Energy fluctuations, Thouless energy, and conductance in the Anderson model in the ballistic and diffusive regimes

E. Cuevas* and E. Louis
Departamento de Física Aplicada, Universidad de Alicante, Apartado 99, E-03080 Alicante, Spain

M. Ortuño
Departamento de Física, Universidad de Murcia, E-30071 Murcia, Spain

J. A. Vergés
Instituto de Ciencia de Materiales de Madrid, Consejo Superior de Investigaciones Científicas, Cantoblanco, E-28049 Madrid, Spain
(Received 26 March 1997; revised manuscript received 16 June 1997)

We perform a numerical calculation of long-range energy fluctuations of two- and three-dimensional Anderson models in the ballistic and diffusive regimes. In calculating the energy fluctuations, averages are taken over different realizations of disorder, and not over energy windows at different levels. For windows of width E smaller than the critical energy E_c, fluctuations follow the logarithmic behavior characteristic of random matrix theory (RMT), no matter the degree of disorder. For energies higher than E_c, fluctuations are nearly constant and below RMT in the ballistic case, and they are higher than RMT and increase with energy in the diffusive case. The results allow a reasonably accurate estimate of E_c. The expected behavior of the critical energy with the system size and energy is reproduced by our numerical results. An efficient implementation of Kubo's formula has been used to calculate the conductance of the system. In the diffusive regime the numerical results for the adimensional conductance are in reasonable agreement with the numerical results for E_c. It is also shown that the asymptotic expression derived by Altshuler and Shklovskii for fluctuations in the diffusive regime gives results much smaller than those reported here. [S0163-1829(97)02748-3]

I. INTRODUCTION

Spectral correlations of disordered quantum systems play a fundamental role in the study of quantum chaos and in the physics of mesoscopic systems. The short-range statistical properties of the quantum level spectra of disordered systems with extended states are adequately described by means of random matrix theory (RMT).[1,2] In the absence of a magnetic field, when time-reversal symmetry is preserved, these short-range properties are closely simulated by the Gaussian orthogonal ensemble (GOE).[2,3] In particular, there is a repulsion between nearest levels, whose spacings follow Wigner-Dyson statistics.[2,4–6] This is true for diffusive as well as for ballistic systems, provided that the typical value of the disorder energy exceeds the mean level spacing, which implies that the energy spectra differ drastically from the clean case. The ballistic regime is an intermediate regime (rather than a crossover point) between the diffusive and the clean limits.[7,8]

To characterize the statistical properties of the spectra it is customary to map each real spectrum $\{\epsilon_i\}$ onto the unfolded spectrum $\{E_i\}$ through $E_i=\overline{N(\epsilon_i)}$, where $N(\epsilon_i)$ is the number of levels up to an energy ϵ_i, and the overline denotes the average over different disorder realizations. The spectrum $\{E_i\}$ has on the average a constant mean spacing equal to 1. After rescaling, the variance of the number of levels $N(E)$ is calculated in an energy window $[E_F, E_F+E]$ (E_F being the Fermi energy). For the GOE, this variance, which is independent of E_F, is equal to[2,6]

$$\Sigma^2(E)=\frac{2}{\pi^2}\left\{\ln(2\pi E)+\gamma+1+\tfrac{1}{2}\mathrm{Si}^2(\pi E)-\frac{\pi}{2}\mathrm{Si}(\pi E)-\cos(2\pi E)-\mathrm{Ci}(2\pi E)+\pi^2 E\left[1-\frac{2}{\pi}\mathrm{Si}(2\pi E)\right]\right\}, \tag{1}$$

where $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the sine and cosine integrals, respectively, and $\gamma=0.5772\ldots$ is Euler's constant. Note that as, for sufficiently large E, the dominant term in Eq. (1) is the logarithm, the dependence of Σ^2 on E is much weaker than the linear dependence expected for uncorrelated energy levels. Long-range fluctuations of the GOE are thus very small, a characteristic known as spectral rigidity.

Disordered systems with extended states can be either diffusive or ballistic depending on whether the elastic mean free path l is smaller or larger than the linear size of the system L, respectively. The spectra of quantum Hamiltonians of diffusive systems, even though they obey Wigner-Dyson statistics, exhibit much larger long-range fluctuations than the GOE (Refs. 3 and 9) for energies greater than a critical energy, known as the Thouless energy.[10] This critical energy ϵ_c^D is associated with the inverse transport time through the system, and is proportional to the Fermi velocity v_F and the elastic mean free path l, and inversely proportional to L^2,

$$\epsilon_c^D=\frac{\hbar v_F l}{dL^2}, \tag{2}$$

where d is the dimensionality of the system. For energies greater than the Thouless energy, Al'tshuler and Shklovskii obtained that the fluctuations are given by[3]

$$\Sigma^2(E)=c_d\left(\frac{E}{E_c^D}\right)^{d/2}, \tag{3}$$

with $c_2=1/4\pi^2$ and $c_3=2^{1/2}/6\pi^3$. E_c^D is the critical energy in normalized units, which will be calculated below.

In ballistic systems ($l>L$), there is also a critical energy ϵ_c^B limiting the applicability of RMT.[8,11] This critical energy loses its meaning as an inverse transport time $t_f^{-1}=v_F/L$ through the system, which now is disorder independent. Atland, Gefen, and Montambaux[8] predicted that this critical energy is given by

$$\epsilon_c^B=\frac{\hbar v_F}{l}. \tag{4}$$

It plays a role analogous to the Thouless energy for diffusive systems. This ballistic regime is valid for critical energies larger than the average level spacing, below which the clean limit sets in.

In this work we carry out a systematic numerical analysis of the energy spectra of the Anderson model. We vary the disorder energy W and the size of the sample in order to consider both diffusive and ballistic systems. We will ascertain that the elastic mean free path is indeed the key parameter as long as the long-range energy-level fluctuations are concerned. We also show that the limiting value of logarithmic dependence of the energy fluctuations of ballistic systems corresponds to the critical energy ϵ_c^B. In contrast to the case of diffusive systems, the fluctuations of ballistic systems are smaller than those of the GOE for energies greater than the critical one, as already anticipated in Ref. 7. The numerical results for the fluctuations are used to compute the critical energy both in the ballistic and diffusive cases. The results for E_c follow qualitatively the behavior predicted by the expressions given above. In the diffusive case the adimensional conductance is also calculated and shown to be similar (in magnitude) to the critical energy, as expected.[8,12] The numerical results for E_c allow us to show that Eq. (3) gives fluctuations much smaller than the numerical results reported in this work.

II. MODEL AND METHODS

A. Hamiltonian

We compute the energy spectrum of the Anderson Hamiltonian

$$H=\sum_i w_i c_i^\dagger c_i+\sum_{\langle ij\rangle} t_{ij}c_i^\dagger c_j, \tag{5}$$

where the operator c_i destroys an electron on site i, and t_{ij} is the hopping integral between sites i and j (the symbol $\langle ij\rangle$ restricts the sum to nearest-neighbor sites). We take $t_{ij}=t=-1$ and consider square and cubic lattices for two-dimensional (2D) and 3D systems, respectively. Calculations have been carried out on clusters of sizes up to $L=60$, for 2D systems, and up to $L=16$, for 3D systems. The energies of the atomic levels w_i are randomly chosen between $-W/2$ and $W/2$. For each value of the disorder W, we consider at least 1000 different realizations. The Schwarz algorithm for symmetric band matrices[13] was used to compute the whole spectrum. We have checked that for all sets of parameters W and L considered, spacings among nearest-neighbor levels are distributed according to Wigner-Dyson statistics. For very small values of W this is not always the case. The analysis of the results in this limit is also made difficult by finite-size oscillations in the density of states.[14]

B. Fluctuations

As remarked in Sec. I, the standard way of measuring $\Sigma^2(E)$ consists first, of transforming real spectra onto unfolded spectra: each real spectrum $\{\epsilon_i\}$ is mapped onto an unfolded spectrum $\{E_i\}$ through $E_i=\overline{N(\epsilon_i)}$, where $\overline{N(\epsilon)}$ is the averaged number of levels up to an energy ϵ. Second, the variance of the number of levels found in an interval of fixed length E is directly obtained from the unfolded spectra. In this section, we present results for the variance of the number of levels obtained by a different numerical procedure which gives no fluctuations in the clean limit (see below). Of course, we have checked that overall features are procedure independent, i.e., our method coincides with the standard one when subtleties are ignored. Nevertheless, fine details of the number variance are better given by our more direct definition than by the standard procedure.

For each value of W and L, an energy window around ϵ_F is defined: $[\epsilon_F-\delta/2,\epsilon_F+\delta/2]$. Random energies ϵ_1 and ϵ_2 within this interval are chosen. The mean number of levels within interval $[\epsilon_1,\epsilon_2]$ is given by

$$\overline{N}=\langle N(\epsilon_1)-N(\epsilon_2)\rangle_W, \tag{6}$$

where $N(\epsilon)$ gives the total number of states below energy ϵ for a generic disorder realization; $\langle\ \rangle_W$ indicates averaging over disorder configurations. In the same way, the mean of the squared number of levels in this energy interval is given by

$$\overline{N^2}=\langle[N(\epsilon_1)-N(\epsilon_2)]^2\rangle_W. \tag{7}$$

The variance of the number of levels contained in the energy interval $[\epsilon_1,\epsilon_2]$ is simply

$$\Sigma^2(\overline{N})=\overline{N^2}-\overline{N}^2. \tag{8}$$

This provides a value of $\Sigma^2(\epsilon_F,E)$ (note that $E=\overline{N}$). The sequence is repeated a large number of times for randomly selected energy intervals $[\epsilon_1,\epsilon_2]$ within $[\epsilon_F-\delta/2,\epsilon_F+\delta/2]$, until a relatively smooth value is obtained for the variance of the number of levels *averaged* over energy intervals containing the same number of levels. The last step implies averaging over the selected energy region around ϵ_F.

Note that only fluctuations induced by disorder are taken into account by our method. Therefore, the number fluctuation is strictly zero for a single spectrum. On the other hand, the standard method can be used even in this situation. The variance is defined by

$$\Sigma^2(E)=[N(E_F+E)-N(E_F)-E]^2. \tag{9}$$

The subtlety within this definition is the way followed for unfolding just one spectrum. Sometimes, the asymtotic form of $N_{\epsilon\to\infty}(\epsilon)$ (Weyl formula) (Ref. 4) is used even in the lower part of the spectrum, and number fluctuations are measured relative to $N_{\epsilon\to\infty}(\epsilon)$. Most frequently, a large number of levels is used to measure the average level spacing, and then fluctuations relative to the mean value are measured in smaller intervals. In this way, it has been proved that crystalline spectra are uncorrelated up to some energy extent, and consequently, the $\Sigma^2(E)$ statistics is Poisson:[15]

$$\Sigma^2_{\text{Poisson}}(E)=E. \tag{10}$$

Let us remark that our procedure is quite appropriate for disordered systems, as in our case. We have checked that our numerical procedure recovers both the analytical results for random series of energy levels [see Eq. (10)] and for the eigenvalue series of matrices belonging to the GOE [see Eq. (1)]. The logarithmic dependence on E makes this value much smaller than the one corresponding to uncorrelated spectra.

C. Mean free path and critical energy

The ratio l/L is the parameter which controls whether a system is diffusive or ballistic, and hence the type of critical energy and the fluctuations above this energy. The mean free path is approximately given by[16,17]

$$l\approx v_F\tau_F=\frac{\hbar v_F}{2|\mathrm{Im}\Sigma(\epsilon_F)|}, \tag{11}$$

where τ_F is the relaxation time at the Fermi energy. In the present case the Fermi velocity v_F can be directly obtained from the dispersion relation, which in two dimensions is $\epsilon_{\mathbf{k}}=-2[\cos(k_x)+\cos(k_y)]$. The result is

$$\hbar v_F=\langle 2\sqrt{\sin^2(k_x^F)+\sin^2(k_y^F)}\rangle, \tag{12}$$

where the average is carried out over the Fermi surface. In three dimensions, an additional term (k_z) has to be inserted in the square root. On the other hand, the self-energy $\Sigma(\epsilon_F)$ is roughly proportional to the diagonal element of the unperturbed Green's function,[16]

$$\Sigma(\epsilon_F)\approx\langle w_i^2\rangle G_0(i,i;\epsilon_F). \tag{13}$$

i indicates a lattice site and w_i its random diagonal energy. For the Anderson model, we arrive at

$$l\approx\frac{6\hbar v_F}{W^2\pi\nu(\epsilon_F)}, \tag{14}$$

where $\nu(\epsilon_F)$ is the density of states at the Fermi energy. This result qualitatively reproduces the coherent potential approximation results reported in Ref. 17. The parameter l/L depends on the disorder W and on the length of the system L via the product $A=W^2L$.

The critical energy in normalized units is calculated by multiplying the Thouless energy, Eq. (2), by the density of states and by L^d. In the diffusive regime this takes the form

$$E_c^D=\frac{6\hbar^2v_F^2}{Ad\pi}L^{d-1}. \tag{15}$$

Note that in the diffusive regime the critical energy in normalized units E_c coincides with the adimensional conductance of the system.[8,12] The corresponding result for the ballistic case is

$$E_c^B=\frac{A\pi\nu^2(\epsilon_F)}{6}L^{d-1}. \tag{16}$$

If the Fermi velocity is replaced by Eq. (12) and the density of states at the Fermi energy is approximated by $1/4d$, the final expressions for the mean free path and for the critical energy can be written in terms of dimensionless magnitudes only, i.e., disorder energies W are measured in units of the hopping energy t, lengths in units of the lattice constant a, and $\mathbf{k}$ in units of a^{-1}.

D. Conductance

In calculating the conductance of the system we use the standard Kubo formula[18,19] conveniently implemented for the problem at hand.[20] The static electrical conductivity is given by

$$S=\sigma_{xx}(0)=-2\frac{e^2}{h}\mathrm{Tr}[(\hbar\hat{v}_x)\mathrm{Im}\,\hat{G}(E)(\hbar\hat{v}_x)\mathrm{Im}\,\hat{G}(E)], \tag{17}$$

where $\mathrm{Im}\,\hat{G}(E)$ is calculated from the advanced and retarded Green's functions

$$\mathrm{Im}\,\hat{G}(E)=\frac{1}{2i}[\hat{G}^+(E)-\hat{G}^-(E)]. \tag{18}$$

The velocity (current) operator $\hat{v}_x$ is related to the position operator $\hat{x}$ through the equation of motion

$$i\hbar\hat{v}_x=[\hat{H},\hat{x}], \tag{19}$$

where H is the Hamiltonian in Eq. (5).

In carrying out numerical calculations, we connected L^d clusters to semi-infinite leads with L^{d-1} channels, and one atomic orbital per site with the same hopping integral as in the Hamiltonian of Eq. (5). Within the one-electron approximation and if linear response is assumed, the exact form of the electrical field does not matter. Therefore, an abrupt potential drop at one of the two cluster sides provides the simplest numerical implementation of the Kubo formula. In this case, operator $\hat{v}_x$ has finite matrix elements on only two adjacent layers, and Green's functions are just needed for this restricted subset of sites owing to the trace appearing in Eq. (17). Green's functions are given by

$$[E-\hat{H}-\hat{\Sigma}_r(E)-\hat{\Sigma}_l(E)]\hat{G}(E)=I, \tag{20}$$

where $\hat{\Sigma}_r(E)$ and $\hat{\Sigma}_l(E)$ are the self-energies introduced by the semi-infinite right and left leads, respectively.[21] Thus, the evaluation of the matrix elements of $\mathrm{Im}\hat{G}(E)$ is efficiently achieved applying an LU decomposition to the band matrix

$$\langle i|E-\hat{H}-\hat{\Sigma}_r(E)-\hat{\Sigma}_l(E)|j\rangle, \tag{21}$$

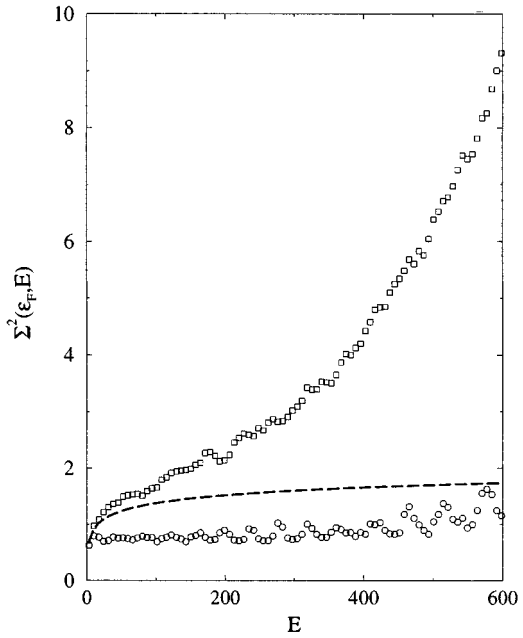

FIG. 1. $\Sigma^2(\epsilon_F,E)$ as a function of E for the 2D Anderson model with $L=50$. The results correspond to $\epsilon_F=-2$ and $W^2L=20$ (circles) and $W^2L=160$ (squares). The thick broken line gives the GOE fluctuations.

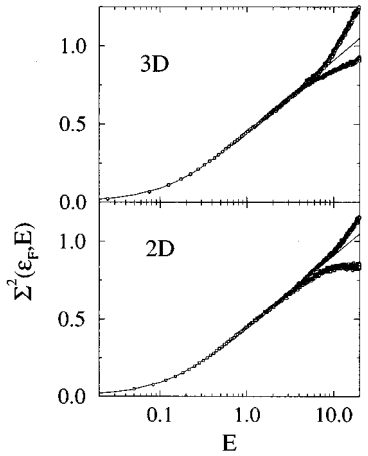

FIG. 2. $\Sigma^2(\epsilon_F,E)$ as a function of E for the Anderson model on a logarithmic scale. The results correspond to $L=50$ and $\epsilon_F=-2$ in two dimensions and $L=10$ and $\epsilon_F=0.0$ in three dimensions. $W^2L=20$ (squares) and $W^2L=160$ (circles). The solid line gives the GOE fluctuations.

where i and j are sites of the L^d clusters. Self-energies are calculated for any energy E using a recurrent algorithm, and retarded or advanced Green's functions are obtained using the corresponding self-energies in Eq. (20). Since lead self-energies add an imaginary part to some diagonal elements of the Hamiltonian, there is no need to add a small imaginary part to the energy E before solving the set of linear algebraic equations. The advantages of the procedures followed here were discussed in detail in Ref. 20.

III. RESULTS

A. Two dimensions

In Fig. 1 we show $\Sigma^2(\epsilon_F,E)$ as a function of energy in normalized units for the 2D Anderson model with $L=50$ and $\epsilon_F=-2$. The squares correspond to the diffusive regime ($W^2L=160$), and the circles to the ballistic case ($W^2L=20$). The ratio l/L for these two cases is approximately 0.2 and 1.7, respectively. The thick broken line gives the logarithmic behavior of the GOE. As expected, in the diffusive case, fluctuations follow the GOE up to the critical energy and then increase beyond the GOE, although at a much lower pace than in the completely uncorrelated case (see above). A critical energy which sets a limit to the applicability of RMT also exists in the ballistic regime, which corresponds to the inverse of the relaxation time (or, equivalently, the imaginary part of the self-energy). For energies higher than the critical one, the fluctuations are smaller than predicted by RMT, and remain almost constant as soon as they deviate from the logarithmic behavior. For large energies a slow increase of the fluctuations with E is observed. This behavior has already been found by several authors in geometrical quantum chaotic billiards.[22,24,23] Our results for intermediate values of W indicate that the transition from the ballistic to the diffusive regimes is smooth. The trends shown in Fig. 1 are very general, and can be better exhibited on a log-linear plot. Figure 2 shows this type of plot for the variance $\Sigma^2(\epsilon_F,E)$ as a function of the normalized energy. The solid line corresponds to the result for the GOE. At small E the curves follow the same behavior as the GOE until they deviate from it to larger fluctuations in the diffusive cases, and to smaller fluctuations in the ballistic cases.

We investigated the variation of the critical energies with W and L. In determining E_c we used curves similar to those plotted in Fig. 2. The criterium we followed is that E_c is the energy at which the numerical results for the fluctuations deviate 2% from the GOE. Instead of varying W and L independently, we keep l/L (and thus the product $A=W^2L$) constant. We choose the two values of A of Figs. 1 and 2, one corresponding to the diffusive regime ($A=160$) and the other to the ballistic regime ($A=20$), and analyze the change in E_c as a function of the remaining free parameter. The results for the critical energy as a function of the system size for the diffusive regime are ilustrated in Fig. 3. The fitted straight line is $E_c^D=0.14L-0.61$. The numerical results for the conductance S are also shown in Fig. 3. The fitted straight line is in this case, $S=0.13L-0.22$. The agreement between the two magnitudes (calculated through completely different methods) is excellent. The results do not, however, agree quantitatively with that given by Eq. (15), namely, $E_c^D=0.03L$. A similar plot for the ballistic case is shown in Fig. 4. The fitted line is now $E_c^B=0.06L-0.12$, which again is rather different from the analytical result of Eq. (16), $E_c^B=0.18L$.

We observe significant variations in E_c with the initial energy ϵ_F. This is a consequence of changes in the Fermi velocity (diffusive case) and in the density of states (ballistic regime), which in the tight-binding model used here depend appreciably on the energy, particularly the former. Numerical results for E_c, multiplied by a constant factor (1.3) in the diffusive case and the square of the Fermi velocity, as a function of the energy are plotted in Fig. 5. Note that E_c is roughly proportional to v_F^2, in qualitative agreement with

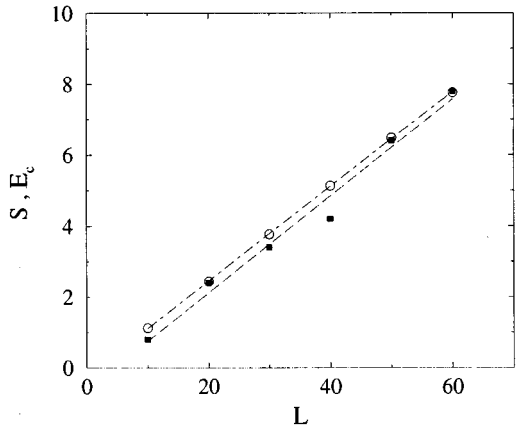

FIG. 3. Critical energy E_c (squares) and conductance S (circles) for the 2D Anderson model as a function of the system size, in the diffusive (D) regime. The results correspond to an energy average over the range $(-2.2,-1.8)$ and $W^2L=160$. The fitted straight lines are $E_c^D=0.14L-0.61$ and $S=0.13L-0.22$.

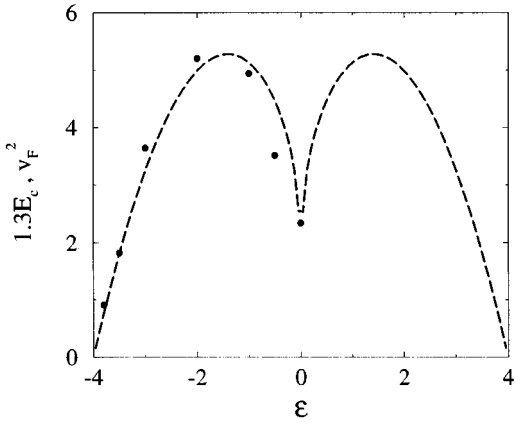

FIG. 5. Critical energy E_c (circles) multiplied by a constant factor 1.3 (see text) and square of the Fermi velocity v_F^2 averaged over the Fermi surface as defined in Eq. (12) (dashed line) as a function of energy, for the 2D Anderson model in the diffusive regime. The results correspond to $L=40$ and $W^2L=160$. Both magnitudes were calculated by averaging for each energy ϵ in the range $(\epsilon-0.2,\epsilon+0.2)$.

Eq. (15). It should be noted, however, that as already pointed out this equation does not quantitatively agree with the numerical results. In fact it gives a numerical factor which is three times larger than that used in Fig. 5, i.e., $\hbar^2 v_F^2=4.2E_c$.

A question of interest is the asymptotic behavior of the fluctuations predicted by Altshuler and Shklovskii[3] in the diffusive case, Eq. (3), for $E\gg E_c^D$ in two dimensions. By using the value of the critical energy reported in Fig. 3, and the same parameters of Fig. 1, we obtain $\Sigma^2(E)\approx 0.004E$. This gives much smaller fluctuations than the numerical results of Fig. 1. A source of errors is the actual value of E_c. However, our numerical results for E_c should be rather accurate, as indicated by the agreement between E_c and the conductance. On the other hand, we note that the linear behavior predicted in Ref. 3 is not seen in Fig. 1.

An important point to note is that the way to obtain such clean results is by averaging over different disorder realizations only, and keeping the actual initial energy fired. The fact that the spectra are not translationally invariant in energy may blur the results if averages over energy windows at different levels are taken, because the region with larger fluctuations can quickly dominate. We have checked, however, that if additional averages over energy windows away from regions where fluctuations are abnormally large (the edges and center of the band) are taken, the present results are not changed.

Referring to the results in the ballistic regime (fluctuations below the GOE), we note that simulations on quantum chaotic billiards (commonly assumed to be ballistic systems) show dissimilar results. In particular, while many authors report fluctuations similar to or larger than the GOE,[25,26,28,29] other studies obtain fluctuations smaller than the GOE,[22,24,27] as found here for the ballistic case. Although one cannot discard technical reasons in explaining these discrepancies, such as the type of variance or the way in which averages are carried out, it is more appealing to ascribe them to differences in the behavior of the different billiards. At this point it is interesting to note that the possibility of a chaotic billiard showing diffusive behavior cannot be discarded.[23]

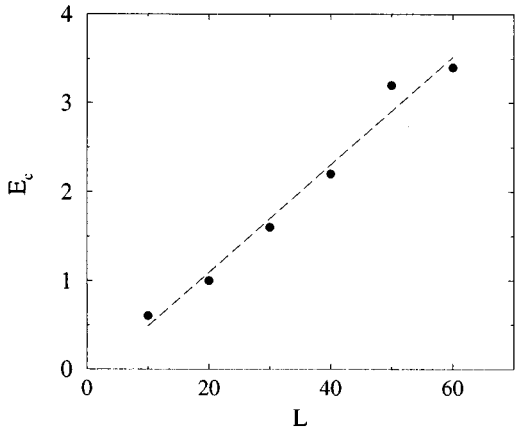

FIG. 4. Critical energy as a function of the system size for the 2D Anderson model in the ballistic (B) regime. The results correspond to an energy average over the range $(-2.2,-1.8)$ and $W^2L=20$. The fitted straight line is $E_c^B=0.06L-0.12$.

B. Three dimensions

3D systems show the same general trends as 2D systems. Figure 6 depicts the variance $\Sigma^2(\epsilon_F,E)$ versus E for the 3D Anderson model with $L=10$ and $\epsilon_F=0.0$. The squares correspond to the diffusive case ($W^2L=160$) and the circles to the ballistic regime ($W^2L=20$). The mean free path in these two cases is $l/L=0.23$ and 1.8, respectively. The broken line gives the logarithmic behavior of the GOE. We note that now fluctuations in the ballistic regime almost coincide with the GOE, indicating that, for similar values of the mean free paths, fluctuations in three dimensional are higher than in two dimensions, and that the crossover to the diffusive regime occurs at smaller values of l/L. At high energies the increase in the fluctuations in the diffusive case with energy slows down probably due to finite-size effects, as already noted in Ref. 20.

On the other hand, our results show that if the product W^2L is kept constant, the critical energy increases as L^2, as

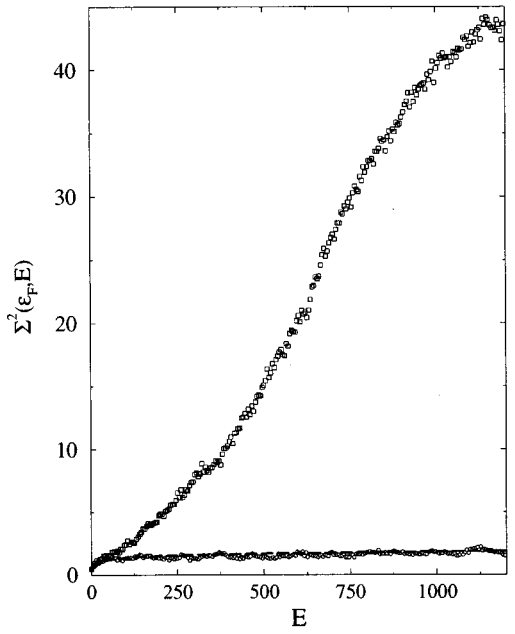

FIG. 6. $\Sigma^2(\epsilon_F,E)$ as a function of E for the 3D Anderson model with $L=10$. The results correspond to $\epsilon_F=0.0$ and $W^2L=20$ (circles) and $W^2L=160$ (squares). The thick broken line gives the GOE fluctuations, which is not clearly seen as it almost coincides with numerical results for the fluctuations in the ballistic case.

predicted by Eq. (15). In calculating E_c we used the same criteria and type of curves (see Fig. 2) of two dimensions. The numerical results for E_c are shown in Fig. 7, along with the conductance S results. The fitted straight lines are $E_c^D=0.075L^2-0.28$ and $S=0.14L^2+0.09$. Now, although the agreement between the two calculations is not as good as in two dimensions, it can be considered as reasonable. Equation (15) in this case gives $E_c^D=0.028L^2$, again significantly smaller than our numerical results for both E_c and S.

As regards the asymptotic behavior predicted in Ref. 3, we note that Eq. (3) gives, for the parameters of Fig. 6, $\Sigma^2(\epsilon_F,E)\approx 4\times 10^{-4}E^{3/2}$, which is much smaller than the numerical results reported in Fig. 6. In a recent study of this

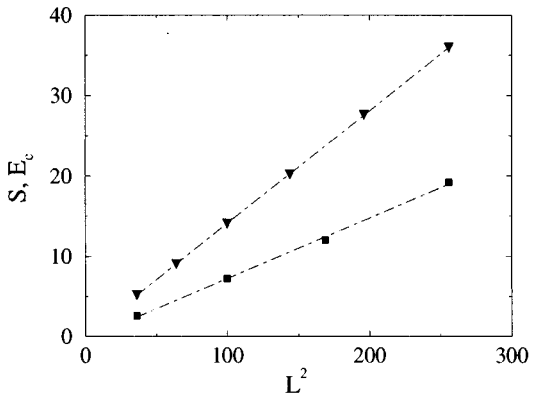

FIG. 7. Critical energy E_c (squares) and conductance S (triangles) for the 3D Anderson model as a function of the system size, in the diffusive (D) regime. The results correspond to an energy average over the range $(-0.2,0.2)$ and $W^2L=160$. The fitted straight lines are $E_c^D=0.075L^2-0.28$ and $S=0.14L^2+0.09$.

system Braun and Montambaux[30] found a reasonable agreement between Eq. (15) and their numerical results (apart from the finite-size effects mentioned above). Again, E_c is a crucial parameter. We note, however, that our results for E_c are supported by our analysis of the conductance; in fact, had we used the conductance instead of E_c (see Fig. 7), even smaller fluctuations would have resulted.

C. Thermodynamic limit

The results discussed above are valid for mesoscopic systems (finite L). The question is whether they will still hold in the thermodynamic limit. The answer to this question depends on how this limit is approached. In the standard procedure one increases the size of the system L, keeping all other parameters constant. In this procedure not only l/L tends to zero, but also the system size L eventually becomes larger than the localization length. As a consequence, all states become localized, and fluctuations follow Poisson's distribution.

The other way to approach the thermodynamic limit is that proposed here, namely, to keep l/L (or W^2L) constant. When this is done, the critical energy increases as L^{d-1} (both in the diffusive and in the ballistic cases). This implies that, although the region where fluctuations follow the GOE becomes relatively narrower as L is increased (note that the total number of levels increases as L^d), its absolute width increases. Consequently, we expect the scenario found for finite L also to be valid in the limit of L tending to infinity. Note, however, that although the limiting procedure proposed here is methodologically useful, the one relevant to the experiments is the standard procedure discussed in the preceeding paragraph.

IV. CONCLUDING REMARKS

We have performed a systematic numerical analysis of energy fluctuations in the Anderson model in two and three dimensions, over a wide range of parameters, in the ballistic and diffusive regimes. Averages were taken over different realizations of disorder. The main conclusions of our study are the following: (i) Both in two and three dimensions, fluctuations follow the GOE up to a critical energy E_c. (ii) Above E_c, fluctuations increase with energy and are larger than the GOE in the diffusive case, and basically constant and smaller than the GOE in the ballistic case. (iii) The numerical results for E_c so obtained qualitatively reproduce the expected behavior, namely, they are proportional to L^{d-1} in the two regimes. (iv) Furthermore, in the diffusive regime E_c (the Thouless energy) is also proportional to the square of the Fermi velocity. (v) In diffusive systems, the adimensional conductance reasonably agrees with E_c, giving support to the results of the present investigation. (vi) The numerical results for E_c were used to show that the asymptotic expressions for the fluctuations in the diffusive regime, reported in Ref. 3, give results much smaller than those found here, both for two and three dimensions.

ACKNOWLEDGMENTS

This work was supported in part by the Spanish DGICYT (Grants Nos. PB96-1118 and PB96-0085). We thank Professor M. Pollak for useful comments and for a critical reading of the manuscript.

*Permanent address: Departamento de Física, Universidad de Murcia, E-30071 Murcia, Spain.

[1] K. B. Efetov, Adv. Phys. **32**, 53 (1983); O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); E. Akkermans and G. Montambaux, *ibid.* **68**, 642 (1992).

[2] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).

[3] B. L. Al'tshuler and B. I. Shklovskii, Zh. Éksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].

[4] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[5] O. Bohigas, in *Chaos and Quantum Physics*, Proceedings of the Les Houches Summer School, Session L11, 1989, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, New York, 1991).

[6] M. L. Mehta, *Random Matrices* (Academic, San Diego, 1991).

[7] A. Altland and Y. Gefen, Phys. Rev. B **51**, 10 671 (1995).

[8] A. Altland, Y. Gefen, and G. Montambaux, Phys. Rev. Lett. **76**, 1130 (1996).

[9] V. E. Kravtsov and I. V. Lerner, Phys. Rev. Lett. **74**, 2563 (1995).

[10] D. J. Thouless, Phys. Rev. Lett. **39**, 1167 (1977).

[11] A. V. Andreev, O. Agam, B. D. Simons, and B. L. Altshuler, Phys. Rev. Lett. **76**, 3947 (1996).

[12] J. T. Edwards and D. J. Thouless, J. Phys. C **5**, 807 (1972).

[13] H. R. Schwarz, Numer. Math. **12**, 231 (1968).

[14] V. Gasparian, E. Louis, and J. A. Vergés, Phys. Rev. B **53**, 6967 (1996).

[15] G. Casati, B. V. Chirikov, and I. Guarneri, Phys. Rev. Lett. **54**, 1350 (1985).

[16] E. N. Economou, *Green's Functions in Quantum Physics* (Springer-Verlag, Berlin, 1983).

[17] P. Sheng, *Introduction to Wave Scattering, Localization, and Mesoscopic Systems* (Academic, San Diego, 1995).

[18] A. Bastin, C. Lewiner, O. Betbeder-Matibet, and P. Nozières, J. Phys. Chem. Solids **32**, 1811 (1971).

[19] S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge, 1995).

[20] J.A. Vergés (unpublished).

[21] S. Datta, *Electronic Transport in Mesoscopic Systems* (Ref. 19), p. 197.

[22] A. Bäcker, F. Steiner, and P. Stifter, Phys. Rev. E **52**, 2463 (1995).

[23] E. Louis, E. Cuevas, J. A. Vergés, and M. Ortuño, Phys. Rev. B **56**, 2120 (1997).

[24] R. Aurich, F. Scheffler, and F. Steiner, Phys. Rev. E **51**, 4173 (1995).

[25] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[26] T. Cheon and T. D. Cohen, Phys. Rev. Lett. **62**, 2769 (1989).

[27] E. B. Bogomolny, B. Georgeot, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **69**, 1477 (1992).

[28] H. Alt, H.-D. Gräf, H. L. Harney, R. Hofferbert, H. Lengeler, C. Rangacharyulu, A. Richter, and P. Schardt, Phys. Rev. E **50**, R1 (1994).

[29] H. Primack and U. Smilansky, Phys. Rev. Lett. **74**, 4831 (1995).

[30] D. Braun and G. Montambaux, Phys. Rev. B **52**, 13 903 (1995).