coupling of the Landau bands. We examined the Hall conductance, since its values, which are quantized in every energy gap, contain quantitative information about the structure of the spectrum. The Landau band coupling changes this structure compared to Hofstadter's butterfly, resulting in dramatical modifications of the Hall conductance. We find the following three principal deviations from the Hall conductance in Hofstadter's butterfly: (i) opening of previously closed gaps; (ii) rearrangement of subbands, including their contributions to the Hall conductance, and (iii) unexpected subband contributions to the Hall conductance. Remarkably, even for weak Landau band coupling these changes can be found. This was explained by the occurrence of crossings of branches of the spectrum, e.g., flatband positions, which are very sensitive to the Landau band coupling.

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[1] R. Peierls, Z. Phys. **80**, 763 (1933); W. Kohn, Phys. Rev. **115**, 1460 (1959); J. Zak, *ibid.* **134**, A1602 (1964); E. Brown, *ibid.* **133**, A1038 (1964); M. Ya.Azbel, Zh. Éksp. Teor. Fiz. **46**, 929 (1964) [Sov. Phys. JETP **19**, 634 (1964)]; D. Langbein, Phys. Rev. **180**, 633 (1969); J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).

[2] L. Landau, Z. Phys. **64**, 629 (1930).

[3] F. Bloch, Z. Phys. **52**, 555 (1929).

[4] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[5] P. G. Harper, Proc. R. Soc. London, Ser. A **68**, 874 (1955).

[6] A. Rauh, Phys. Status Solidi B **69**, K9 (1975).

[7] G. Petschel and T. Geisel, Phys. Rev. Lett. **71**, 239 (1993).

[8] T. Geisel, J. Wagenhuber, P. Niebauer, and G. Obermair, Phys. Rev. Lett. **64**, 1581 (1990).

[9] O. Kühn, P. E. Selbmann, V. Fessatidis, H. L. Cui, and N. J. Horing, Phys. Rev. B **47**, 13 019 (1993); O. Kühn, P. E. Selbmann, V. Fessatidis, and H. L. Cui, J. Phys., Condens. Matter **5**, 8225 (1993).

[10] H. Silberbauer and U. Rössler, Phys. Rev. B **50**, 11 911 (1994).

[11] V. Gudmundsson and R. R. Gerhardts, Phys. Rev. B **54**, 5223 (1996).

[12] R. R. Gerhardts, D. Weiss, and U. Wulf, Phys. Rev. B **43**, 5192 (1991).

[13] T. Schlösser, K. Ensslin, J. P. Kotthaus, and M. Holland, Europhys. Lett. **33**, 683 (1996).

[14] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[15] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[16] R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).

[17] See also A. H. MacDonald, Phys. Rev. B **28**, 6713 (1983); Y. Hatsugai, *ibid.* **48**, 11 851 (1993).

[18] The influence of disorder on the Hall conductance of Hofstadter's butterfly was studied in B. Huckestein and R. N. Bhatt, Surf. Sci. **305**, 438 (1994).

[19] M. C. Chang and Q. Niu, Phys. Rev. B **50**, 10 843 (1994); M. Nielsen and P. Hedegard, *ibid.* **51**, 7679 (1995); P. D. Ye *et al.*, Phys. Rev. Lett. **74**, 3013 (1995), and references therein.

[20] D. Pfannkuche and R. R. Gerhardts, Phys. Rev. B **46**, 12 606 (1992).

[21] M. Kohmoto, Ann. Phys. (N.Y.) **160**, 343 (1985).

[22] N. A. Usov, Zh. Éksp. Teor. Fiz. **94**, 305 (1988) [Sov. Phys. JETP **67**, 2565 (1988)].

[23] A semiclassical derivation of these Hall conductances was obtained in M. C. Chang and Q. Niu, Phys. Rev. B **53**, 7010 (1996).

[24] In order to find the extrema of the energy subbands, we solve Eq. (10) numerically by standard diagonalization techniques for a number of combinations of the phases κ and ϑ, making use of symmetries.

[25] R. Fleischmann, T. Geisel, and R. Ketzmerick, Phys. Rev. Lett. **68**, 1367 (1992).

[26] I. Dana, Y. Avron, and J. Zak, J. Phys. C **18**, L679 (1985).

[27] J. E. Avron and L. G. Yaffe, Phys. Rev. Lett. **56**, 2084 (1986).

[28] P. Středa, J. Phys. C **15**, L717 (1982); **15**, L1299 (1982).

[29] Z. Tesanovic, F. Axel, and B. I. Halperin, Phys. Rev. B **39**, 8525 (1989).

[30] T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).