Coherence in the quasiparticle scattering by the vortex lattice in pure type-II superconductors

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The effect of quasiparticle (QP) scattering by the vortex lattice on the de Haas-van Alphen (dHvA) oscillations in a pure type-II superconductor is investigated within a mean-field asymptotic perturbation theory. Using a two-dimensional (2D) electron-gas model it is shown that, due to a strict phase coherence in the many particle correlation functions, the scattering effect in the asymptotic limit ($\sqrt{E_F/\hbar\omega_c} \gg 1$) is much weaker than what is predicted by the random vortex-lattice model proposed by Maki and Stephen, which destroys this coherence. The coherent many particle configuration is a collinear array of many particle coordinates, localized within a spatial region with size of the order of the magnetic length. The amplitude of the magnetization oscillations is sharply damped just below H_{c2} because of strong 180° out of phase magnetic oscillations in the superconducting condensation energy, which tend to cancel the normal electron oscillations. Within the ideal 2D model used it is found, however, that because of the relative smallness of the quartic and higher order terms in the expansion, the oscillations amplitude at lower fields does not really damp to zero, but only reverses sign and remains virtually undamped well below H_{c2}. This conclusion may be changed if disorder in the vortex lattice, or vortex line motion is taken into account. The reduced QP scattering effect may be responsible for the apparent crossover from a strong damping of the dHvA oscillations just below H_{c2} to a weaker damping at lower fields observed experimentally in several 3D superconductors. [S0163-1829(97)05045-5]

I. INTRODUCTION

Magnetic quantum oscillations have been recently observed in several type-II superconductors below H_{c2}.[1–8] A systematic study of this remarkable effect has been impaired, however, by the lack of a complete quantitative theory of the de Haas-van Alphen (dHvA) effect in the vortex state, analogous to the Lifshitz-Kosevich (LK) theory in normal metals.[9] Such a theory would require a detailed analysis of the effect of the superconducting order parameter on the magnetization oscillations in the vortex state, which turns out to be an extremely subtle theoretical problem.

A common feature reported by all experimental groups so far, which is far from being well understood, has been the observation of an additional damping in the dHvA amplitude below H_{c2}. Several theoretical papers have attributed this attenuation to the broadening of the Landau levels by the inhomogeneous pair potential, building up below H_{c2}. There is, however, a remarkable disagreement among the various theoretical approaches to this problem concerning both the size of the attenuation factor and its detailed dependence on the strength of the field below H_{c2}.

The semiclassical approach, adopted originally by Maki,[10] elaborated later by Stephen,[11] and reviewed very recently by Wasserman and Springford,[12] considered the correction to the quasiparticles (QP) lifetime due to the scattering by the vortex lattice.

For the first harmonic of the oscillatory part of the magnetization below H_{c2} they predicted

$$M_{\rm osc} = M_{n,\rm osc}\exp[-\lambda(\Delta_0, n_F)], \tag{1}$$

with

$$\lambda(\Delta_0, n_F) = \pi^{3/2}\frac{\tilde{\Delta}_0^2}{n_F^{1/2}}, \tag{2}$$

where $M_{n,\rm osc}$ is the corresponding normal electrons contribution to the oscillatory magnetization, $\tilde{\Delta}_0 \equiv \Delta_0/\hbar\omega_c$, Δ_0 is the magnitude of the superconducting (SC) order parameter, ω_c is the cyclotron frequency, and $n_F = E_F/\hbar\omega_c$ is the Landau level index corresponding to the extremal orbit on the Fermi surface.

An exponential damping with a different exponent, i.e., $\lambda \sim \tilde{\Delta}_0/n_F^{1/4}$, has been proposed by Norman *et al.*,[13] who carried out a full quantum mechanical calculation, based on a numerical solution of the Bogoliubov-de Gennes (BdG) equations for the quasiparticles at low temperatures. The numerical computations carried out by these authors were limited, however, to relatively small values of $\sqrt{n_F}$.

A similar approach, invoked by Dukan and Tesanovic,[14] has led to a qualitatively different behavior at very low temperature, that is a power law attenuation of the dHvA oscillations below H_{c2}

$$M_{\rm osc}/M_{n,\rm osc} \sim (k_B T/\Delta_0)^2 + O[(k_B T/\Delta_0)^4]. \tag{3}$$

This result was obtained by considering only the dominant contribution to the dHvA oscillations as originating in the gapless region of the QP spectrum around the Fermi surface. It looks similar to a low-temperature high Δ expansion, emphasizing the opening of the SC gap well below H_{c2}.

In the Maki-Stephen (MS) theory it is assumed that the vortex lattice acts as a random potential for the quasiparticles, and so by averaging over the realizations of the vortex lattice, the QP self-energy acquires a large imaginary part, leading to a strong exponential damping of the dHvA amplitude.

Strictly speaking, however, the scattering by the inhomogeneous pair potential is a highly coherent process, as in multiple Andreev reflection at the interfaces of a 2D periodic array of normal and SC phases.[15]

Thus, in the ideal, self-consistent vortex lattice model the QP self-energy has zero imaginary part. However, the broadening of the Landau levels into real energy bands by the inhomogeneous pair potential should lead to damping of the dHvA oscillations even for an infinite QP lifetime. The term scattering refers to this inhomogeneous broadening effect in the present paper.

The quadratic dependence of $\lambda(\Delta_0, n_F)$ on Δ_0 in Eq. (2) reflects its origin in a perturbation expansion of the the QP self-energy in the mean square order parameter, which is strictly valid for a pure type-II superconductor only at sufficiently high temperatures, when the Lifshitz-Kosevich[9] thermal smearing factor $X \equiv 2\pi^2 k_B T/\hbar\omega_c > 1$.

It is therefore very interesting to compare the result expressed in Eq. (1) to that obtained by Maniv *et al.*,[16,17] who considered Gorkov's expansion of the SC free energy in the small vortex state order parameter near H_{c2}.

Using a semiclassical approximation, valid for $\sqrt{n_F} \gg 1$, these authors have found a quadratic term consistent with Eq. (2), but a quartic term $M^{(4)}_{\rm osc}/M_{n,\rm osc} \sim \tilde{\Delta}_0^4/n_F^{3/2}$, which is smaller than the corresponding term, obtained in MS theory (i.e., $M^{(4)}_{\rm osc}/M_{n,\rm osc} \sim \lambda^2 \sim \tilde{\Delta}_0^4/n_F$), by the factor $1/\sqrt{n_F} \ll 1$.

Very recently, Bruun *et al.*[18] developed an exact numerical scheme for calculating the coefficients of the Gorkov's expansion within the same model used by Maniv *et al.* and found good agreement with the results obtained by Norman *et al.* for small values of n_F. They have also made an estimate of the n_F dependence of the quartic term by using an approximation similar in spirit to the random lattice approximation, and found a result which essentially agrees with that of Maki and Stephen.

In the light of this controversy our purpose in the present paper is to carefully examine the high temperature $X \geqslant 1$, small Δ_0, asymptotic ($\sqrt{n_F} \gg 1$) expansion, in order to elucidate the origin of the disputed n_F dependence. We find that incoherent scattering channels, which generate the dominant contribution in the random lattice approximation, are completely cancelled in the self-consistent, periodic lattice calculation, due to the presence of a strict phase coherence in the four-particle correlation function. The remaining coherent four-particle configuration is a collinear array of four-electron coordinates, localized within a spatial region with size of the order of the magnetic length.

As a result, the inhomogeneous broadening of the Landau levels by the pair potential does not contribute significantly to the damping of the dHvA amplitude just below H_{c2}, as is the case in the MS theory. The dominant damping mechanism in the asymptotic limit, $n_F^{1/2} \gg 1$ is found to arise from the strong, 180° out of phase oscillations of the SC condensation energy with respect to the normal electrons oscillations, as was first proposed in Ref. 17.

The organization of the paper is as follows. In Sec. II we present the general framework of our approach, which is based on Gorkov's expansion of the SC free energy near H_{c2}. In Sec. III we carefully examine the quartic term in the asymptotic limit $\sqrt{n_F} \gg 1$ and justify the main approximation used in our calculation. In Sec. IV we derive simple analytical expressions for the self-consistent order parameter and for the oscillatory magnetization, and verify the validity of our claimed n_F dependence in the asymptotic limit. In Sec. V we discuss the connection of our theory to the other theoretical approaches and compare our predictions to experiment.

II. SMALL ORDER PARAMETER EXPANSION

We consider the quadratic $F_s^{(2)}$ and the quartic $F_s^{(4)}$ terms in the Gorkov-Ginsburg-Landau expansion of the SC free energy in the SC order parameter $\Delta(\vec{r})$.[16] For the sake of simplicity let us assume a 2D electron gas model and neglect the spin degrees of freedom. The former assumption may be justified by noting that the main contribution to the dHvA effect in an isotropic 3D normal electron system comes from the extremal orbit, corresponding to the value $k_z = 0$ of the electron momentum parallel to the field direction, and to the Landau level $n \approx n_F \equiv E_F/\hbar\omega_c$.

Note, however, that the pairing effect responsible for the Cooper instability in a 3D electron gas is dominated by the region $k_z \approx k_F \equiv (2mE_F/\hbar^2)^{1/2}$, $n \approx 0$ near the Fermi surface. Since the focus in the present paper is on the effect of QP scattering by the vortex lattice, we shall ignore, for the sake of simplicity, this aspect of 3D systems in our calculation, but will return to this problem later while discussing our results in connection with experiment.

Thus

$$F_s^{(2)} = \frac{1}{V}\int d^2r |\Delta(\vec{r})|^2 - \int d^2r_1 d^2r_2 K_2(\vec{r}_1, \vec{r}_2) \times \Delta(\vec{r}_1)\Delta^*(\vec{r}_2) \tag{4}$$

and

$$F_s^{(4)} = \frac{1}{2}\int d^2r_1 d^2r_2 d^2r_3 d^2r_4 K_4(\{\vec{r}_i\})\Delta(\vec{r}_1) \times \Delta^*(\vec{r}_2)\Delta(\vec{r}_3)\Delta^*(\vec{r}_4), \tag{5}$$

where V is a BCS interaction constant. The kernels K_2 and K_4 are expressed through the product of two- and four-electron Green's functions $G_0[\vec{r}_l, \vec{r}_{l+1}; (-)^l\omega_\nu]$, $l = 1,2$ with $\vec{r}_3 \equiv \vec{r}_1$, and $l = 1,\ldots,4$ with $\vec{r}_5 \equiv \vec{r}_1$, respectively, in magnetic field [in our case $\vec{H} = (0,0,H)$]. Here ω_ν is the Matzubara frequency. The magnetic field breaks translational symmetry of the BCS Hamiltonian and Green's function. However, owing to the gauge symmetry the Green's function can be represented in a factorized form: $G_0(\vec{r}_l, \vec{r}_{l+1}; \omega_\nu) = g(\vec{r}_l, \vec{r}_{l+1})\tilde{G}_0(\rho_l, \omega_\nu)$,[19] where the reduced Green's function

$$\widetilde{G}_0(\rho_l,\omega_\nu)=\frac{1}{2\pi a_H^2}e^{-\rho_l^2/4}\sum_n \frac{L_n(\rho_l^2/2)}{i\omega_\nu-\omega_c(n+1/2)+\mu} \tag{6}$$

depends only on relative coordinates $\vec{\rho}_l=(\vec{r}_{l+1}-\vec{r}_l)/a_H$, $\Sigma\vec{\rho}_l\equiv 0$, $a_H=\sqrt{c\hbar/eH}$. In this expression L_n is the Lagaurre polynomial of order n, and μ is the chemical potential. The noninvariant gauge factor in the symmetric gauge has the form $g(\vec{r}_l,\vec{r}_{l+1})=\exp[-i\varepsilon_{ik}(r_{l,i}+r_{l+1,i})\rho_{l,k}/4a_H]$, $\varepsilon_{lk}=-\varepsilon_{kl}$ is antisymmetric tensor in 2D space. From now on we shall express all spatial variables, except for $\vec{r}_i$, in units of the magnetic length.

In the quasiclassical limit ($\sqrt{n_F}\gg 1$) the Green's function $\widetilde{G}_0(\rho_l,\omega_\nu)$ has two different types of behavior. Near turning point $\rho_l\simeq 2r_F$ ($r_F=\sqrt{2n_F}$ is the cyclotron radius) it is a smooth function, and in the intermediate region $\rho_l<2r_F$ it is a sharply oscillating function.

Using asymptotic for Lagaurre polynomial L_n at $n\to\infty$ (Ref. 20) and the Poisson summation formula for the sum over n in Eq. (6) one can show that $\widetilde{G}_0(\rho_l,\omega_\nu)\sim 1/n_F^{1/3}$ at $4r_F^2-\rho_l^2\ll(16n_F/3)^{1/3}$, and

$$\widetilde{G}_0(\rho_l,\omega_\nu)=-\frac{i\varepsilon_{\omega_\nu}J(\omega_\nu)}{(2\pi)^{1/2}a_H^2\omega_c}\times\frac{\exp\{i\varepsilon_{\omega_\nu}[n_F(\phi+\sin\phi)]-\phi|\omega_\nu|/\omega_c\}}{\rho_l^{1/2}(r_F^2-\rho_l^2)^{1/4}}, \tag{7}$$

where $J^{-1}(\omega_\nu)=1-\exp(2\pi i\varepsilon_{\omega_\nu}n_F-2\pi|\omega_\nu|/\omega_c)$, $\sin\phi=(\rho_l/r_F)[1-(\rho_l/2r_F)^2]^{1/2}$, $\varepsilon_{\omega_\nu}=\mathrm{sgn}(\omega_\nu)$, at $\rho_l<2r_F$. Note that this expression is a generalization of the Green's function obtained in Ref. 16 for $\rho_l/2r_F\ll 1$. The similarity to the short distance limit is apparent since the analytic continuation of the Lagaurre polynomial asymptotic in the complex n plane gives rise to short distance approximation at $|n|\to\infty$.

Combining the gauge factors we obtain the dependence of K_2 and K_4 on the two- and four-particle center of mass coordinates $\vec{r}=(\Sigma_{l=1}{}^2\vec{r}_l)/2a_H$ and $\vec{R}=(\Sigma_{l=1}^4\vec{r}_l)/4a_H$, respectively,

$$K_2(\vec{r}_1,\vec{r}_2)=\exp(-i\varepsilon_{lk}r_l\rho_k)\,\frac{1}{\beta}\sum_\nu \widetilde{K}_{2,\nu}(\rho), \tag{8}$$

$$K_4(\{\vec{r}_i\})=\exp(-4i\varepsilon_{lk}R_lQ_k)\,\frac{1}{\beta}\sum_\nu \widetilde{K}_{4,\nu}(\{\rho_l\}), \tag{9}$$

where $\vec{\rho}\equiv\vec{\rho}_1$, $\vec{Q}=-(\vec{\rho}_1-\vec{\rho}_2+\vec{\rho}_3-\vec{\rho}_4)/8$ and

$$\widetilde{K}_{2,\nu}(\rho)=\widetilde{G}_0(\rho,-\omega_\nu)\widetilde{G}_0(\rho,\omega_\nu), \tag{10}$$

$$\widetilde{K}_{4,\nu}(\{\rho_l\})=\widetilde{G}_0(\rho_1,-\omega_\nu)\widetilde{G}_0(\rho_2,\omega_\nu)\widetilde{G}_0(\rho_3,-\omega_\nu)\times\widetilde{G}_0(\rho_4,\omega_\nu). \tag{11}$$

For the order parameter we use the Abrikosov form. In symmetric gauge

$$\Delta(\vec{r})=\Delta_0\exp(-y^2+ixy)\sum_{n=-\infty}^{\infty}\exp\left[-(1-i\gamma)\left(\frac{\pi n}{a_x}\right)^2+2i\frac{\pi n}{a_x}(x+iy)\right]. \tag{12}$$

The numbers γ and a_x are arbitrary. It can be shown that this quasiperiodical form results from gauge symmetry as a solution minimizing free energy.[16,21] The order parameter is normalized as

$$\Delta_0^2=\frac{\sqrt{2/\pi}}{a_x a_H^2 N}\int d^2r|\Delta(\vec{r})|^2, \tag{13}$$

where N is the number of vortices.

Using expression (12) for the order parameter in Eq. (4) for the quadratic term we get after integrating over the two-particle center of mass coordinate $\vec{r}$

$$F_s^{(2)}=\left(\frac{1}{V}-\mathcal{A}\right)\pi a_H^2N\Delta_0^2, \tag{14}$$

where $\mathcal{A}\propto(1/\beta)\Sigma_\nu\int d^2\rho e^{-\rho^2/2}\widetilde{K}_{2,\nu}(\rho)$. The Gaussian factor restricts the effective integration region by the distances $\rho\sim 1$. This important fact means the loss of coherence in electron pair propagation over distances much larger than the magnetic length.

The localization of the electron correlation functions within a region of a size of the order of the order of the magnetic length can be shown to exist, in quasiclassical limit, also for many electron configurations. Here we discuss only four-particle correlations. Integration of Eq. (5) over $\vec{R}$ gives rise to the expression

$$F_s^{(4)}=\int d^2Qe^{-4Q^2}\int d^2Sd^2T\mathcal{D}(\vec{S},\vec{T})\,\frac{1}{\beta}\sum_\nu \widetilde{K}_{4,\nu}(\{\rho_l\}), \tag{15}$$

where $\vec{S}\equiv\frac{1}{4}(\vec{\rho}_3-\vec{\rho}_1)$, $\vec{T}\equiv\frac{1}{4}(\vec{\rho}_4-\vec{\rho}_2)$. The function $\mathcal{D}(\vec{S},\vec{T})$ includes the summation over vortices

$$\mathcal{D}(\vec{S},\vec{T})\propto Ne^{4i(S_xT_y+S_yT_x)}\times\sum_{m_1,m_2}e^{-2i\gamma m_1m_2+4im_1T_x+4im_2S_x}\times\exp[-(m_1+2S_y)^2-(m_2+2T_y)^2]. \tag{16}$$

For convenience, we denoted $m_1=\pi(n_2-n_1)/a_x$, $m_2=\pi(n_3-n_1)/a_x$, where n_i is the summation index corresponding to the order parameter $\Delta(\vec{r}_i)$ in Eq. (12). Note that only a 2D sum remains in Eq. (16) from the original 4D sum; the free double sum is equal to the number of vortices N, while the summation in Eq. (16) is invariant under the vortex number shift transformation $n_i\to n_i+n_0$. Note also that the combination of gauge factors from the order parameters and Green's function causes integral over center mass coordinate $\vec{R}$ to vanish if $n_1-n_2+n_3-n_4\neq 0$.

It is clear from Eq. (16) that the lattice sum in $\mathcal{D}(\vec{S},\vec{T})$ is dominated by the lattice point $m_1\simeq -2S_y$, $m_2\simeq -2T_y$. Taking only this term in the sum into account we obtain that

$$\mathcal{D}(\vec{S},\vec{T}) \propto N e^{-4i(2\gamma S_y T_y + S_y T_x + S_x T_y)}. \tag{17}$$

The lattice function $\mathcal{D}(\vec{S},\vec{T})$ being multiplied by the kernel $\widetilde{K}_{4,\nu}(\{\rho_l\})$ determines the free energy distribution in the space of relative electron coordinates.

III. ASYMPTOTIC LOCAL APPROXIMATION

In this section we analyze in detail the relative importance of all different spatial regions contributing to the multiple integral, Eq. (15), which determines the quartic term in the asymptotic limit $\sqrt{n_F} \gg 1$. Every four-electron configuration in this integral is defined by the three vectors $(\vec{Q},\vec{S},\vec{T})$. Since the kernel $\widetilde{K}_{4,\nu}(\vec{Q},\vec{S},\vec{T})$ is a bounded function at $|\vec{Q}| \gg 1$, we may conclude from Eq. (15) that the main contribution to the free energy comes from the the region $|\vec{Q}| \leqslant 1$, and so assume in what follows that $|\vec{Q}| \leqslant 1$. To estimate the integrals over $\vec{S}$ and $\vec{T}$ we separate the entire domain of integrations into three regions according to the behavior of the Green's function [Eq. (6)] and the function $\mathcal{D}(\vec{S},\vec{T})$.

(1) The turning point region $\rho_i \simeq 2r_F$, for which in the asymptotic limit $\sqrt{n_F} \gg 1$,

$$\rho_{1,3} = 2|\vec{Q} \pm \vec{S}| \simeq 2S \simeq 2r_F = 2\sqrt{2n_F} \gg 1,$$

$$\rho_{2,4} = 2|\vec{Q} \mp \vec{T}| \simeq 2T \simeq 2r_F = 2\sqrt{2n_F} \gg 1. \tag{18}$$

The size of this region is of the order $\Delta\rho_i = \Delta S = \Delta T \sim 1/r_F^{1/3}$. It is characterized by a smooth behavior of all Green's functions $\widetilde{G}_0(\rho_i) \sim \widetilde{G}_0(2r_F) \sim 1/n_F^{1/3}$ and kernel $\widetilde{K}_{4,\nu} \sim [\widetilde{G}_0(2r_F)]^4 \sim 1/n_F^{4/3}$.

(2) In the intermediate region $1 \ll \rho_i < 2r_F$ variables S and T are still large: $S \gg 1$, $T \gg 1$ but essentially less than r_F. Here the Green's function is approximated by the expression Eq. (7). To simplify the considerations we note that the oscillating phase factor in Eq. (7) can be written as

$$n_F(\phi + \sin\phi) = \sqrt{2n_F}\,\xi(\rho/2r_F)\rho, \tag{19}$$

where $\pi/4 \leqslant \xi(x) \leqslant 1$ at $0 \leqslant x \leqslant 1$. For our purpose we may take $\xi(x) = 1$. Substituting $\rho_{1,3} \simeq 2S$, $\rho_{2,4} \simeq 2T$, the kernel $\widetilde{K}_{4,\nu}$ in this region can be transformed to

$$\widetilde{K}_{4,\nu}(\vec{Q},\vec{S},\vec{T}) \propto \frac{\exp[i4\sqrt{2n_F}(S-T)]}{ST(r_F^2 - S^2)^{1/2}(r_F^2 - T^2)^{1/2}}. \tag{20}$$

(3) In the third region all variables $\vec{Q},\vec{S},\vec{T}$ are of the order of the magnetic length: $Q,S,T \sim 1$. Here $\mathcal{D}(\vec{S},\vec{T})$ becomes a smooth function and the strong oscillations of the integrand in Eq. (15) arise only from $\widetilde{K}_{4,\nu}$. Since $\xi(x) \to 1$ at $x \to 0$ the exact short distance asymptotic is

$$\widetilde{K}_{4,\nu}(\{\rho_i\}) \propto \frac{1}{n_F(\rho_1\rho_2\rho_3\rho_4)^{1/2}} \times \exp[i\sqrt{2n_F}(\rho_1 - \rho_2 + \rho_3 - \rho_4)]. \tag{21}$$

Let us first consider the case when both variables $\vec{S},\vec{T}$ belong to the first region [i.e., region (1,1)]: Replacing $\widetilde{K}_{4,\nu}(\vec{Q},\vec{S},\vec{T})$ by a constant of the order $\sim 1/n_F^{4/3}$ we can estimate the free energy, Eq. (15), for a given Matsubara frequency in polar coordinate system $\vec{S} \equiv (S,\alpha_s)$, $\vec{T} = (T,\alpha_t)$ as

$$F_{s,\nu}^{(4)} \propto \frac{1}{n_F^{4/3}} \int_{r_F - \Delta\rho}^{r_F} S\,dS\,T\,dT \int_0^{2\pi} d\alpha_s\, d\alpha_t\, e^{-4iST\phi(\alpha_s,\alpha_t)}, \tag{22}$$

where $\phi(\alpha_s,\alpha_t) = \gamma\cos(\alpha_s - \alpha_t) + \sin(\alpha_s + \alpha_t) - \gamma\cos(\alpha_s + \alpha_t)$. After simple integration over α_s and α_t by the stationary phase method, which is justified by the very large values of ST, Eq. (22) transforms to

$$F_{s,\nu}^{(4)} \propto \frac{1}{n_F^{4/3}} \int_{r_F - \Delta\rho}^{r_F} dS\,dT\, e^{-4iST\phi_s^{\pm}} \sim \frac{1}{n_F^{7/3}}, \tag{23}$$

where $\phi_s^{\pm} = \sqrt{1+\gamma^2} \pm \gamma$ are the values of $\phi(\alpha_s,\alpha_t)$ at the stationary points.

Let us next consider the case when both S and T belong to the intermediate region [i.e., region (2,2)]: Here $\widetilde{K}_{4,\nu}(\{\rho_i\})$, defined by Eq. (20) as $\mathcal{D}(\vec{S},\vec{T})$, is a sharply oscillating function with frequency of order of $\sqrt{n_F}$. Substituting Eqs. (20), (17) into Eq. (15) we obtain for $1 \ll S$, $T < r_F$

$$F_{s,\nu}^{(4)} \propto \int \frac{d^2S\,d^2T\,\exp\{4i[\sqrt{2n_F}(S-T) - 2\gamma S_y T_y - S_y T_x - S_x T_y]\}}{ST(r_F^2 - S^2)^{1/2}(r_F^2 - T^2)^{1/2}}. \tag{24}$$

The integrals over angle variables are identical to previously considered ones and give a factor of the order $1/ST$. Thus free energy (24) reduces to

$$F_{s,\nu}^{(4)} \propto \int (dS\,dT\,\exp\{4i[\sqrt{2n_F}(S-T) \pm ST\phi_s^{\pm}]\})/ST(r_F^2 - S^2)^{1/2}(r_F^2 - T^2)^{1/2}. \tag{25}$$

Taking for the smooth preexponential factor its value at a point $S \sim \sqrt{n_F}$, $T \sim \sqrt{n_F}$ and noting that there is no stationary point for the integrals over S and T, we get the following estimate for free energy in the intermediate region: $F_{s,\nu}^{(4)} \sim 1/n_F^3$.

It is clear that if one of the variables, for example, S, is from the turning point region and the other T, from the intermediate region [i.e., region (1,2)], the free energy $F_{s,\nu}^{(4)}$ will be proportional to a factor of the order $\sim 1/n_F^{8/3}$. This result follows from the fact that the product of two of the

Green's functions is proportional to $\sim 1/n_F^{2/3}$, the product of the other two is proportional to $\sim 1/n_F$, and the integration over S and T yields the factor $\sim 1/n_F$. The integration over the angles α_s and α_t produces the factor $\sim 1/ST$ which cancels the same factor in the numerator of the integrand (15).

Let us consider now the four-electron configuration when one of the variables S or T, e.g., T, is of the order of the magnetic length $T\sim 1$ [i.e., region (3)]. In contrast to previous cases, $S,T\gg 1$, where the oscillations with $\vec{Q}$ can be neglected, in this case an additional small factor arises from the integration over $\vec{Q}$. It is partially cancelled by the large factor arising from the integration over $\vec{S}$ if it is from the turning point region. Assuming that $S\sim r_F$ and $T\sim 1$ [i.e., region (1,3)], the estimate of the free energy (15) is given by the formula

$$F_{s,\nu}^{(4)}\propto \frac{1}{n_F^{7/6}}\int d^2Qe^{-4Q^2}\int d^2Sd^2T \times\exp\left\{-4i\left[\frac{1}{2}\sqrt{2n_F}(|\vec{Q}-\vec{T}|+|\vec{Q}+\vec{T}|) +2\gamma S_yT_y+S_yT_x+S_xT_y\right]\right\}. \tag{26}$$

The integral over $\vec{Q}$ can be divided into two regions, $Q\leqslant T$ and $Q\geqslant T$. The main contribution comes from $Q\leqslant T$, where the above exponential factor does not depend on the component $Q^{\parallel}$, parallel to $\vec{T}$. For the small transverse component $Q^{\perp}$ we get

$$|\vec{Q}-\vec{T}|+|\vec{Q}+\vec{T}|\simeq 2T+\left(\frac{1}{\rho_2}+\frac{1}{\rho_4}\right)(Q^{\perp})^2. \tag{27}$$

Thus the integral over $Q^{\perp}$ is proportional to $\sim 1/n_F^{1/4}$ and the integral over $Q^{\parallel}$ gives $\int_0^T dQ^{\parallel}\sim T$. Performing the integration over angles α_s and α_t gives, as previously, $1/ST$ and taking into account that integral over S from a smooth function can be estimated by the area of the integration region $S\Delta S\sim r_F\Delta\rho\sim n_F^{1/3}$, we get for $Q\leqslant T$ that

$$F_{s,\nu}^{(4)}\sim \frac{1}{n_F^{19/12}}\int TdT\,\exp[-4i\sqrt{n_F}T(1\pm\phi_s^{\pm})]\sim\frac{1}{n_F^{31/12}}. \tag{28}$$

In the second region $Q\geqslant T$, the integral over $\vec{Q}$ leads to a contribution smaller by the factor $n_F^{1/4}$ than Eq. (28). This is because the phase (27) does not depend on $T^{\parallel}$ and we can neglect the quadratic in $T^{\perp}$ term in comparison with the linear one in Eq. (26).Thus the integral over $\vec{Q}$ can be estimated here as $\sim 1/n_F$. If $\vec{S}$ belongs to the intermediate region [i.e., considering the region (2,3)], the contribution to the free energy will be even smaller because of the Green's function oscillations and smaller preexponential factor.

Finally, in the short distance region $\rho_i\leqslant 1$ the smooth function $\mathcal{D}(\vec{S},\vec{T})$ is of the order one and we should calculate the integrals in Eq. (15) from the kernel $\widetilde{K}_{4,\nu}(\{\rho_i\})$ Eq. (21). The phase factor of $\widetilde{K}_{4,\nu}$ which is proportional to

$$\Phi(\vec{Q},\vec{S},\vec{T})=\rho_1-\rho_2+\rho_3-\rho_4 =2\{|\vec{Q}+\vec{S}|-|\vec{Q}-\vec{T}|+|\vec{Q}-\vec{S}|-|\vec{Q}+\vec{T}|\} \tag{29}$$

at fixed Q has a set of stationary points $S\leqslant Q$, $T\leqslant Q$ for collinear vectors $\vec{Q},\vec{S},\vec{T}$. Going back to $\vec{\rho}_i$ variables we conclude that this configuration is equivalent to the propagation of odd particles in a single direction $\vec{n}$ and even particles in the opposite direction:

$$\vec{\rho}_1=\rho_1\vec{n},\quad \vec{\rho}_2=-\rho_2\vec{n},\quad \vec{\rho}_3=\rho_3\vec{n},\quad \vec{\rho}_4=-\rho_4\vec{n}, \tag{30}$$

where $\vec{n}$ is an arbitrary unit vector. Note that the special role of the configuration (30) follows from the fact that for this configuration $\Sigma\vec{\rho}_i=\Phi(\rho_i)\vec{n}\equiv 0$ in the integration region.

It should be emphasized that in the resulting coherent configuration the correlation among all four particles is essential. The phase factor Eq. (29) and stationary point equations cannot be factorized.

Expanding $\Phi(\vec{Q},\vec{S},\vec{T})$ in the coordinates $S^{\perp}$ and $T^{\perp}$ perpendicular to $\vec{n}$, and noting that from the definition of $\vec{Q}$ follows $\vec{n}=-\vec{Q}/Q$, we can reduce Eq. (29) to

$$\Phi(\vec{Q},\vec{S},\vec{T})=2\left[\left(\frac{1}{\rho_1}+\frac{1}{\rho_3}\right)(S^{\perp})^2+\left(\frac{1}{\rho_2}+\frac{1}{\rho_4}\right)(T^{\perp})^2\right], \tag{31}$$

where $\rho_1+\rho_3=\rho_2+\rho_4=4Q$. Now performing the integration of the kernel $\widetilde{K}_{4,\nu}(\{\rho_i\})$, given in Eq. (21), over $S^{\perp}$ and $T^{\perp}$ we obtain a factor $\sim(1/\sqrt{n_F})[(\rho_1\rho_2\rho_3\rho_4)^{1/2}/Q]$, which gives rise to the ν dependent free energy $F_{s,\nu}^{(4)}\sim 1/n_F^{3/2}$.

This result completes our analysis: it implies that in the asymptotic limit $\sqrt{n_F}\gg 1$ the dominant contributions to the quartic term of the free energy originate in the short distance region only. Thus, using the short distance approximation for $\widetilde{K}_{4,\nu}$, taking into account its dependence on ω_ν, and restoring the exact form of $\mathcal{D}(\vec{S},\vec{T})$, Eq. (16), the quartic term of the SC free energy can be written in the form

$$F_s^{(4)}\propto\frac{1}{n_F^{3/2}}\sum_{\nu}q_\nu^2\int dQe^{-4Q^2-4\alpha_\nu Q}\int_{-Q}^{Q}dSdTd\theta \times\exp[-4(S^2+T^2)\sin^2\theta+4iST\,\sin(2\theta)] \times\sum_{m_1,m_2}\exp[-m_1^2-m_2^2-2i\gamma m_1m_2+4i(m_1T+m_2S) \times\cos\theta-(m_1S+m_2T)\sin\theta], \tag{32}$$

where the angle θ describes the direction of the unit vector $\vec{n}$, $2q_\nu=J(-\omega_\nu)J(\omega_\nu)$, and $\alpha_\nu=2(2\nu+1)a_H/\zeta$, with $\zeta=\hbar v_F/(\pi k_BT)$.

This free energy is determined by collinear, essentially four particles configurations with the size of the order of magnetic length. The resulting expression Eq. (32) agrees with the quartic term $F_s^{(4)}$ derived previously by Maniv *et al.*[16] Our present considerations justify the used approximation and clarify the geometry of coherent configurations.

It should be emphasized that the random vortex-lattice approximation, used in Ref. 11, gives rise to a markedly different result, namely, $F_{s,\nu}^{(4)} \sim 1/n_F$. The reason for the disagreement is due to the averaging over random vortex lattice[11] which leads to factorization of the multiple products of pair potentials into products of pair correlation functions only.

For example, the quartic term in this approximation becomes

$$\begin{aligned}\langle \Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\Delta(\vec{r}_3)\Delta^*(\vec{r}_4)\rangle &\propto \langle \Delta(\vec{r}_1)\Delta^*(\vec{r}_2)\rangle\langle \Delta(\vec{r}_3)\Delta^*(\vec{r}_4)\rangle \\ &\propto \exp\left[-\frac{1}{2}(\rho_1^2+\rho_3^2)\right]\exp\{-i[\zeta(2,1)+\zeta(4,3)]\},\end{aligned}$$

where $\zeta(2,1)=(x_2+x_1)(y_2-y_1)$ is the (Landau) gauge factor of $G_0(\vec{r}_1,\vec{r}_2;\omega_\nu)$, and $\langle\cdots\rangle$ stands for averaging over vortex distributions. The corresponding free energy is given by

$$\begin{aligned}F_{s,\nu}^{(4)} \propto \int \prod d^2 r_i \widetilde{K}_{4,\nu}(\{\rho_i\})\exp\left[-\frac{1}{2}(\rho_1^2+\rho_3^2)\right] \\ \times\exp\left[\frac{i}{2}[(x_4-x_2)(y_3-y_1)-(x_3-x_1)(y_4-y_2)]\right].\end{aligned} \tag{33}$$

In the important region of integration $\rho_1,\rho_3 \leqslant 1$, the constraint $\Sigma_{l=1}^4 \vec{\rho}_l = 0$, implies that $\vec{\rho}_2 \approx -\vec{\rho}_4$, or alternatively $\vec{r}_1 \sim \vec{r}_2$ and $\vec{r}_3 \sim \vec{r}_4$. Thus the imaginary exponent in Eq. (33) is always of the order unity or smaller in the important region of integrations, and so the total gauge factor is a smooth function. Consequently, only two distances in the kernel $\widetilde{K}_{4,\nu}$ are restricted to the size of the order of magnetic length, allowing two others to be arbitrary. Thus, in contrast to our result, where all long range configurations of electron pairs in the ordered vortex lattice interfere destructively (i.e., appear as incoherent scattering channels), the smoothing of rapidly oscillating gauge factors in the MS theory introduces a huge incoherent contribution to the SC free energy.

Let us estimate now the quartic term within this approximation. Substituting for the Green's functions $\widetilde{G}_0(\rho_1,-\omega_\nu)$ and $\widetilde{G}_0(\rho_3,-\omega_\nu)$ their approximants in the short distance region, and omitting the smooth gauge factor, we get for the free energy $F_{s,\nu}^{(4)}$, after integrating over the center of mass coordinates:

$$\begin{aligned}F_{s,\nu}^{(4)} \propto \frac{1}{n_F^{1/2}} \int \frac{\prod d^2\rho_i}{(\rho_1\rho_3)^{1/2}} \delta\left(\sum_{l=1}^4 \vec{\rho}_l\right) \widetilde{G}_0(\rho_2,\omega_\nu) \\ \times \widetilde{G}_0(\rho_4,\omega_\nu)\exp(i\sqrt{2n_F}(\rho_1+\rho_3)) \\ \times\exp\left[-\frac{1}{2}(\rho_1^2+\rho_3^2)\right].\end{aligned} \tag{34}$$

The main contribution to Eq. (34) arises from the region $\rho_2 \simeq 2r_F$, $\rho_4 \simeq 2r_F$. Allowing $\vec{\rho}_2$, $\vec{\rho}_4$ to vary independently within the turning point region, $\vec{\rho}_1$, $\vec{\rho}_3$ are not independent variables; taking $\vec{\rho}_1$ as the third independent variable of integration, and noting that $\vec{\rho}_1 \simeq -\vec{\rho}_3$, we have

$$F_{s,\nu}^{(4)} \sim \frac{1}{n_F^{1/2}} \int d^2\rho_1 \frac{e^{2i\sqrt{2n_F}\rho_1}}{\rho_1}\left[\int d^2\rho \widetilde{G}_0(\rho,\omega_\nu)\right]^2. \tag{35}$$

Now, since $\int \widetilde{G}_0(\rho)d^2\rho \sim 1$, and the integration over $\vec{\rho}_1$ yields the factor $\sim 1/\sqrt{2n_F}$, the resulting n_F dependence is $1/n_F$, in agreement with Ref. 11.

IV. SELF-CONSISTENT ORDER PARAMETER

The local approximation, verified in the previous section, becomes very transparent if we rewrite the free energy Eq. (32) as a functional of the order parameter $\Delta(\vec{r})$. After some straightforward, but combersome calculations one can show that Eq. (32) is equivalent to

$$\begin{aligned}F_s^{(4)} \propto \frac{1}{n_F^{3/2}} \int d^2R \int d\theta dQ f(Q) e^{-4Q^2} \int_{-Q}^{Q} dS \\ \times \int_{-Q}^{Q} dT \Delta[\vec{R}+(S+T)\vec{n}]\Delta^*[\vec{R}+(S-T)\vec{n}] \\ \times \Delta[\vec{R}-(S+T)\vec{n}]\Delta^*[\vec{R}-(S-T)\vec{n}],\end{aligned} \tag{36}$$

where $f(Q)=\Sigma_\nu q_\nu^2 e^{-4\alpha_\nu Q}$, and $\Delta(\vec{R})$ is defined by Eq. (12). Since $|S|$, $|T| \leqslant Q \leqslant 1$, the expression (36) can be considered as averaging of the four-order parameter product over a region with radius of the order of the magnetic length. The additional averaging over the direction of $\vec{n}$ in Eq. (32) leads to a completely local expression plus a nonlocal correction, i.e.,

$$F_s^{(4)} = B\int d^2R|\Delta(\vec{R})|^4 + F_{s,\mathrm{nloc}}^{(4)}, \tag{37}$$

where $B \propto (1/n_F^{3/2}) \int dQ f(Q) e^{-4Q^2} \int_{-Q}^{Q} dS \int_{-Q}^{Q} dT e^{-2(S^2+T^2)}$. The nonlocal correction $F_{s,\mathrm{nloc}}^{(4)}$ is numerically small since it arises from high (i.e, fourth and higher) order terms in the cumulant expansion of the exponential in Eq. (32) (see Ref. 16).

This result is of fundamental importance since it shows that the well known, fully local form of the Ginzburg-Landau free energy functional in the low field regime near $T_c(H=0)$ is basically valid also in the opposite, high magnetic field regime near $H_{c2}(T=0)$. This locality is closely related to the coherence effect discussed above. For example, in the random lattice approximation, discussed in the previous section, the dominant contribution to $F_s^{(4)}$ is extremely nonlocal.

Neglecting the small nonlocal correction, the total SC free energy, up to fourth order in Δ_0, can be turned into the following one-parameter variational form:[16]

$$f_s \equiv \frac{F_s}{N\pi a_H^2} = \mathcal{D}_{2D}\left[-\widetilde{\alpha}\Delta_0^2 + \frac{\widetilde{B}}{(\pi k_B T_c)^2}\Delta_0^4\right], \tag{38}$$

where $\mathcal{D}_{2D}=m_c/2\pi\hbar^2$ (i.e., the 2D single electron density of states),

$$\tilde{\alpha}=2\frac{a_H}{\zeta}\sum_{\nu=0}^{\nu_D}\ \mathrm{Re}(q_\nu)\gamma_\nu-1/g \tag{39}$$

with $\gamma_\nu=\int_0^\infty d\rho e^{-\alpha_\nu\rho-(1/2)\rho^2}$, $g=V\mathcal{D}_{2D}$, and $\nu_D\equiv(T_D/2T-1)$, where T_D is the Debye temperature.

The coefficient $\tilde{B}$, of the quartic term can be readily obtained from Eq. (37) (after replacing Q with $\rho/2$):

$$\tilde{B}=\beta_A\frac{a_H}{\zeta}\left(\frac{a_H}{\xi_0}\right)^2\sum_{\nu}^{\nu_D}\ \mathrm{Re}(q_\nu^2)\delta_\nu, \tag{40}$$

with $\delta_\nu\equiv 2\pi\int_0^\infty d\rho e^{-2\alpha_\nu\rho-\rho^2}\,\mathrm{erf}^2(\rho/\sqrt{2})$, $\xi_0\equiv\hbar v_F/\pi k_B T_c$, and β_A is the geometrical factor of the Abrikosov lattice.[16]

The key parameters, which control the crossover from the low field to the high field regime are a_H/ζ and $X=2\pi^2 k_B T/\hbar\omega_c$; they are connected by

$$X=2\pi(2n_F)^{1/2}\left(\frac{a_H}{\zeta}\right), \tag{41}$$

which means that in the asymptotic limit considered here, our high-temperature regime, $X\sim 1$, of the quantum magnetic oscillations domain is still in the low-temperature regime of the SC-normal phase boundary, since $a_H/\zeta\sim 1/\sqrt{n_F}\ll 1$.

In this case $q_\nu\approx 2$ for all ν, and the coefficient, $\tilde{\alpha}$, of the quadratic term, can be calculated from Eq. (39) by dividing the sum over the Matzubara frequencies ν into two regions: (1) $\alpha_\nu\ll 1$, namely, $\nu\ll\nu_{\max}\equiv(\zeta/a_H)/2\sqrt{2}$, and (2) $\nu\geqslant\nu_{\max}$. The contribution from the first region is $\sqrt{\pi}\int_0^1 e^{x^2}[1-\mathrm{erf}(x)]dx\approx 1.147$, while the sum in the second region leads to the familiar logarithmic expression $\Sigma_{\nu=\nu_{\max}}^{\nu_D}1/(\nu+1/2)\approx\ln[\sqrt{2}(T_D/T)(a_H/\zeta)]$, provided that the Debye cutoff temperatue $T_D\equiv(2\nu_D+1)T$ is much larger than $(2\nu_{\max}+1)T$. The last condition may be rewritten in a more transparent form, i.e., $(k_B T_D/\hbar\omega_c)^2\gg n_F/2\pi^3$.

Combining the contributions from the two regions we find

$$\tilde{\alpha}\approx\ln\left[\frac{a_H}{\sqrt{2}\xi(0)}\right], \tag{42}$$

where $\xi(0)\equiv 0.18\hbar v_F/k_B T_c\approx 0.56\xi_0$.

Now consider the coefficient $\tilde{B}$, of the quartic term. Again, we divide the Matzubara sum into the same two regions. In the first, where $\nu\ll\nu_{\max}$, each term δ_ν is independent of ν so that $\Sigma_{\nu=0}^{\nu_{\max}}\delta_\nu\approx 4\nu_{\max}\int_0^\infty d\rho e^{-\rho^2}\rho^2 d\rho\approx 0.63(\zeta/a_H)$, whereas the second region yields $\Sigma_{\nu=\nu_{\max}}^{\infty}[1/2(2\nu+1)a_H/\zeta]^3\approx\frac{1}{16}\zeta/a_H$. Combining these results we find that the sum over Matzubara frequencies changes significantly the n_F dependence of the quartic term with respect to the individual $F_{s,\nu}^{(4)}$ terms, since $\Sigma_\nu\delta_\nu\approx 0.69(\zeta/a_H)\sim n_F^{1/2}$.

We thus find that $\tilde{B}/(\pi k_B T_c)^2\approx 1.38/E_F\hbar\omega_c$, so that

$$f_s\approx\frac{\hbar\omega_c}{2\pi a_H^2}\left[-\tilde{\Delta}_0^2\ln\left(\frac{a_H}{\sqrt{2}\xi(0)}\right)+\frac{1.38}{n_F}\tilde{\Delta}_0^4\right]. \tag{43}$$

It should be emphasized, here again, that the n_F dependence of both the quartic and the quadratic terms in f_s above differs by the large factor $\sqrt{n_F}$ from the indevidual terms $F_{s,\nu}^{(4)}, F_{s,\nu}^{(2)}$ because of the sum over the Matzubara frequencies.

Using expression (43), the self-consistent mean field order parameter is given by

$$\tilde{\Delta}_0^2=0.36 n_F\ln\left(\frac{a_H}{\sqrt{2}\xi(0)}\right). \tag{44}$$

This expression is identical to the well-known high field limit of the Gorkov-Ginzburg-Landau SC order parameter.[12] Indeed, at magnetic fields H near $H_{c2}(0)=\phi_0/2\pi\xi(0)^2$, $\phi_0=ch/2e$, where $a_H\approx\sqrt{2}\xi(0)$, we have

$$E_F\hbar\omega_c\approx\frac{(\pi k_B T_c)^2}{2(a_H/\xi_0)^2}\approx 0.78(\pi k_B T_c)^2,$$

so that Eq. (44) reduces to the well-known result

$$\Delta_0\approx 1.7 k_B T_c\left[\ln\left(\frac{a_H}{\sqrt{2}\xi(0)}\right)\right]^{1/2}\approx 1.7 k_B T_c[1-H/H_{c2}(0)]^{1/2}. \tag{45}$$

Interestingly, the n_F dependence of the self-consistent Δ_0 obtained in Eq. (44) for $H\approx H_{c2}$ determines a small parameter

$$x\equiv\frac{\tilde{\Delta}_0^2}{n_F}\approx 0.36[1-H/H_{c2}(0)],$$

which is seen to be the expansion parameter in the perturbation theory leading to Eq. (43). This observation will be further discussed in the next section.

Note that in deriving the above expressions for the self-consistent order parameter we have neglected the oscillatory compomemts of the SC free energy, which should add an oscillatory contribution to the order parameter.[16,18] In the high-temperature limit considered, this oscillatoray term is much smaller than the nonoscillatory one, except for a very narrow region near H_{c_2}.[27]

Let us consider now the magnetization oscillations; the dominant contribution to the superconducting part can be obtained by differentiating the density of states factors q_ν in the free energy (38) with respect to magnetic field, namely;

$$M_{s,\mathrm{osc}}\propto-\sum_\nu\ \frac{\partial f_s}{\partial q_\nu}\frac{\partial q_\nu}{\partial H}. \tag{46}$$

Explicitly we have

$$M_{s,\mathrm{osc}}\approx 2\mathcal{D}_{2D}\frac{a_H}{\zeta}\Delta_0^2\sum_{\nu=0}^{\nu_D-1}\left[\gamma_\nu-\left(\frac{\Delta_0}{\pi k_B T_c}\right)^2\left(\frac{a_H}{\zeta}\right)\delta_\nu q_\nu\right]\frac{\partial q_\nu}{\partial H}. \tag{47}$$

For $X\geqslant 1$, $\partial q_\nu/\partial H\approx-(8\pi n_F/H)\sin(2\pi n_F)e^{-(2\nu+1)X}$, so that the sum over ν is limited by the thermal damping factor to the first few terms only. This contrasts the nonoscillatory magnetization, which picks up contributions from many Matzubara frequencies.

Thus the first harmonic of the oscillatory magnetization, M_{osc}, just below H_{c2} can be written as[17]

$$\widetilde{M}_{\rm osc}\equiv\frac{\phi_0}{E_F}M_{\rm osc}\approx\widetilde{M}_{n,\rm osc}\left[1-\frac{\pi^{3/2}\widetilde{\Delta}_0^2}{n_F^{1/2}}+\frac{\sqrt{2}\pi^{3/2}\beta_A\widetilde{\Delta}_0^4}{n_F^{3/2}}\right], \tag{48}$$

where $\beta_A\approx 1.16$ for a triangular lattice, and $\widetilde{M}_{n,\rm osc}\equiv(X/e^X)\sin(2\pi n_F)$ is the normal electrons oscillatory magnetization.[25]

In the expansion (48) there are two scales of order parameter Δ_0. Near H_{c2}, where $\Delta_0^2\ll(\hbar\omega_c)^{3/2}E_F^{1/2}$ (i.e., $\widetilde{\Delta}_0^2\ll n_F^{1/2}$, which means that $\ln[a_H/\sqrt{2}\xi(0)]\sim 1/n_F^{1/2}$), the attenuation of the magnetization oscillations amplitude occurs as the result of the electron pairing. Here the contribution of the many electron coherent configurations is negligible. Far from H_{c2}, where $\Delta_0^2\sim\hbar\omega_c E_F$ (i.e., $\widetilde{\Delta}_0^2\sim n_F$ so that $\ln[a_H/\sqrt{2}\xi(0)]\sim 1$) the quadratic and the quartic terms in the free energy (and magnetization) are comparable. It can be shown[22] that the higher order terms in this expansion are determined by the parameter $\Delta_0^2/\hbar\omega_c E_F=\widetilde{\Delta}_0^2/n_F$. In the region where this parameter is of the order unity or larger the SC state is a highly correlated many electron-pair configuration, which is quite different from the condensate of electron pairs, dominating the SC free energy just below H_{c2}.

V. CONCLUSION

The results of the last two sections enable us now to critically discuss the various theoretical approaches to the problem of the intrinsic attenuation of the dHvA oscillations in the vortex state, and the relevance of our model to real experiments. It is, first of all, clear that the assumption of disordered vortex lattice, and the consequent averaging over the random pair potential configurations, which greatly simplified the analysis in the MS theory,[10,11] replaces the many electron correlation function with a product of pair correlation functions, and so greatly overestimates the QP scattering effect in the asymptotic limit $n_F^{1/2}\gg 1$. In fact, up to the second order in $\widetilde{\Delta}_0$, our result [Eq. (48)] is identical to that obtained by MS [Eq. (1)]. The higher order terms, however, differ substantially; our quartic term is $1/n_F^{1/2}\ll 1$ smaller than that obtained by expanding the exponential in Eq. (1) up to second order in $\lambda(\Delta_0,n_F)$.

This result reflects a very interesting phenomenon: In the ground Landau level approximation for the condensate of Cooper pairs, the quadratic term in the free energy expansion is known[16] to be completely independent of the vortex lines distribution. Therefore, it has nothing to do with the broadening of the Landau levels by the inhomogeneous pair potential in the vortex state. Indeed, in the standard expression[26] for the SC free energy in terms of the dressed electron Green's function (or the QP Green's function) the entire series of self-energy corrections is multiplied by a second order factor in Δ. Consequently, the quartic term is the lowest order correction to the free energy, which contains the scattering effect. It may be, therefore, concluded that in the asymptotic limit of the 2D model used here, the scattering effect is much weaker than what predicted by any theory consistent with the random vortex lattice approximation.[11,10]

The structure of our expression for the free energy Eq. (43) as well as for the oscillatory magnetization Eq. (48) suggests that the small expansion parameter in the theory is $x\equiv\widetilde{\Delta}_0^2/n_F$ rather than $\widetilde{\Delta}_0^2/n_F^{1/2}$, as suggested by Eq. (1). The full expansion should therefore read

$$\widetilde{M}_{\rm osc}\approx\widetilde{M}_{n,\rm osc}[1-\pi^{3/2}\sqrt{n_F}x\Theta(x)], \tag{49}$$

where at $x\ll 1$ the function $\Theta(x)$ has an expansion $\Theta(x)\approx 1-\sqrt{2}\beta_A x$.

Now the expression within the square brackets in Eq. (49) vanishes at $x\Theta(x)=1/\pi^{3/2}n_F^{1/2}$. Thus a sign inversion of the magnetic oscillations amplitude takes place at $x\approx 1/\pi^{3/2}n_F^{1/2}\ll 1$, where $\Theta(x)\approx 1$, i.e., well within the range of validity of our expansion.[13,23]

One therefore expects that in a 2D superconductor the dHvA amplitude will reverse sign due to pairing at a certain field $H_{\rm inv}$ below H_{c2}, and remains virtually undamped well below the point of inversion. This conclusion may be changed if disorder in the vortex lattice, or vortex line motion is taken into account, as indicated by the MS result. However, the application of the MS model to real disordered vortex lattices should be considered very cautiously since the effect of disorder has not been introduced self-consistently there.

The crossover to the low-temperature power-law behavior, obtained by Dukan *et al.*, is reflected in our theory by the breakdown of perturbation theory at very low temperature. At such low temperatures, the LK thermal smearing parameter $X\ll 1$, and our expansion does not exist for all magnetic fields since the density of states parameter q_ν diverges as

$$q_\nu=\frac{2}{\{1-\exp[(2\nu+1)X]\}}\sim\frac{2}{(2\nu+1)^2X^2} \tag{50}$$

when a Landau level crosses the Fermi energy with half integer filling factor n_F.

Under this condition, and for sufficiently small Δ_0, the SC pairing is restricted to a single Landau level, and the QP energies are close to the diagonal elements of the BdG Hamiltonian in the Landau levels representation, i.e.,[24,13]

$$E_{\vec{k},n}=\sqrt{[\hbar\omega_c(n+1/2-n_F)]^2+|\Delta_{n,n}(\vec{k})|^2} \tag{51}$$

which is not an analytical function of Δ_0^2 at the Fermi surface. This also explains the linear dependence of $\lambda(\Delta_0,n_F)$ on Δ_0, obtained by Norman *et al.*[13] for small $\widetilde{\Delta}_0$ at low temperatures.

It is interesting to note that in our expansion the quadratic and the quartic terms for each Matzubara frequency ν are proportional to q_ν and q_ν^2, respectively. Thus, the expansion parameter is actually $x\sim[\Delta_0^2/(\hbar\omega_c)^2 n_F]q_0$.

In the high-temperature limit $X\gg 1$, where $q_0\approx 2$, it reduces to the temperature independent value $x\sim\widetilde{\Delta}_0^2/n_F$ used above. In the very low-temperature limit $X\ll 1$ it diverges with $(1/T)^2$, i.e., $x\sim[\Delta_0/\pi k_B T]^2/n_F$. The breakdown of the small Δ expansion, resulting from this divergence at sufficiently low temperatures, seems to be related to the emergence of an opposite, high Δ expansion in the small parameter $1/x\sim(\pi k_B T/\Delta_0)^2$, as obtained by Dukan and Tesanovic[14] [see Eq. (3)] in the low-temperature limit.

The application of the theory developed in the present paper to real experimental situations is not a straightforward

matter; in addition to the influence of disorder in the vortex lattice and vortex line fluctuations on the QP scattering, discussed above, the 3D nature of the single electron band structure could also play an important role. The importance of the latter effect may be appreciated by noting that in contrast to the 2D model studied here, in a 3D electron system, e.g., with a spherical Fermi surface, Cooper pairs in low Landau levels (i.e., for $n,n' \approx 0$) and with large longitudinal momenta (i.e., near $k_z = k_z' = k_F$), have the largest contribution to the SC condensation energy. This region is far away from the extremal orbit $k_z = 0$, $n = n_F$, which dominates dHvA oscillations.

As a result, in addition to the QP near the extremal orbit, their counterparts with small cyclotron orbits (i.e., for $n \ll n_F$) and large longitudinal momenta k_z, should also contribute significantly to the SC free energy in this case. The relatively strong sensitivty of QP with small cyclotron orbits to scattering by the vortex lattice, as implied by the large damping parameter λ found in Ref. 13, may indicate that the QP scattering effect in 3D systems is stronger than in the equivalent 2D systems. An effective parameter $n_F^* \leqslant n_F$ may be therefore introduced to take into account such an increase in the QP scattering effect.

Most of the SC materials in which clear dHvA oscillations were observed in the vortex state, such as V_3Si, Nb_3Sn, YNi_2B_2C, and $NbSe_2$, are essentially 3D systems with complex band structures and nonspherical Fermi surfaces. One therefore expects characteristic values of n_F^* smaller than n_F in these materials.

Furthermore, the nonspherical Fermi surfaces, combined with some unavoidable deviations from perfect crystaline order, should lead to some finite distribution around each dHvA frequency. This should be compared to the effective range of frequency modulation $(\Delta F)_{\rm inv} = H_{c2}H_{\rm inv}/4(H_{c2} - H_{\rm inv})$, associated with the expression within the square brackets in Eq. (49), which does not exceed 15 T.

Thus it is not surprising that the measured signal does not exhibit a fine structure like the sign inversion predicted in our ideal 2D electron gas model.

The organic superconductor $\kappa-(ET)_2Cu(NCS)_2$ seems at first sight a good candidate for testing the predictions of our theory, due to the quasi-2D nature of its electron band structure. Unfortunately, the transition from the normal to the SC state observed experimentally in this material is very broad,[8] extending far below the estimated value of $H_{\rm inv}$, which is found to be very close to the mean field value of H_{c2} in this material. This is not surprising since the low-dimensional nature of this compound and the low temperatures used in the dHvA experiments can lead to strong quantum flucuations in the phase of the order parameter[28,29] and so to the breakdown of the mean field approximation used in our theory.

The relatively weak QP scattering, predicted in the present paper, seems to be confirmed, however, by the majority of the experiments performed so far: According to our theory it should lead to a significant deviation of the experimentally measured amplitude from the Maki-Stephen-Wasserman fitting formula [see Eqs. (1),(2)] in the region where the leading SC effect exceeds the zeroth order (i.e., normal electron) term, i.e, for $H \geqslant H_{\rm inv}$. In this region the above qualitative analysis indicates that the damping of the dHvA oscillations may be described by a parameter $\tilde{\Delta}_0^2/n_F^*$ smaller than the characteristic MS parameter $\lambda \sim \tilde{\Delta}_0^2/\sqrt{n_F}$. Such a crossover from a relatively strong damping just below H_{c2}, described well by the MS fitting formula, to a weaker damping at lower fields, was indeed observed in almost all experiments carried out so far.[4,5,7]

Furthermore, from the available experimental data two different characteristic slopes of the corresponding Dingle plot can be clearly distinguished. Our estimations show that the experimental crossover field $H_{\rm cross}$ from one slope to another is in a good agreement with the calculated inversion field $H_{\rm inv}$. In particular, we have obtained for V_3Si($F = 1570$ T): $H_{\rm cross} \sim 12.5$ T, $H_{\rm inv} \sim 13.8$ T; for YNi_2B_2C($F = 511$ T): $H_{\rm cross} \sim 4.5$ T, $H_{\rm inv} \sim 6$ T; and for Nb_3Sn($F = 581$ T): $H_{\rm cross} \sim 11.4$ T, $H_{\rm inv} \sim 13.7$ T.

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