Specific-heat amplitude ratio near a Lifshitz point

Ibraheem Nasser*
Physics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

Afaf Abdel-Hady and R. Folk†
Institute of Theoretical Physics, University of Linz, Linz, Austria

(Received 30 September 1996)

The specific-heat amplitude ratio A_+/A_- (SHAR) in the neighborhood of an m-type Lifshitz point has been calculated in the Gaussian approximation. The crossover behavior of the SHAR between the Lifshitz behavior and the usual isotropic critical behavior is considered. This crossover turns out to depend on the temperature as well as the anisotropy of the dispersion. Renormalization-group calculations in one-loop order of the SHAR generalize the result for the usual isotropic critical point. [S0163-1829(97)04125-8]

I. INTRODUCTION

In the vicinity of a critical point, the amplitudes of the power laws characterizing divergent quantities, like susceptibility and specific heat, are nonuniversal. However, from the two-scale factor hypothesis,[1] it follows that certain combinations of these amplitudes are universal, i.e., they depend only on the quantities characterizing the universality class to which the critical point belongs. One important example is the amplitude ratio of the divergence of the specific heat above and below the critical point (SHAR). Its asymptotic value has been calculated for different universality classes. Here we consider the SHAR for a Lifshitz point.[2] The Lifshitz point critical behavior constitutes a universality class on its own and one expects the SHAR to be different from its value at the usual critical point. The renormalization-group theory calculates these asymptotic values.

In the region further away from the critical point, one expects that, although fluctuations are present, their effect may be approximated by taking them into account only in quadratic order, neglecting the fourth-order interaction of the fluctuations. This is the region of application of the so-called Gaussian model.[3] One may also calculate the SHAR within this region and compare with experiments. This has been done for superconductors[4] and suggested for magnetic systems.[5] Since the amplitude ratios within this region are further away from the critical point, one does not expect them to be universal, but they may depend on the nonuniversal parameters of the Gaussian model.[4] In this paper we consider the SHAR in the Gaussian region near a Lifshitz point and the crossover of SHAR to its value at the usual isotropic critical point. A new feature comes into play since the divergence of the specific heat in the Gaussian region of a Lifshitz point ($\alpha_{LG}=[4+m/2-d]/2$) is different from the corresponding divergence for the usual critical point ($\alpha_{IG}=[4-d]/2$). Therefore the SHAR becomes temperature dependent in the crossover region.

The crossover between the Lifshitz behavior and the usual isotropic critical behavior in the nonordered phase has been studied in (Ref. 6) and the crossover function of the specific heat was calculated in one-loop order. The results of these calculations above the phase-transition line might be helpful for the understanding of the experimental results in systems whose phase diagram show a Lifshitz point.[7] Examples studied extensively are, e.g., ferroelectrics[8] or the magnetic system as MnP.[9] For this magnet the SHAR has been measured and a value $A_+/A_-=0.65$ has been found.[10] This amplitude ratio is considered theoretically for a Lifshitz point. With the help of the renormalization-group theory[11] and by using field-theoretical methods,[12] we calculate the asymptotic value of SHAR in one-loop order. As we shall see, because of mathematical difficulties, the analytic results do not go beyond one-loop order. This limits our prediction of the SHAR.

The paper is arranged as follows: In Sec. II we introduce the theoretical model for calculating SHAR at the Lifshitz point using the Gaussian model. A comparison between pure Lifshitz point and the pure Gaussian isotropic point is made. In Sec. III, the crossover, within the Gaussian model between the Lifshitz and isotropic points, will be introduced and discussed. Section IV will be devoted to a review of generalized renormalization-group procedure and the calculation of the SHAR, which is followed by a discussion of the experimental result of Ref. 10 in the light of our results in Sec. IV. Some important formulas will be given in the appendixes which follow our discussion. Through our paper, we will use the subscripts + and − for the values above and below T_c, respectively.

II. GAUSSIAN MODEL AT THE LIFSHITZ POINT

A. Above T_c

In the following we will briefly recapitulate the calculation of the specific heat within the Gaussian approximation for the Lifshitz point. Let us consider the partition function

$$Z=\int_{\varphi} e^{-H[\varphi]}, \tag{1}$$

where the Landau-Ginzburg-Wilson (LGW) free-energy functional, describing the critical behavior of a system that exhibits an m-fold Lifshitz point, can be approximated by

$$H[\varphi]=\frac{1}{2}\int_k (r+p^2+q^4)\varphi(\mathbf{k})\varphi(-\mathbf{k}). \tag{2}$$

Here $\varphi(\mathbf{k})$ represents the order parameter, e.g., magnetization, polarization, etc. The d-dimensional wave vector $\mathbf{k}$ is decomposed into $\mathbf{q}$ and $\mathbf{p}$ components of dimensions m and $(d-m)$, respectively. In Eq. (2) we used the notations $\int_k=\int[d^{d-m}pd^mq/(2\pi)^d]$, and $r=(T-T_c)/T_c$ as the reduced temperature.

The specific heat is computed according to the definition

$$C=-T\frac{\partial^2 F}{\partial T^2}, \tag{3}$$

where F is the free energy per unit volume and is related to Z by the relation $F=-(1/V)\ln Z$. Calculating the functional integral we obtain

$$C_+=\int\frac{d^{d-m}pd^mq}{(2\pi)^d}\frac{1}{(r+p^2+q^4)^2}=I_2(r,g=0,h=1). \tag{4}$$

See Appendix A for detailed definitions of the integral I_2. After performing the integration in Eq. (4) and only keeping the most singular part, the specific heat reads

$$C_+=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)\Gamma\left(\frac{\tilde{\epsilon}}{2}\right)r^{-\tilde{\epsilon}/2}=A_+r^{-\tilde{\epsilon}/2}, \tag{5}$$

where $\tilde{\epsilon}=4+m/2-d$ and the specific heat exponent α_{LG} is equal to $\tilde{\epsilon}/2$

B. Below T_c

Below T_c one has to observe the finite value of the order parameter. In order to get a finite partition function and a phase with broken symmetry one has to go one step further in the expansion and take into account the fourth-order term. Expanding the fluctuation around the order parameter by means of

$$\varphi(\mathbf{k})=\langle\varphi\rangle_{\mathbf{k}=0}+\varphi(\mathbf{k}), \tag{6}$$

one also obtains a quadratic k-independent contribution from the fourth-order term, which leads to a factor of 2 in front of the r term. In the present case, the Gaussian H reads

$$H[\varphi]=\frac{1}{2}\int_k (2|r|+p^2+q^4)\varphi(\mathbf{k})\varphi(-\mathbf{k}), \tag{7}$$

and one immediately calculates the most singular part of the specific heat as

$$C_-=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)\Gamma\left(\frac{\tilde{\epsilon}}{2}\right)2^2|r|^{-\tilde{\epsilon}/2}=A_-|r|^{-\tilde{\epsilon}/2}. \tag{8}$$

The factor 2^2 comes from the shift of $|r|$ to $2|r|$ at $T<T_c$ and the application of Eq. (3)

It is straightforward to generalize our procedure for n components of the order parameter. If one takes into account that, below T_c, the leading singular contribution comes from the parallel fluctuations of the order parameter only, then this will introduce a factor of n into the ratio, i.e.,

$$\frac{C_+}{C_-}=\frac{A_+}{A_-}=\frac{n}{2^{d/2-m/4}}. \tag{9}$$

This may be compared with the ratio of the purely isotropic Gaussian point (formally $m=0$)

$$\frac{A_+}{A_-}=\frac{n}{2^{d/2}}. \tag{10}$$

Note that this value of the ratio was found from the singular part of the specific heat which goes like $r^{-\epsilon/2}$, where $\epsilon=4-d$ and the specific-heat exponent α_{IG} is equal to $\epsilon/2$.

III. CROSSOVER WITHIN GAUSSIAN MODEL

In this case the Landau-Ginzburg-Wilson (LGW) free-energy functional is written as

$$H[\varphi]=\frac{1}{2}\int_k (r+p^2+gq^2+hq^4)\varphi(\mathbf{k})\varphi(-\mathbf{k}), \tag{11}$$

with the same definitions as in Eq. (2), but with g and h as parameters of the dispersion. H contains now the limits of a pure Lifshitz behavior $(g=0)$ and a pure isotropic behavior ($h=0$, the finite value of g may be scaled away by choosing a proper scale of the wave vector components $\mathbf{q}$). We have to keep in mind what we mentioned earlier, that below T_c the parameter r has to be replaced by $2|r|$. The specific heat above T_c reads

$$C_+=\int\frac{d^{d-m}pd^mq}{(2\pi)^d}\frac{1}{(r+p^2+gq^2+hq^4)^2}=I_2(r,g,h). \tag{12}$$

The I_2 integral is related to the hypergeometric function ${}_2F_1(2-d+m/2,3/2-d/2;3/2-d/2+m/2;Q)$ with

$$Q(r)=\frac{\tilde{g}-2\sqrt{r}}{\tilde{g}+2\sqrt{r}}$$

and can be calculated in a closed form.[13] Its analytic properties will be explained in more detail in Appendix A.

The amplitude ratio becomes (we generalize to an n-component order parameter)

$$\frac{A_+}{A_-}=\frac{n}{2^2}\frac{I_2(r,g,h)}{I_2(2|r|,g,h)}. \tag{13}$$

In Appendix A we calculate the limiting values of this expression. The limit of pure Lifshitz behavior is obtained for $g=0$. On the other hand, in the limit $h\to 0$ or $r\to 0$ at finite g, we recover Eq. (10). For general dimensions of d, the ratio remains dependent on all the parameters. Note however that the limiting values do not depend on the values of h or g. The independence property is to be expected in the asymptotic region, but we are in the Gaussian region where the amplitudes may be (and in our case are) nonuniversal.

Let us now discuss the physical case of $d=3$. Then SHAR simplifies considerably because of the property ${}_2F_1(-1+m/2,0;m/2;Q)=1$, and we have

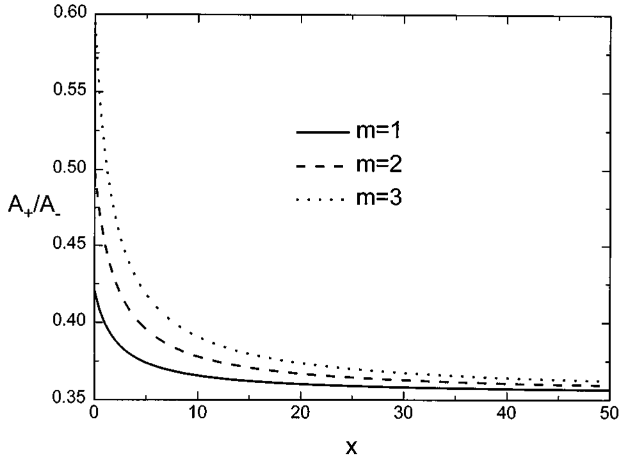

FIG. 1. The specific heat amplitude ratio in the Gaussian model, as a function of $x=\tilde{g}/\sqrt{r}$, for different values of m. $x=0$ and $x\rightarrow\infty$ correspond to the Lifshitz point value and the usual isotropic critical point, respectively.

$$\frac{A_+}{A_-}=\frac{n}{2\sqrt{2}}\left(\frac{x+2\sqrt{2}}{x+2}\right)^{m/2}. \tag{14}$$

We introduced the ''scaling variable'' $x=g/\sqrt{hr}$. Figure 1 shows our calculation result of the SHAR as a function of x for $m=1,\ 2$, and 3. One can see the pronounced effect of small values of x on the shape of the SHAR. At $x=0$, all the curves start at the Lifshitz point value (e.g., $A_+/A_-=0.420$ for $m=1$). With increasing x, the SHAR tends to decrease to the usual isotropic point value ($A_+/A_-=0.354$).

One may ask if this simple form extrapolates to other dimensions too, so we are proposing an empirical formula for general d, x, and m, which covers both limits and is exact at $d=3$,

$$\frac{A_+}{A_-}=\frac{1}{2^{d/2}}\left(\frac{x+2\sqrt{2}}{x+2}\right)^{m/2}. \tag{15}$$

The formula has been checked for different values of d, x, and m. A maximal error of 4%, compared with the exact formula Eqs. (13) and (A7), was found.

IV. ASYMPTOTICS AT THE LIFSHITZ POINT

Approaching T_c the Gaussian model breaks down and the fourth-order term of the fluctuations have to be taken into account. Thus the LGW free-energy functional reads

$$H[\varphi]=\frac{1}{2}\int_k(r_0+p^2+h_0q^4)\varphi(\mathbf{k})\varphi(-\mathbf{k})$$
$$+\frac{u_0}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\varphi(\mathbf{k}_3)\varphi(\mathbf{k}_4)\delta\left[\sum k_i\right]. \tag{16}$$

The coefficients in the LGW functional are unrenormalized ones (as before in the Gaussian model), marked by the subscript zero. The renormalization proceeds along the same lines as in Ref. 14. In particular we have, after introducing the momentum scale μ,

$$r_0=Z_\varphi^{-1}Z_r r,\quad u_0=\mu^\epsilon Z_\varphi^{-2}Z_u u K_{d-m,m}^{-1},$$
$$h_0=\mu^{-2}Z_\varphi^{-1}Z_h h,\varphi_o=Z_\varphi^{1/2}\varphi \tag{17}$$

In one-loop order $Z_\varphi=Z_h=1$, and the perturbational expansion parameter, for example, Z_u and Z_r, is performed in the effective interaction $\tilde{u}=\frac{1}{4}\Gamma(m/4)uh^{-m/4}$ (see below). Explicitly,

$$Z_r=1+\frac{1}{2}\frac{\tilde{u}}{\tilde{\epsilon}},\quad Z_u=1+\frac{3}{2}\frac{\tilde{u}}{\tilde{\epsilon}}.$$

Within the renormalization-group theory the specific heat C_+ above T_c is given by the cummulant[12] and calculated from the vertex function $\Gamma^{(0,2)}$ by

$$C_+=\left\langle\frac{1}{2}\varphi^2\frac{1}{2}\varphi^2\right\rangle=-\Gamma_+^{(0,2)}. \tag{18}$$

and below T_c by[15]

$$C_-(r,u,\mu)=-\Gamma_-^{(0,2)}+\frac{(\Gamma_-^{(1,1)})^2}{\Gamma_-^{(0,2)}}. \tag{19}$$

In order to obtain $C_\pm$ quantities using the field-renormalization-group procedure, we have to calculate the renormalized vertex function $\Gamma_\pm^{(0,2)}$, by solving the renormalization-group equations (RGE's) in the form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_u(\Omega)\frac{\partial}{\partial u}+\zeta_r(\Omega)\left\{2+r\frac{\partial}{\partial r}\right\}\right.$$
$$\left.+[2-\zeta_\varphi+\zeta_h(\Omega)]h\frac{\partial}{\partial h}\right]\Gamma_\pm^{(0,2)}(r,\Omega,\mu)=\mu^{-\epsilon}B(\Omega,\mu). \tag{20}$$

In Eq. (20), we have defined $\zeta_i(\Omega)=\mu(\partial/\partial\mu)\ln Z_i^{-1}|_0$, $i=\varphi,\ r,\ h$, and $\beta_u=\mu(\partial/\partial\mu)u|_0$. $\Omega\equiv(h,u)$, the symbol $|_0$ indicates that all derivatives are to be taken at fixed bare parameter r_0, h_0, and u_0. The inhomogeneity of Eq. (20), which is related to the additive renormalization of the specific heat, has the form

$$B(\Omega,\mu)=-\mu^\epsilon Z_r^2\mu\frac{d}{d\mu}Z_r^{-2}[Z_r^2\Gamma_B^{(0,2)}(k=0)]_{\text{sing}}. \tag{21}$$

Using the method of characteristics,[16] the general solution of Eq. (20) reads[6]

$$\Gamma_{\pm}^{(0,2)}(r,\Omega,\mu)=(\mu l)^{-\epsilon}e^{2\int_1^l(d\rho/\rho)\zeta_r(\rho)}\times\left\{-F_{\pm}\left(\frac{r(l)}{\mu^2 l^2},\quad u(l),\quad h(l)\right)+\int_1^l\frac{d\rho}{\rho}B(\rho)e^{\int_1^l(d\rho'/\rho')[2\zeta_r(\rho')-\epsilon]}\right\},\tag{22}$$

where l is an arbitrary flow parameter to be chosen suitably, and $F_{\pm}$ is the dimensionless vertex function. The functions $r(l)$, $u(l)$, and $h(l)$ fulfill the equations

$$l\frac{dr(l)}{dl}=\zeta_r r(l),\quad l\frac{du(l)}{dl}=\beta_u,\quad l\frac{dh(l)}{dl}=(2-\zeta_\varphi+\zeta_h)h.\tag{23}$$

Here we are going to adopt the scaling procedure of our previous work[14] which leads us to a scaling solution of the RGE's in the form

$$\Gamma_{\pm}^{(0,2)}(r,h,u,\mu)=(\mu l)^{-\tilde{\epsilon}}h^{-(m/4)}e^{\int_1^l(d\rho/\rho)\{2\zeta_r(\rho)-m/4[\zeta_\varphi(\rho)-\zeta_h(\rho)]\}}\left\{-\tilde{F}_{\pm}\left(\frac{r(l)}{\mu^2 l^2},\tilde{u}(l)\right)+\int_1^l\frac{d\rho}{\rho}\tilde{B}(\tilde{u})e^{\int_1^l(d\rho'/\rho')\{2\zeta_r(\rho)-m/4[\zeta_\varphi(\rho)-\zeta_h(\rho)]-\tilde{\epsilon}\}}\right\},\tag{24}$$

with the flow equations

$$l\frac{d\tilde{u}(l)}{dl}=h^{-m/4}\beta_u-\frac{m}{4}\tilde{u}(2-\zeta_\varphi+\zeta_h)=\beta_{\tilde{u}}.$$

Due to the behavior of the Lifshitz point, the susceptibility is characterized by two correlation lengths diverging differently[17] and this leads to two different ν exponents, $\nu_\perp$ and $\nu_\parallel$, perpendicular and parallel to the m-dimensional $\mathbf{q}$ direction.

Usually one chooses $r(l)/l^2=1$. This choice leads to the connection between the temperatures $r_{\pm}=|T_{\pm}-T_c|/T_c$ and the flow parameter $l=l(r_{\pm})$ via the solution of the flow equation for $r(1)=r_{\pm}$. So, at the fixed point, the specific heat is given by

$$C_{\pm}(r^*,u^*)\sim l_{\pm}^{-\alpha(2-\zeta_r^*)}\left\{F_{\pm}+\frac{B^*}{\alpha(2-\zeta_r^*)}\right\},\tag{25}$$

where

$$\alpha=\frac{\tilde{\epsilon}-2\zeta_r^*+m/4\,(\zeta_\varphi^*-\zeta_h^*)}{2-\zeta_r^*}.$$

By taking the relations $l_+\sim r_+^{1/(2-\zeta_r^*)}$ and $l_-\sim(-2r_-)^{1/(2-\zeta_r^*)}$ into account, and defining the amplitude ratio $A_{\pm}$ through the relation

$$C_{\pm}=A_{\pm}|r|^{-\alpha},\quad r\to\pm 0,\tag{26}$$

then the amplitude ratio is $[\nu_\perp=(2-\zeta_r^*)^{-1}]$ calculated as

$$\frac{A_+}{A_-}=2^{\alpha}\frac{\tilde{B}^*\nu_\perp+\alpha\tilde{F}_+^*}{\tilde{B}^*\nu_\perp+\alpha\tilde{F}_-^*}.\tag{27}$$

This generalizes the expression for the SHAR derived by Dohm[18] for the usual isotropic system.

For the Lifshitz point, the scaling functions $\tilde{F}_{\pm}$ and $\tilde{B}$ have to be calculated by the perturbation theory for $n=1$. In one-loop order one recovers the same results as for the usual isotropic φ^4 model, the only difference being that the effective fourth-order interaction appears. In particular it is found (see Appendix B) that $\tilde{B}^*=(1/2)K_{d-m,m}$, $\tilde{F}_+^*=-(1/4)K_{d-m,m}$ and $\tilde{F}_-^*=(3/\tilde{u}^*-1)K_{d-m,m}$. The exponents in one loop order are $\alpha_L=\tilde{\epsilon}/6$ and $\nu_\perp=1/2(1+\tilde{\epsilon}/6)$ and the fixed-point value $\tilde{u}^*=(2/3)\tilde{\epsilon}$. Thus we end up with the result

$$\frac{A_+}{A_-}=\frac{2^{\alpha_L}}{4}.\tag{28}$$

Let us compare this with the SHAR for an isotropic case ($d=4-\epsilon$) and $n=1$, which is known as[18]

$$\frac{A_+}{A_-}=\frac{2^{\alpha_I}}{4}(1+\epsilon).\tag{29}$$

The ϵ term results from a two-loop calculation. In order to get this correction of order ϵ, the value in Eq. (29) of α_I $(=\epsilon/6-29\epsilon^2/324)$ and u^* has to be known to the order of ϵ^2. In the case of the Lifshitz point, α_L and $\tilde{u}^*$ are only known to the order of $\tilde{\epsilon}$.

V. DISCUSSION

Our calculations of the crossover behavior in the Gaussian approximation are valid on approaching the Lifshitz point from the side of the transition curve between the paramagnetic and ferromagnetic phase. This is just the region where the experiments of Ref. 10 in MnP were performed. Note that in comparing with the experiment our theoretical temperature field has to be identfied with the magnetic field perpendicular to the easy axis and the specific heat with the corresponding magnetic susceptibility. One expects, from

TABLE I. Specific-heat amplitude ratio for $d=3$, and $m=n=1$. The value of the corresponding exponent α is given in the bracket, the values for the isotropic case are taken from (Ref. 19).

	Isotropic	Lifshitz
Gaussian	0.35(0.500)	0.42 (0.750)
One-loop	0.28(0.167)	0.30 (0.25)
Two-loop	0.53(0.077)	
Field theory	0.541	
Experiment	0.63–0.48	0.65 (Ref. 10).

measurements at different temperatures $T>T_L$ and the magnetic field H approaching the critical value H_c on the paraferro line, to observe a crossover between Ising behavior and Lifshitz behavior. This has been found in the experiments (see Fig. 5 in Ref. 10). Qualitatively it corresponds to the crossover function displayed in Fig. 1 valid in the Gaussian region.

Regarding the absolute values of the SHAR, we have collected them in Table I for $d=3$, and $m=n=1$. All values are obtained so far for the usual isotropic critical point as well as for the Lifshitz point. In one-loop order the only difference to the isotropic case comes from the different specific-heat exponents α. This leads to an increase of the SHAR by roughly 7% (second row of Table I). A much larger increase of 20% can be seen in the Gaussian region (first row of Table I). In the isotropic case, the next order in ϵ doubles the SHAR compared to the lowest order. We expect a similar effect for the Lifshitz point. Taking the hyperscaling law and experimental values for the exponents $\nu_\parallel$ and $\nu_\perp$, a value of $\alpha_L=0.49$ (compared to values of $\alpha_I\approx 0.1$) was estimated.[10] This would give a value of the SHAR without the $\tilde{\epsilon}$ correction of 0.35. Near the Lifshitz point, in fact different specific-heat exponents were measured, above and below, and this was attributed to the closeness of the first-order phase transition between the ferromagnetic and fan phase. Several other problems arise in extracting the SHAR, e.g., that one has to subtract the background values of the specific heat in order to get the singular part. Nevertheless, so far, the experimental result for the SHAR seem to be in agreement with the theoretical estimates made from our results. Further investigations both on the experimental side as well as on the theoretical side, seems to be necessary, although a two-loop calculation is expected to be outside the analytical possibilities.

ACKNOWLEDGMENTS

I.N. would like to express his thanks for the warm hospitality extended by the Institute of Theoretical Physics at Linz during his visit in the summer of 1996. KFUPM support is also gratefully acknowledged.

APPENDIX A: LIFSHITZ INTEGRAL

Our main interest in this appendix is the calculation of the integral $I_2(r)$, which we used to calculate the specific heat and SHAR, and study its behavior at different limits. The Lifshitz integral can be expressed in the form

$$I_S(r)=(2\pi)^{-d}\int d^{d-m}p\int d^mqG^S(r),\quad S=1,2, \tag{A1}$$

where G^S is the Gaussian propagator and equal to $(r+p^2+gq^2+hq^4)^{-S}$, with $I_2(r)=-\partial I_1(r)/\partial r$. Other arguments, such as g and h, have been suppressed for simplicity.

Using standard techniques[6,13] for the evaluation of the integral $I_1(r)$ one has

$$I_1(r)=Dt_1(r)t_2(r), \tag{A2}$$

where

$$t_1(r)=[\tilde{g}+2\sqrt{r}]^{-2a},\quad t_2(r)={}_2F_1(2a,2a-c+1;c;Q),$$

$$Q(r)=\frac{\tilde{g}-2\sqrt{r}}{\tilde{g}+2\sqrt{r}}, \tag{A3}$$

$D=2^{4a-2}h^{-m/4}K_{d-m,m}\Gamma(\beta)B(m/2,2a)$, $a=\beta-m/4$, $c=\beta+1/2$, $\tilde{g}=g/\sqrt{h}$, $\beta=1-(d-m)/2$, and $K_{d-m,m}=\Gamma(1-\beta)S_{d-m}S_m$, with $S_d=[2^{d-1}\pi^{d/2}\Gamma(d/2)]^{-1}$ is the standard geometrical factor of the integration. The functions Γ, B, and ${}_2F_1$ are the Euler's gamma, normal beta, and hypergeometric functions,[20] respectively. Differentiating Eq. (A2) we get

$$I_2(r)=-D\{t_1(r)t_2'(r)+t_1'(r)t_2(r)\}, \tag{A4}$$

with

$$t_1'(r)=\frac{\partial t_1(r)}{\partial r}=-\frac{2a}{\sqrt{r}}[\tilde{g}+2\sqrt{r}]^{-2a-1}, \tag{A5}$$

$$t_2'(r)=\frac{\partial t_2(r)}{\partial r}=\frac{2a(2a-c+1)}{c}\times{}_2F_1[2a+1,2a-c+2;c+1;Q(r)]Q'(r), \tag{A6}$$

and

$$Q'(r)=\frac{\partial Q(r)}{\partial r}=\frac{-2\tilde{g}}{(\tilde{g}+2\sqrt{r})^2}.$$

Finally, the exact form of the integral's ratio reads

$$\frac{I_2(r)}{I_2(2r)}=\frac{t_1(r)t_2'(r)+t_1'(r)t_2(r)}{t_1(2r)t_2'(2r)+t_1'(2r)t_2(2r)}. \tag{A7}$$

In the following, Eq. (A7) will be used to deduce the exact limiting cases

(i) $d=3$.

For this case Eq. (A7) simplifies considerably, since the second argument of ${}_2F_1$ will be equal to zero. Consequently $t_2(r)=t_2(2r)=1$, and $t_2'(r)=t_2'(2r)=0$. Then Eq. (A7) gives

$$\frac{I_2(r)}{I_2(2r)}=\frac{t_1'(r)}{t_1'(2r)}=\frac{\frac{(1-m/2)}{\sqrt{r}}[\widetilde{g}+2\sqrt{r}]^{-m/2}}{\frac{(1-m/2)}{\sqrt{2r}}[\widetilde{g}+2\sqrt{2r}]^{-m/2}}$$

$$=\sqrt{2}\left(\frac{\widetilde{g}+2\sqrt{2r}}{\widetilde{g}+2\sqrt{r}}\right)^{m/2} \tag{A8}$$

$$=\sqrt{2}\left(\frac{x+2\sqrt{2}}{x+2}\right)^{m/2}, \tag{A9}$$

where $x=\widetilde{g}/\sqrt{r}$ has been introduced.

(ii) $g=0$.

In this case Q' and $t_2'(r)=0$, hence from Eqs. (A7) and (A5) one can find

$$\frac{I_2(r)}{I_2(2r)}=\frac{t_1'(r)}{t_1'(2r)}=\sqrt{2}\left(\frac{\widetilde{g}+2\sqrt{2r}}{\widetilde{g}+2\sqrt{r}}\right)^{3-d+m/2}$$

$$=\sqrt{2}\left(\frac{x+2\sqrt{2}}{x+2}\right)^{3-d+m/2}. \tag{A10}$$

Equation (A10) covers the pure Lifshitz case when $x\rightarrow 0$, see Eq. (8).

(iii) $r=0$.

This case is problematic and should be handled with care. The main question here is how to extract the r dependent of the I_2 integral as $r\rightarrow 0$. The answer is the following: because of the special form of the second argument of the hypergeometric function in Eq. (A3), it can be replaced by the equivalent form [see Ref. 20, Eq. (15.3.19)]

$$I_1(r)=D(2\widetilde{g})^{-2a}{}_2F_1\left(a,a+\frac{1}{2};c;1-\frac{4r}{\widetilde{g}^2}\right). \tag{A11}$$

Differentiating Eq. (A11) gives

$$I_2(r)=\frac{2a(2a+1)}{\widetilde{g}^2c}D(2\widetilde{g})^{-2a}$$

$$\times{}_2F_1\left(a+1,a+\frac{3}{2};c+1;1-\frac{4r}{\widetilde{g}^2}\right),$$

and using the standard formula[20]

$${}_2F_1(\nu,b;\delta;1-t)=t^{\delta-\nu-b}{}_2F_1(\delta-\nu,\delta-b;\delta;1-t)$$

in our case, where $\delta-\nu-b=c-2a-3/2=-2+d/2$, one obtains

$$\frac{I_2(r)}{I_2(2r)}=\frac{(4r/\widetilde{g}^2)^{-2+d/2}}{(8r/\widetilde{g}^2)^{-2+d/2}}$$

$$\times\frac{{}_2F_1(c-a,c-a-1/2;c+1;1-4r/\widetilde{g}^2)}{{}_2F_1(c-a,c-a-1/2;c+1;1-8r/\widetilde{g}^2)}$$

$$=\frac{1}{2^{-2+d/2}}. \tag{A12}$$

Equation (A12) covers the pure isotropic case when $g>0$, see Eq. (9). By similar arguments, the limit $h\rightarrow 0$ can be treated and the result for the isotropic case is recovered.

APPENDIX B: RENORMALIZATION GROUP CALCULATIONS

In this appendix, we will use the ϵ-expansion method to formulate our integrals and calculate their related function. In the limit of $g\rightarrow 0$, Eq. (A2) gives

$$I_1(r,h)=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)\Gamma\left(\frac{\widetilde{\epsilon}}{2}-1\right)h^{-m/4}r^{1-\widetilde{\epsilon}/2}. \tag{B1}$$

To extract the pole from Γ function in the above equation, we can use the relation[20]

$$\Gamma\left(\frac{\widetilde{\epsilon}}{2}-1\right)=\frac{\Gamma(\widetilde{\epsilon}/2+1)}{\widetilde{\epsilon}/2(\widetilde{\epsilon}/2-1)}\approx-\frac{2}{\widetilde{\epsilon}},$$

then

$$I_1(r,h)=-\frac{1}{4}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)h^{-m/4}\frac{1}{\widetilde{\epsilon}}r^{1-\widetilde{\epsilon}/2}. \tag{B2}$$

Also,

$$I_2(r,h)=-\frac{\partial I_1(r,h)}{\partial r}$$

$$=\frac{1}{4}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)h^{-m/4}\frac{(1-\widetilde{\epsilon}/2)}{\widetilde{\epsilon}}r^{-\widetilde{\epsilon}/2}. \tag{B3}$$

To calculate the specific heat we have to calculate the renormalized vertex function $\hat{\Gamma}_{\pm}^{(0,2)}$.

For $T>T_c$

$$\hat{\Gamma}_{+}^{(0,2)}=Z_r^2\Gamma_{B+}^{(0,2)}-[Z_r^2\Gamma_B^{(0,2)}]_{\text{sing}}$$

$$=Z_r^2\left[-\frac{1}{2}I_2(r_o,h_o)\right]-[Z_r^2\Gamma_B^{(0,2)}]_{\text{sing}}, \tag{B4}$$

where the integral $I_2(r,h)$ is defined in Eq. (B3), then

$$\hat{\Gamma}_{+}^{(0,2)}=-\frac{1}{2\widetilde{\epsilon}}K_{d-m,m}\left[1-\frac{\widetilde{\epsilon}}{2}-\frac{\widetilde{\epsilon}}{2}\ln\left(\frac{r}{\mu^2}\right)+O(\widetilde{\epsilon}^2)\right]$$

$$+\frac{1}{2\widetilde{\epsilon}}K_{d-m,m}. \tag{B5}$$

Using the condition $r/\mu^2=1$ we have

$$\widetilde{F}_{+}=-\hat{\Gamma}_{+}^{(0,2)}=-\frac{1}{4}K_{d-m,m}. \tag{B6}$$

For $T<T_c$

$$\hat{\Gamma}_{-}^{(0,2)}=Z_r^2\Gamma_{B-}^{(0,2)}-[Z_r^2\Gamma_{B+}^{(0,2)}]_{\rm sing}$$

$$=Z_r^2\left[\frac{3}{u_o}+2I_2(-2r_o)\right]-[Z_r^2\Gamma_{B+}^{(0,2)}]_{\rm sing}, \qquad \text{(B7)}$$

and using Eq. (19) we finally get

$$\tilde{F}_{-}=K_{d-m,m}\left(1+\frac{\tilde{u}}{\tilde{\epsilon}}\right)\left\{\frac{3}{\tilde{u}}-\frac{9}{2\tilde{\epsilon}}+\frac{2}{\tilde{\epsilon}}\right.$$

$$\left.-1-\ln\left(-\frac{2r}{\mu^2}\right)+\frac{\tilde{\epsilon}}{2}\ln\left(-\frac{2r}{\mu^2}\right)\right\}-\frac{1}{2\tilde{\epsilon}}K_{d-m,m}$$

$$=\left[\frac{3}{\tilde{u}}-1\right]K_{d-m,m}, \qquad \text{(B8)}$$

where the condition $-2r/\mu^2=1$ has been used. The inhomogeneity follows from Eq. (21) and the last part of Eq. (B5),

$$\tilde{B}=\frac{1}{2}K_{d-m,m}.$$

[*]Electronic address: imnasser@dpc.kfupm.sa.
[†]Electronic address: folk@tphys.uni-linz.ac.at.

[1] P. Hohenberg, A. Aharony, B. Halperin, and E. Sieggia, Phys. Rev. B **13**, 2986 (1976).
[2] R. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975).
[3] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976), Chap. 3.
[4] J. Annett, M. Randeria, and S. Renn, Phys. Rev. B **38**, 4660 (1988).
[5] Y. T. Millev, Phys. Rev. B **52**, 12 798 (1995).
[6] I. Nasser and R. Folk, Phys. Rev. B **52**, 15 799 (1995).
[7] W. Selke, in *Phase Transition and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic, London, 1992), Vol. 15.
[8] Yu. Vysochanskii and V. Slivka, Sov. Phys. Usp. **35**, 123 (1992).
[9] Y. Shapira, *Multicritical Phenomena*, Vol. 106 of *NATO Advanced Study Institute, Series B: Phyiscs* (Plenum, New York, 1984), p. 53.
[10] V. Bindilatti, C. Becerra, and N. Oliveira, Phys. Rev. B **40**, 9412 (1989).
[11] K. G. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974).
[12] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, 2nd ed. (World Scientific, Singapore, 1984).
[13] I. Nasser, Int. J. Theor. Phys. **35**, 481 (1996). There is a misprint in Eq. (8) of this paper, the second argument of the ${}_2F_1$ function should read $\beta-m/2+1/2$ instead of $\beta-m/4+1/2$. Also, in the prefactor $\bar{g}$ should be replaced by $\tilde{g}$.
[14] A. Abdel-Hady and R. Folk, Phys. Rev. B **54**, 3851 (1996).
[15] I. Lawrie, J. Phys. A **18**, 1141 (1985).
[16] J. Binney, N. Dowrick, A. Fisher, and M. Newman, *The Theory of Critical Phenomena, An Introduction to the Renormalization Group* (Oxford University Press, Oxford, 1992).
[17] R. Folk and G. Moser, Phys. Rev. B **47**, 13 992 (1993).
[18] V. Dohm, in *Application of Field Theory to Statistical Mechanics*, edited by L. Garrido (Springer, Berlin, 1984), p. 2630.
[19] V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transition and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic, London, 1991), Vol. 14.
[20] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 15.