Collective modes in supersolid ^{4}He

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We derive the hydrodynamic equations of motion of solid and supersolid ^{4}He, that describe the collective modes of these phases. In particular, the usual hydrodynamics is modified in such a way that it leads to the presence of a propagating instead of a diffusive defect mode. The former is appropriate for a quantum crystal and observed in recent experiments. Furthermore, we find that in supersolid helium there are two additional modes associated with the superfluid degrees of freedom. The observation of these additional modes is a clear experimental signature of the supersolid phase. [S0163-1829(97)04341-5]

I. INTRODUCTION

The low-temperature behavior of the strongly interacting quantum liquid ^{4}He has been a subject of experimental and theoretical research for decades. In 1908 helium was first liquified by Kamerlingh Onnes and in 1911 he discovered a sharp maximum in the density at what is now commonly called the λ point.[1] After that, a number of macroscopic quantum phenomena such as superfluid flow, second sound, the fountain effect, and quantized vortices were observed. Phenomenological theories were developed and justified from a microscopic point of view.[2–4] Also, the famous Kosterlitz-Thouless transition was first observed in thin superfluid helium films.[5]

In the solid phase of ^{4}He, which is reached only at low temperature and high pressure (see Fig. 1), one also expects to observe macroscopic quantum phenomena because of the large zero point vibration of the atoms about their equilibrium position.[6] Because of this, solid helium has been termed a quantum solid.

In such a solid, the interstitials and vacancies are effectively delocalized due to their ability to tunnel through the potential barriers. At low temperatures these point defects then form a weakly interacting Bose gas. Furthermore, the large zero point motion results in an unusually rapid exchange rate of nearest-neighbor atoms, which may lead to large ring exchanges between the helium atoms.[3] Bose-Einsein condensation of the defects or exchange processes of the lattice atoms may then open two routes to a new phase of matter at low temperature in which long-range crystalline and superfluid order coexist. This is called the supersolid phase.[7]

Theoretically the existence of such a phase has since long been anticipated.[8–10] However, it was only recently claimed to have been observed experimentally that three-dimensional solid ^{4}He is a spatially ordered superfluid, or supersolid, at sufficiently low temperatures and densities. The experiments leading to this claim were performed by Lengua and Goodkind, who measured the attenuation and velocity of sound in solid ^{4}He for relatively high purities and low atomic densities of the quantum crystal.[11] The temperature dependence of the attenuation revealed a coupling to thermally activated excitations, consistent with the existence of a propagating mode in the gas of point defects that is expected to be present in a quantum crystal.[8,12] Furthermore, assuming the speed of sound of the defect mode to depend on the density of defects in the same way as in a dilute Bose gas, they found a relation between the temperature dependence of the phase velocity and that of the defect density. To consistently interpret their data they then had to assume a macroscopic population of the zero momentum state of the point defects, i.e., a Bose-Einstein condensation of the point defects. Thus the phase diagram of ^{4}He in three dimensions would be qualitatively given by Fig. 1. Following a certain trajectory in this phase diagram, ^{4}He may undergo a transition from the normal phase to the superfluid phase at some temperature T_λ and subsequently from the superfluid to the supersolid phase at a temperature T_c. As mentioned above, the possibility of superfluid flow in a solid has since long been anticipated theoretically. Andreev and Lifschitz were the first to attempt to derive the hydrodynamics of a supersolid by including the effect of Bose-Einstein condensation of the defects on the hydrodynamics of an ideal crystal.[8] In addition to this pioneering work, Liu has more recently presented a thorough discussion of the Andreev and Lifschitz hydrodynamics.[13]

However, it was pointed out by Martin *et al.* that the conventional treatment of the hydrodynamic equations for a classical crystal, which does not include defects, is neccesarily incomplete since it yields the wrong number of modes.[14] They identified the missing mode as a mode in the defect density. This implies that the hydrodynamics of Andreev and Lifshits is also incomplete, because it does not include the

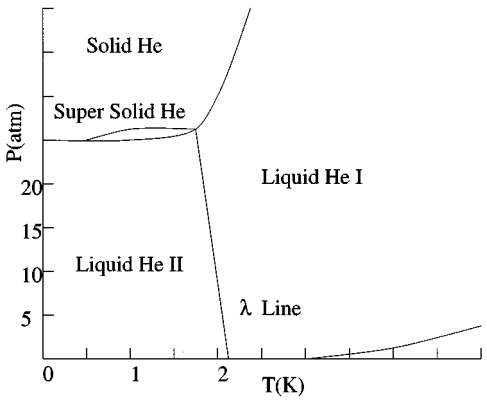

FIG. 1. Tentative sketch of the phase diagram of ^{4}He.

noncondensed defects and as a result does not lead to the required defect mode in the normal state of the crystal. In addition, Martin *et al.* assume diffusive dynamics for their defect mode. This seems to be appropriate for a classical but not for a quantum crystal, where the defect mode is expected to be a propagating mode, as is confirmed by the experiments of Lengua and Goodkind.

Recently Stoof *et al.*, in respons to experiments with submonolayer superfluid helium film,[15] derived the hydrodynamic equations for an isotropic supersolid in two dimensions which did include propagating behavior of the crucial defect mode.[12] Moreover, the longitudinal part of the solid hydrodynamics derived by these authors turns out to be identical to the system of two coupled wave equations that Lengua and Goodkind used to accurately model their data. However, to apply these promising results to the experiments with solid helium, we have to extend them in two ways. First of all we have to consider a three-dimensional system, and second of all we have to take into account the anisotropy of solid ^{4}He, which is a hexagonally closed packed (hcp) crystal. Thus we hope to justify from a microscopic point of view the phenomenological equations that succesfully explained the propagation of sound in solid ^{4}He and led to the first claim of a supersolid phase in this system.

The paper is organized as follows. In Sec. II the hydrodynamic equations describing a normal solid with point defects will be derived. This is achieved by deriving an action describing a solid with dislocations, using methods developed by Kleinert.[16] From this action we obtain the interaction between phonons and a point defect, by seeing the point defect as a limiting case of a dislocation. Also, a more microscopic point of view is presented and dissipation is included. In Sec. III we then add a superfluid degree of freedom to our hydrodynamic equations in the usual way and in Sec. IV we discuss the experiment by Lengua and Goodkind in the light of our results. It should be noted that in order to understand this experiment it is not neccessary to include temperature flucuations into our considerations and we will neglect them in the rest of this article. We conclude with a discussion and outlook in Sec. V.

II. HYDRODYNAMICS OF SOLIDS WITH POINT DEFECTS

In this section we derive the hydrodynamics of a solid with point defects. This will be done by first considering the action describing phonons and their interaction with dislocation loops. We then obtain the interaction of phonons with vacancies and interstitials by shrinking a dislocation loop to zero radius and using a dipolelike approximation. Next, we add the dynamics of the point defects. The structure of the resulting theory is much like that of an electron interacting with electromagnetic fields. An intuitive microscopic picture of point defects is also presented which leads to an alternative derivation of the action descibing a solid with point defects. Finally, the hydrodynamic equations are derived.

A. Gauge theory of phonons and dislocations

We start with deriving a gauge theory that describes the solid phase at long wavelengths. To describe point defects we only have to include dislocations into our theory. Therefore we can ignore higher gradient elasticity, which would be needed if we also wanted to describe disclinations. This section is closely related to previous work done by Kleinert[16,17] but differs from it in the following aspects. First, we do not include higher gradient elasticity. Second, we consider the more general case of anisotropic solids and third, we explicitly remove the unphysical gauge degrees of freedom in the resulting theory of ''quantum defect dynamics.''

The Euclidian action for a solid with dislocations of arbitrary crystaline symmetry is given by[16,17]

$$S[u_i]=\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\left\{\frac{\rho}{2}(\partial_\tau u_i-\beta_i)^2 + \frac{1}{2}\left(u_{ij}-\frac{\beta_{ij}+\beta_{ji}}{2}\right)c_{ijkl}\left(u_{kl}-\frac{\beta_{kl}+\beta_{lk}}{2}\right)\right\}, \quad (1)$$

where $u_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$ is the strain tensor, c_{ijkl} is the elasticity tensor whose structure is determined by the specific symmetry of the crystal under consideration, and ρ is the average mass density. This is the most general quadratic action compatibel with the symmetries of the crystal and the requirement that the Hamiltonian of the system transforms under a Gallilean transformation $(\mathbf{u},t)\rightarrow(\mathbf{u}+\mathbf{v}t,t)$ as $H\rightarrow H+\mathbf{p}\cdot\mathbf{v}+M\mathbf{v}^2$, with $\mathbf{p}$ the total momentum of the crystal and M it's total mass. The latter determines the form of the kinetic energy.

The dislocations mentioned above are topological defects, which exist because the displacement field is multivalued. In much the same way, vortices in a superfluid are a consequence of the multivaluedness of the phase field.[16] The multivaluedness of a displacement field describing a dislocation becomes apparent when writing down what can be seen as the definition of a dislocation, namely,

$$\oint_C du_i=b_i. \quad (2)$$

Here, C is a contour enclosing a dislocation line $\mathcal{L}$ and $\mathbf{b}$ is the so-called Burgers vector measuring the strength of the dislocation. This equation can be written in a differential form as

$$\varepsilon_{ijk}\partial_j\partial_k u_l=b_i\delta_l(\mathbf{x},\mathcal{L})\equiv\alpha_{il}, \quad (3)$$

where α_{il} is called the dislocation density. It is analogous to Ampére's law $\varepsilon_{ijk}\partial_j B_k=J_i$ and states that the displacement field is nonintegrable along the line $\mathcal{L}$ because here the dislocation gives a δ function contribution. If the line $\mathcal{L}$ is parametrized by $\mathbf{x}(s)$, the δ function along $\mathcal{L}$ is defined by

$$\delta_i(\mathbf{x},\mathcal{L})=\int ds\,\frac{\partial x_i(s)}{\partial s}\,\delta[\mathbf{x}-\mathbf{x}(s)]. \quad (4)$$

If we would use in our calculations these multivalued displacement fields, the action would be given by Eq. (1) with $\beta_i=\beta_{ij}=0$. However, to perform a path integral over the u_i it is much more convenient to use a singlevalued displacement field which takes values on the real axis. The unphysical singular contributions to the derivatives of a singlevalued displacement field which describes dislocations in a solid are compensated by substracting the quantities β_{ij} and

β_i. The relation between these quantities and the dislocation line $\mathcal{L}$ is conveniently visualized by the Volterra construction, which we now briefly explain. Given a solid without imperfections, a dislocation can be created by removing from this solid a volume $\mathcal{V}$ and drawing the boundary of the volume together, thus forming a surface $\mathcal{S}$ with boundary $\mathcal{L}$, and restoring the crystaline symmetry everywhere except at this boundary $\mathcal{L}$.

The single-valued displacement field which describes a dislocation created by this construction is discontinuous across the surface $\mathcal{S}$ with a jump in the displacement field that is equal to the Burgers vector **b**. This discontinuity gives a δ function contribution to the gradient of the displacement field which is called the plastic distortion and is given by

$$\beta_{ij}=b_j\delta_i(\mathbf{x},\mathcal{S})=b_j\int_{\mathcal{S}}dS_i\,\delta[\mathbf{x}-\mathbf{x}(u,v)]. \tag{5}$$

The integral measure is defined by $dS_i=\varepsilon_{ijk}\partial_u x_j\partial_v x_k\,du\,dv$ if the surface $\mathcal{S}$ is parametrized by $\mathbf{x}(u,v)$. Furthermore, if the dislocation line $\mathcal{L}$ is moving with a speed **v**, the time derivative of the displacement field gives a δ function contribution of $\beta_i=v_j\beta_{ji}$. If we are not on a dislocation line, the physical values of the spatial and time derivatives of the displacement field should be continuous and are therefore given by

$$(\partial_i u_j)^{\rm phys}=\partial_i u_j-\beta_{ij},$$
$$(\partial_\tau u_j)^{\rm phys}=\partial_\tau u_j-\beta_j. \tag{6}$$

The value of these physical quantities equals what one would get by using the multivalued version of the displacement field to calculate the spatial and time derivatives.[18] We thus see that the action introduced at the beginning of this section is indeed just the classical action for a perfect crystal straightforwardly generalized to include dislocations.[19,20]

To be able to actually calculate the interaction between the phonons and the defects we write the action in a canonical form. We do this by introducing two new fields by means of a Hubbard-Stratonovich transformation.[21] Physically these field are the stress tensor σ_{ij} and momentum density p_i which are canonical to $u_{ij}^{\rm phys}\equiv(\partial_i u_j)^{\rm phys}-(\partial_j u_i)^{\rm phys}$ and $(\partial_\tau u_i)^{\rm phys}$, respectively. It amounts to adding to the Lagrangian density in Eq. (1) the quadratic terms

$$\frac{1}{2\rho}[p_i-i\rho(\partial_\tau u_i)^{\rm phys}]^2$$

and

$$\frac{1}{2}(\sigma_{ij}+iu_{gh}^{\rm phys}c_{ghij})c_{ijkl}^{-1}(\sigma_{kl}+ic_{klmn}u_{mn}^{\rm phys}).$$

Furthermore, to obtain the path integral representation of the partition function $\mathcal{Z}$ we have to add functional integrals over the momentum density p_i and the stress tensor σ_{ij}. Note that the action containes only the symmetric part of the stress tensor, and we should therefore only perform the path integral over the symmetric part of σ_{ij}. Note also that c_{ijkl}^{-1} is symmetric under the exchanges $i\leftrightarrow j$ and $k\leftrightarrow l$, and is defined by

$$c_{ijkl}c_{klmn}^{-1}\equiv\frac{1}{2}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}). \tag{7}$$

The action we find after these transformations reads

$$S[p_i,\sigma_{ij},u_i]=\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\Big\{\frac{p_i^2}{2\rho}+\sigma_{ij}c_{ijkl}^{-1}\sigma_{kl}-ip_i(\partial_\tau u_i)^{\rm phys}+i\sigma_{ij}u_{ij}^{\rm phys}\Big\}. \tag{8}$$

We now integrate out the displacement field, which leads to the constraints

$$\partial_\tau p_j=\partial_i\sigma_{ij}. \tag{9}$$

This is Newton's law. In order to automatically satisfy these constraints we rewrite the fields p_i and σ_{ij} in terms of new fields A_{ij} and F_{ij} by

$$\sigma_{ij}=\partial_\tau F_{ij}+\varepsilon_{ikl}\partial_k A_{lj},$$
$$p_j=\partial_i F_{ij}. \tag{10}$$

Substituting these in the interaction, i.e., the last two terms in the right-hand side of Eq. (8), and performing some partial integrations, we find that this part of the action can be written in terms of the dislocation density α_{ij} and dislocation current density $J_{mlj}\equiv v_m\alpha_{lj}$, as

$$S_{\rm int}[A_{ij},F_{ij}]=\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\{-iA_{ij}\alpha_{ij}-iF_{ij}\varepsilon_{iml}J_{mlj}\}. \tag{11}$$

In the process of rewriting the action we have ended up with to many degrees of freedom. These unphysical degrees of freedom manifest themselves in the fact that the new fields A_{ij} and F_{ij} are gauge fields. Indeed, the expressions for σ_{ij} and p_i are invariant under the gauge transformations

$$F_{ij}\to F_{ij}+\varepsilon_{ikl}\partial_k\Lambda_{lj},$$
$$A_{lj}\to A_{lj}+(\partial_\tau\Lambda_{lj}-\partial_l\Lambda_{\tau j}). \tag{12}$$

At first sight one might therefore think that the gauge freedom removes 12 degrees of freedom. However, we note that these gauge transformations are themselves invariant under a gauge transformation, which reduces the number of gauge degrees of freedom. Indeed, the gauge transformations are invariant under

$$\Lambda_{lj}\to\Lambda_{lj}+\partial_l\lambda_j,$$
$$\Lambda_{\tau j}\to\Lambda_{\tau j}+\partial_\tau\lambda_j. \tag{13}$$

As a result the gauge freedom in Eq. (12) only removes $12-3=9$ degrees of freedom in the expressions for p_i and σ_{ij}. Therefore we are left with $18-9=9$ degrees of freedom in the fields A_{ij} and F_{ij}, which is exactly what we expect because there are 3 degrees of freedom present in p_i, 9 in σ_{ij}, and the constraints in Eq. (9) remove 3 of these. Note that we should also demand σ_{ij} to be symmetric, which will remove another 3 degrees of freedom. This means that

we end up with 6 physical degrees of freedom, corresponding to the usual 6 phonon modes.

In order to extract physically relevant information we will have to remove the gauge-degrees of freedom, i.e., fix the gauge. Before we embark on this problem, however, we will prove the following equalities which we will need later on when deriving the hydrodynamic equations of motion for a solid with point defects. They are

$$\langle u_{ij}^{\rm phys}\rangle = i c_{ijkl}^{-1}\langle \sigma_{kl}\rangle,$$

$$\langle (\partial_\tau u_i)^{\rm phys}\rangle = -\frac{i\langle p_i\rangle}{\rho}. \tag{14}$$

The proof is given by adding to the action in Eq. (1) source terms proportional to the currents K_{ij} and K_i:

$$\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\{K_{ij}u_{ij}^{\rm phys}+K_i(\partial_\tau u_i)^{\rm phys}\}.$$

Expectation values of $f[u_{ij}^{\rm phys},(\partial_\tau u_i)^{\rm phys}]$ are now easily calculated as

$$\langle f[u_{ij}^{\rm phys},(\partial_\tau u_i)^{\rm phys}]\rangle = f\left(\frac{\partial}{\partial K_{ij}},\frac{\partial}{\partial K_i}\right)\ln\,\mathcal{Z}(K_{ij},K_i)|_{K_{ij}=K_i=0}, \tag{15}$$

where $\mathcal{Z}(K_{ij},K_i)$ denotes the partition function with nonvanishing source terms. Again performing a Hubbard-Stratonovich transformation we get

$$S[u_i]=\int_0^{\hbar\beta} d\tau\int d\mathbf{x}\left\{\frac{p_i^2}{2\rho}+\frac{1}{2}\sigma_{ij}c_{ijkl}^{-1}\sigma_{kl}-ip_i\times\left((\partial_\tau u_i)^{\rm phys}+\frac{K_i}{\rho}\right)+i\sigma_{ij}(u_{ij}^{\rm phys}+c_{ijkl}^{-1}K_{kl})-\frac{K_i^2}{2\rho}-K_{ij}c_{ijkl}^{-1}K_{kl}\right\}. \tag{16}$$

Equation (14) now follows by differentiation.

After this digression, we return to the elimination of the nonphysical degrees of freedom present in the action $S=S_0+S_{\rm int}$ and reexamine Eq. (12). As mentioned, this gauge transformation is invariant under the transformations in Eq. (13). We use the latter invariance to choose a gauge in which $\Lambda_{\tau j}=0$, which is always possible by letting λ_j satisfy

$$\lambda_j=-\int_0^\tau \Lambda_{\tau j}(\tau')d\tau'. \tag{17}$$

In this gauge our original gauge transformation reduces to

$$F_{ij}\to F_{ij}+\varepsilon_{ikl}\partial_k\Lambda_{lj},$$

$$A_{lj}\to A_{lj}-\partial_\tau\Lambda_{lj}. \tag{18}$$

In order for σ_{ij} to be symmetric, we use part of this residual gauge freedom to choose F_{ij} symmetric. In addition, we introduce the fields χ_{ij} by means of

$$A_{lj}=\varepsilon_{jmn}\partial_m\chi_{ln}. \tag{19}$$

If we now take χ_{ln} to be symmetric, σ_{ij} will also be symmetric. In terms of these fields the free action S_0 becomes

$$S_0[F_{ij},\chi_{ij}]=\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\left\{\frac{(\partial_i F_{ij})^2}{2\rho}+\frac{1}{2}(\partial_\tau F_{ij}+\varepsilon_{ikl}\varepsilon_{jmn}\partial_k\partial_m\chi_{ln})c_{ijkl}^{-1}\times(\partial_\tau F_{kl}+\varepsilon_{kpq}\varepsilon_{jrs}\partial_p\partial_r\chi_{qs})\right\}. \tag{20}$$

At this point F_{ij} and χ_{ij} both contain 6 degrees of freedom. We are thus left with $12-6=6$ nonphysical degrees of freedom which somehow correspond to 6 degrees of freedom in Λ_{ij}.

To eliminate the remaining unphysical degrees of freedom, we expand the Fourier transform of the fields F_{ij} and χ_{ij} in the helicity basis $\{e_{ij}^{(s,h)}\}$.[16] If we take a direct product of momentum space $\mathcal{P}$ with itself, i.e., $\mathcal{P}\otimes\mathcal{P}$, the helicity basis is defined as the irreducible representations of the rotation group in this space. From group theory we know that they form a complete set.[22] Hence we can develop a given tensor field in this basis leading to

$$\chi_{ij}(\mathbf{k})=\sum_{s,h} e_{ij}^{(s,h)}(\hat{\mathbf{k}})\chi^{(s,h)}(\mathbf{k}),$$

$$F_{ij}(\mathbf{k})=\sum_{s,h} e_{ij}^{(s,h)}(\hat{\mathbf{k}})F^{(s,h)}(\mathbf{k}). \tag{21}$$

Because of the symmetry of χ_{ij} and F_{ij} the six nonzero components are $(s,h)=\{(0,0),(2,0),(2,\pm1),(2,\pm2)\}$. To identify the surviving physical helicity components, we note that the expression $\varepsilon_{ikl}\varepsilon_{jmn}\partial_k\partial_m\chi_{ln}$ is symmetric, traceless and invariant under the transformation

$$\chi_{ln}\to\chi_{ln}+\partial_l\xi_n+\partial_n\xi_l. \tag{22}$$

If we choose as a basis in Fourier space $\{\hat{\mathbf{k}}_n,e_n^{(1,1)}(\hat{\mathbf{k}}),e_n^{(1,-1)}(\hat{\mathbf{k}})\}$ and develop ξ_n in terms of this basis, this transformation reads up to a factor k

$$\begin{aligned}\hat{k}_l\xi_n+\hat{k}_n\xi_l &= 2\hat{k}_l\hat{k}_n\xi^{(0)}+(\hat{k}_l e_n^{(1,1)}+\hat{k}_n e_l^{(1,1)})\xi^{(1)}+(\hat{k}_l e_n^{(1,-1)}+\hat{k}_n e_l^{(1,-1)})\xi^{(-1)}\\ &=\frac{2}{\sqrt{3}}(\sqrt{2}e_{ln}^{(2,0)}+e_{ln}^{(0,0)})\xi^{(0)}+\sqrt{2}e_{ln}^{(2,1)}\xi^{(1)}+\sqrt{2}e_{ln}^{(2,-1)}\xi^{(-1)}.\end{aligned} \tag{23}$$

From this expression we see that if we choose a new basis in which to develop χ_{ln} given by

$$\{e_{ln}^{(2,2)},e_{ln}^{(2,-2)},e_{ln}^{(2,1)},e_{ln}^{(2,-1)},e_{ln}^{L},e_{ln}^{L'}\}, \tag{24}$$

where

$$e_{ln}^{L}=\frac{1}{\sqrt{3}}(-e_{ln}^{(2,0)}+\sqrt{2}e_{ln}^{(0,0)})=\frac{1}{\sqrt{2}}(\delta_{ln}-\hat{k}_l\hat{k}_n),$$

$$e_{ln}^{L'}=\frac{1}{\sqrt{3}}(\sqrt{2}e_{ln}^{(2,0)}+e_{ln}^{(0,0)})=\hat{k}_l\hat{k}_n, \tag{25}$$

the components of χ_{ln} corresponding to $\{(2,1),(2,-1),L'\}$ are unphysical and disappear from the action because they correspond to a gauge transformation. The coordinate transformation from the old to the new basis is unitary and hence the new basis is also orthonormal and complete in the space of symmetric second rank tensors. In addition the elements $\{(2,2),(2,-2),L\}$ satisfy

$$k_l e_{ln}^{(L)}=k_l e_{ln}^{(2,2)}=k_l e_{ln}^{(2,-2)}=0. \tag{26}$$

This means there are only 3 dynamical degrees of freedom left in $\partial_i F_{ij}$ corresponding to the helicity components $\{(2,1),(2,-1),L'\}$.

We now Fourier transform the action and expand the fields F_{ij} and χ_{ij} in terms of the basis in Eq. (24). We get

$$\begin{aligned}S_0[F_{ij},\chi_{ij}]=\int_0^{\hbar\beta}d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\Bigg\{&\frac{1}{2\rho}|ik_i(e_{ij}^{(2,1)}F^{(2,1)}\\&+e_{ij}^{(2,-1)}F^{(2,-1)}+e_{ij}^{L'}F^{L'})|^2\\&+\frac{1}{2}\Bigg[\sum_{s,h}e_{ij}^{(s,h)}\partial_\tau F^{(s,h)}+k^2(e_{ij}^{(2,2)}\chi^{(2,2)}\\&+e_{ij}^{(2,-2)}\chi^{(2,-2)}-e_{ij}^{L}\chi^{L})\Bigg]^*c_{ijkl}^{-1}[ij\leftrightarrow kl]\Bigg\},\end{aligned} \tag{27}$$

where $[ij\leftrightarrow kl]$ denotes the part between brackets with the indicated interchange of indices. Up to now we have removed all but three unphysical degrees of freedom. By introducing χ_{ij} and choosing a gauge in which F_{ij} and χ_{ij} are both symmetric, the stress tensor was made symmetric. Hence we were left with 12 degrees of freedom. Then we identified 3 unphysical gauge degrees of freedom in χ_{ij} corresponding to the helicity components $\{(2,1),(2,-1),L'\}$, which reduced the remaining number degrees of freedom to 9. Hence we expect a residual gauge freedom to be present in the above action corresponding to three unphysical degrees of freedom. As is apparent in the expression for S_0, this is indeed the case and the remaining gauge freedom corresponds to

$$\begin{aligned}F^{(s,h)}&\rightarrow F^{(s,h)}+k^2\Lambda^{(s,h)},\\ \chi^{(s,h)}&\rightarrow\chi^{(s,h)}-\partial_\tau\Lambda^{(s,h)},\end{aligned} \tag{28}$$

for $(s,h)=\{(2,2),(2,-2),L\}$. We will see below that this gauge freedom is also present in $S_{\rm int}$. To remove this remaining freedom we introduce three new, invariant, fields

$$\chi'^{(2,2)}=\chi^{(2,2)}+\frac{\partial_\tau F^{(2,2)}}{k^2},$$

$$\chi'^{(2,-2)}=\chi^{(2,-2)}+\frac{\partial_\tau F^{(2,-2)}}{k^2},$$

$$\chi'^{L}=\chi^{L}-\frac{\partial_\tau F^{L}}{k^2}. \tag{29}$$

If we furthermore realize that

$$k_i e_{ij}^{(2,1)}=\frac{k}{\sqrt{2}}e_j^{(1,1)},$$

$$k_i e_{ij}^{(2,-1)}=\frac{k}{\sqrt{2}}e_j^{(1,-1)},$$

$$k_i e_{ij}^{L'}=k\hat{k}_j, \tag{30}$$

we obtain the final expression for S_0 which contains precisely 6 dynamical degrees of freedom.

Before explicitly writing down S_0 we introduce a new compact notation which also simplifies the algebraic manipulations involved in the remainder of this article. We define the quantities

$$\vec{F}=\begin{pmatrix}F^{(2,1)}\\F^{(2,-1)}\\F^{L'}\end{pmatrix};\quad \vec{\chi}'=\begin{pmatrix}\chi'^{(2,2)}\\\chi'^{(2,-2)}\\-\chi'^{L}\end{pmatrix};$$

$$\vec{e}_{ij}^{(1)}=\begin{pmatrix}e_{ij}^{(2,1)}\\e_{ij}^{(2,-1)}\\e_{ij}^{L'}\end{pmatrix};\quad \vec{e}_{ij}^{(2)}=\begin{pmatrix}e_{ij}^{(2,2)}\\e_{ij}^{(2,-2)}\\e_{ij}^{L}\end{pmatrix} \tag{31}$$

and

$$\begin{aligned}A_{\mu\nu}&=(\vec{e}_{ij}^{(1)})_\mu c_{ijkl}^{-1}(\vec{e}_{kl}^{(1)})_\nu\\ B_{\mu\nu}&=(\vec{e}_{ij}^{(1)})_\mu c_{ijkl}^{-1}(\vec{e}_{kl}^{(2)})_\nu\\ C_{\mu\nu}&=(\vec{e}_{ij}^{(2)})_\mu c_{ijkl}^{-1}(\vec{e}_{kl}^{(2)})_\nu\end{aligned};\quad A'=\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}. \tag{32}$$

Indices refering to the abstract vector space introduced above, are denoted by Greek symbols to distinguish them from their real space counterparts. This allows us to write S_0 as

$$\begin{aligned}S_0[\vec{F},\vec{\chi}']=\frac{1}{2}\int_0^{\hbar\beta}d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\\ \times\begin{pmatrix}\vec{F}\\\vec{\chi}'\end{pmatrix}^*\cdot\begin{pmatrix}\frac{A'k^2}{2\rho}-A\partial_\tau^2&-k^2B^\dagger\partial_\tau\\k^2B\partial_\tau&k^4C\end{pmatrix}\cdot\begin{pmatrix}\vec{F}\\\vec{\chi}'\end{pmatrix}.\end{aligned} \tag{33}$$

Note the minus signs, which arise from partial integration.

Next, we also have to write the interaction in terms of the physical fields $\vec{F}$ and $\vec{\chi}'$. The interaction in terms of F_{ij} and χ_{ij} is given by

$$S_{\rm int}[\chi_{ij},F_{ij}]=\int d\tau\int d\mathbf{x}\{-i\varepsilon_{jmn}\partial_m\chi_{in}\alpha_{ij}-iF_{ij}\varepsilon_{iml}J_{mlj}\}. \tag{34}$$

It is not immediately obvious from this equation that the interaction can be rewritten in terms of $\vec{F}$ and $\vec{\chi}'$. Therefore we will show this explicitly. After partially integrating the first part of Eq. (34), the field χ_{ij} interacts with $\varepsilon_{jmn}\partial_m\alpha_{ij}=\varepsilon_{jmn}\partial_m\varepsilon_{ikl}\partial_k\beta_{lj}$ which is symmetric and trace-

less and therefore has only the helicity components $\{(2,2),(2,-2),L'\}$. Next we rewrite the second term term $-iF_{ij}\varepsilon_{iml}J_{mlj}$ as

$$-iF_{ij}\varepsilon_{iml}J_{mlj}=-iF_{ij}\Bigg[\left(\delta_{il}-\frac{\partial_i\partial_l}{\partial^2}\right)\left(\delta_{kj}-\frac{\partial_k\partial_j}{\partial^2}\right)\varepsilon_{lmn}J_{mnk}+\frac{\partial_i\partial_l}{\partial^2}\varepsilon_{lmn}J_{mnj}+\left(\delta_{il}-\frac{\partial_i\partial_l}{\partial^2}\right)\times\frac{\partial_k\partial_j}{\partial^2}\varepsilon_{lmn}J_{mnk}\Bigg]. \tag{35}$$

The second and third term on the right-hand side of the above equation will give an interaction with $F^{(2,1)}$, $F^{(2,-1)}$, and $F^{(L')}$ as is obvious from the fact that the contractions $\partial_i F_{ij}$ and $\partial_j F_{ij}$ annihilate the components $\{(2,2),(2,-2),L\}$. The first term together with the interaction term involving χ_{ij} will reduce to an interaction with the new fields introduced in Eq. (29), i.e., $\vec{\chi}'$. To see this we use

$$\partial^2\delta_{ij}-\partial_i\partial_j=\varepsilon_{iml}\partial_m\varepsilon_{lkj}\partial_k, \tag{36}$$

and substite this in the first term on the right-hand side of Eq. (35). We obtain

$$\begin{aligned}
&-i\int d\mathbf{x}\,\frac{F_{ij}}{\partial^4}(\varepsilon_{ipq}\partial_p\varepsilon_{qrl}\partial_r)(\varepsilon_{kst}\partial_s\varepsilon_{tvj}\partial_v)\varepsilon_{lmn}J_{mnk}\\
&=i\int d\mathbf{x}\left(\varepsilon_{qpi}\partial_p\varepsilon_{tvj}\partial_v\frac{F_{ij}}{\partial^4}\right)\varepsilon_{kst}\partial_s\varepsilon_{qrl}\partial_r\varepsilon_{lmn}J_{mnk}\\
&=-i\int d\mathbf{x}\left(\varepsilon_{qpi}\partial_p\varepsilon_{tvj}\partial_v\frac{F_{ij}}{\partial^4}\right)\varepsilon_{kst}\partial_s\partial_\tau\alpha_{qk}\\
&=i\int d\mathbf{x}\left(\varepsilon_{qpi}\partial_p\varepsilon_{tvj}\partial_v\frac{\partial_\tau F_{ij}}{\partial^4}\right)\varepsilon_{kst}\partial_s\alpha_{qk},
\end{aligned} \tag{37}$$

where we have used

$$\varepsilon_{qrl}\partial_r\varepsilon_{lmn}J_{mkl}=\partial_r(\delta_{qm}\delta_{rn}-\delta_{qn}\delta_{rm})v_m\alpha_{nk}=\partial_\tau\alpha_{qk}. \tag{38}$$

We see that there is indeed only an interaction with the (2,2), $(2,-2)$, and L components of F_{ij}. Together with the expansion for $-i\varepsilon_{jmn}\partial_m\chi_{in}\alpha_{ij}$ these precisely form the fields $\chi'^{(s,h)}$. Inserting all this into $S_{\rm int}$, the interaction between the dislocations and the physical fields describing the phonon modes finally becomes

$$\begin{aligned}
S_{\rm int}[\vec{F},\vec{\chi}']=i\int d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\{&(e_{ln}^{(2,2)}\chi'^{(2,2)}+e_{ln}^{(2,-2)}\chi'^{(2,-2)}\\
&+e_{ln}^{L}\chi'^{L})\varepsilon_{jmn}[ik_m\alpha_{lj}]^*-(e_{ij}^{(2,1)}F^{(2,1)}\\
&+e_{ij}^{(2,-1)}F^{(2,-1)}-e_{ij}^{(L')}F^{(L')})\hat{k}_i\hat{k}_l\varepsilon_{lmn}J_{mnj}^*\\
&+(e_{ij}^{(2,1)}F^{(2,1)}+e_{ij}^{(2,-1)}F^{(2,-1)}\\
&-e_{ij}^{(L')}F^{(L')})\varepsilon_{ipq}\hat{k}_p\hat{k}_k\hat{k}_j\hat{k}_mJ_{mqk}^*\}.
\end{aligned} \tag{39}$$

From this interaction between phonons and a dislocation loop we now want to derive the interaction between phonons and a point defect. To do so, we consider a point defect to be a dislocation loop that shrinks to zero radius. The Volterra construction shows that one creates a dislocation by removing a volume $\mathcal{V}$ from the crystal. When shrinking the dislocation loop, this volume finally ends up being the volume of a single atom. In this way one can thus remove or add the volume of a single atom, which results in creating a vacancy or an interstitial. Alternatively, one can take the long wavelength limit, in which the dislocation will effectively look similar to a point defect. However, naively applying the above procedure yields zero interaction between the point defects and the phonons because a point defect has no Burgers vector. Indeed, assuming that the wavelength of the fluctuations of the phonon fields are much larger than the radius of a dislocation allows for the action to be coarse grained. The physical fields $\vec{F}$ and $\vec{\chi}$ then interact with something proportional to

$$\int_{\mathcal{V}}d\mathbf{x}\alpha_{ij}.$$

However, the above quantity is zero and therefore there is no interaction with the phonon fields in a first approximation. We thus conclude that in order to find an interaction we need to have a gradient of the fields $\vec{F}$ and $\vec{\chi}$ over the radius of a dislocation loop. This is analogous to the well-known multipole expansion in electrodynamics, where a dipole has no netto charge but interacts with the gradient of the electromagnetic potentials. Assuming the physical fields to be slowly varying over the region of nonzero dislocation density we can perform a gradient expansion that leads to an interaction of the form

$$S_{\rm int}=i\int_0^{\hbar\beta}d\tau\int\frac{d\mathbf{k}}{(2\pi)^2}\{k^2\vec{a}\cdot\vec{\chi}'N_\Delta^*+k\vec{F}\cdot M\cdot\vec{J}^*\}, \tag{40}$$

containing an extra factor of k. The defect density is denoted by N_Δ, $\vec{J}=(J^{(0)},J^{(1)},J^{(-1)})$ are the helicity components of the defect current density $\mathbf{J}$, and $M_{\mu\nu}$ is a matrix. If we have n defects located at $\{\mathbf{x}^{(n)}\}$, the defect density N_Δ and defect current density J_i are given by

$$N_\Delta(\mathbf{x})=\sum_n q^{(n)}\delta(\mathbf{x}-\mathbf{x}^{(n)}),$$

$$J_i(\mathbf{x})=\sum_n i\partial_\tau\mathbf{x}^{(n)}q^{(n)}\delta(\mathbf{x}-\mathbf{x}^{(n)}). \tag{41}$$

In general there can be both vacancies and interstitials present, and the charge q distinguishes between vacancies $(q=-1)$ and interstitials $(q=1)$. Thus, the netto defect density N_Δ is in fact the difference between the interstitial density and the vacancy density $N_\Delta^{\rm int}-N_\Delta^{\rm def}$, which is conserved because defects and interstitials can locally only be created in pairs. The defect density and defect current density therefore satisfy a continuity equation

$$\partial_\tau N_\Delta=i\partial_i J_i. \tag{42}$$

Up to this point, no specific crystaline symmetry has been assumed. However, the explicit form of the interaction is determined by the symmetry of the crystal under consider-

ation. This symmetry constrains the coefficients $\vec{a}$ and the matrix M. In principle it should be possible to expicitly take the limit of a dislocation loop shrinking to zero in the interaction given by Eq. (39). The symmetry of the crystal is then contained in α_{ij} and J_{ijk}, because the Burgers vector can only be a lattice vector. There are however subtilities involved in doing this, and the vector $\vec{a}$ and matrix M will therefore be determined from symmetry considerations. Because we are especially interested in the behavior of solid ^{4}He, we consider from this point on the special case of a hexagonally close packed (hcp) crystal structure. The associated symmetry group is $\mathcal{C}_{6h}$, which contains rotations about the c axis and reflections in the ab plane. The elasticity tensor c_{ijkl} for this symmetry containes five constants, the analogs of the Lamé constants λ and μ in the isotropic case.

In what follows we need the field equations for $\vec{F}$ and $\vec{\chi}'$ which follow from the complete action $S_0+S_{\rm int}$ and are given by

$$\partial_\tau^2\vec{F}=\frac{k^2}{2\rho}P\cdot\left(A'\cdot\vec{F}+i\frac{2\rho}{k}M\cdot\vec{J}\right)-iQ^\dagger\cdot\vec{a}\,\partial_\tau N_\Delta,$$

$$\partial_\tau\vec{\chi}'=\frac{1}{2\rho}Q\cdot\left(A'\cdot\vec{F}+i\frac{2\rho}{k}M\cdot\vec{J}\right)-\frac{i}{k^2}R\cdot\vec{a}\,\partial_\tau N_\Delta, \quad (43)$$

where

$$\begin{pmatrix}P & Q^\dagger\\ Q & R\end{pmatrix}\cdot\begin{pmatrix}A & B^\dagger\\ B & C\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \quad (44)$$

To determine the form of the interaction between the phonons and the defects we calculate the stress tensor σ_{ij} resulting from a single point defect. We then note that from the symmetry of the crystal and the fact that a single point defect has no orientation it follows that σ_{ij} has to be invariant under the symmetry operations of $\mathcal{C}_{6h}$, i.e., rotations around the c axis and reflections in the ab plane. Denoting a particular symmetry operation by S, we get

$$S_{ik}S_{jl}\sigma_{kl}(S^{-1}\mathbf{x})=\sigma_{ij}(\mathbf{x}), \quad (45)$$

which in Fourier language reads

$$S_{ik}S_{jl}\sigma_{kl}(S^{-1}\mathbf{k})=\sigma_{ij}(\mathbf{k}). \quad (46)$$

In the case of a static defect we can solve for σ_{ij}, using the field equations for $\vec{\chi}'$. These can be derived from Eq. (43) and read

$$\vec{\chi}'=-\frac{i}{k^2}C^{-1}\cdot\vec{a}N_\Delta. \quad (47)$$

With the notation introduced before χ_μ and a_ν are vectors, and $C_{\mu\nu}$ is a matrix given by Eq. (32). Inserting this into the equation for σ_{ij} with $\vec{F}=0$, i.e., considering a static defect, we get

$$\sigma_{ij}=k^2\vec{e}_{ij}^{(2)}\cdot\vec{\chi}'=-i\vec{e}_{ij}^{(2)}\cdot C^{-1}\cdot\vec{a}N_\Delta. \quad (48)$$

This means that Eq. (46) translates into

$$i\{[S_{ik}S_{jl}\vec{e}_{kl}^{(2)}(S^{-1}\hat{\mathbf{k}})]\cdot[C^{-1}(S^{-1}\hat{\mathbf{k}})]$$
$$-[\vec{e}_{kl}^{(2)}(\hat{\mathbf{k}})]\cdot[C^{-1}(\hat{\mathbf{k}})]\}\cdot\vec{a}=0. \quad (49)$$

Next we are going to translate this equation into a restriction on the coefficients $\vec{a}$ in the interaction. In order to do so, we must choose a particular form for the thus far unspecified helicity basis. On each point of the unit sphere we choose three orthonormal vectors. One in the radial direction, i.e., $\hat{\mathbf{k}}$, the other two in such a way that the vector fields we get in this way are invariant under rotations about the c axis. This cannot be done for the entire sphere and the points $(k_a,k_b,k_c)=\{(0,0,1),(0,0,-1)\}$ are excluded. Therefore the only points where Eq. (49) is not, by construction, automatically satisfied for rotations about the c axis, are indeed these two points. We only treat $(k_a,k_b,k_c)=(0,0,1)$, because the other point does not lead to any additional restrictions. In this point we choose

$$\mathbf{e}^{(1)}=(1,0,0),$$

$$\mathbf{e}^{(2)}=(0,1,0),$$

$$\mathbf{e}^{(3)}=(0,0,1)=\hat{\mathbf{k}}. \quad (50)$$

This means that here our helicity basis becomes

$$e_{ij}^L=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij},$$

$$e_{ij}^{(2,2)}=\frac{1}{2}\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij},$$

$$e_{ij}^{(2,-2)}=\frac{1}{2}\begin{pmatrix}1 & -i & 0\\ -i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij}. \quad (51)$$

Each of these three matrices transforms according to an irreducible representation of $\mathcal{C}_{6h}$. Thus, according to Schur's lemma,[22] they do not mix under the operator c_{ijkl}, which means that $C_{\mu\nu}=0$ if $\mu\neq\nu$. We notice that these matrices transform under rotations over an angle α about the c axis [we are only considering the point (0,0,1)] as

$$e_{ij}^L(\hat{\mathbf{k}})\to e_{ij}^L(\hat{\mathbf{k}}),$$

$$e_{ij}^{(2,2)}(\hat{\mathbf{k}})\to e^{i2\alpha}e_{ij}^{(2,2)}(\hat{\mathbf{k}}),$$

$$e_{ij}^{(2,-2)}(\hat{\mathbf{k}})\to e^{-i2\alpha}e_{ij}^{(2,-2)}(\hat{\mathbf{k}}). \quad (52)$$

It now follows that

$$C_{\mu\nu}^{-1}(S^{-1}\hat{\mathbf{k}})=C_{\mu\nu}^{-1}(\hat{\mathbf{k}}). \quad (53)$$

Thus for infinitesimal rotations Eq. (49) reduces to

$$\begin{pmatrix}i2\alpha e_{ij}^{(2,2)}\\ -i2\alpha e_{ij}^{(2,-2)}\\ 0\end{pmatrix}\cdot\begin{pmatrix}C_{11}^{-1} & 0 & 0\\ 0 & C_{22}^{-1} & 0\\ 0 & 0 & C_{33}^{-1}\end{pmatrix}\cdot\begin{pmatrix}a^{(2,2)}\\ a^{(2,-2)}\\ a^L\end{pmatrix}=0. \quad (54)$$

This equation has to be valid for all values of i and j. Therefore we get

$$a^{(2,2)}C_{11}^{-1}-a^{(2,-2)}C_{22}^{-1}=0, \quad \text{for } i=j,$$

$$a^{(2,2)}C_{11}^{-1}+a^{(2,-2)}C_{22}^{-1}=0, \quad \text{for } i\neq j. \tag{55}$$

The only solution to these equations is

$$a^{(2,2)}=a^{(2,-2)}=0, \tag{56}$$

which means that our interaction in first instance reduces to

$$S_{\rm int}[\vec{F},\vec{\chi}']=\int_0^{\hbar\beta}d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\{k^2a^L\chi^L N_\Delta^*+k\vec{F}\cdot M\cdot\vec{J}^*\}. \tag{57}$$

Next we must determine the form of the matrix M. This is done by demanding the following equality to be valid:

$$i\rho\partial_\tau\langle(\partial_i u_i)^{\rm phys}\rangle=i\rho\partial_i\langle(\partial_\tau u_i)^{\rm phys}\rangle. \tag{58}$$

If the displacement field is single valued and continuous everywhere, the left- and right-hand side of this equation are two equivalent expressions for $-i\partial_\tau\delta\rho$. Therefore it should be valid when there are only point defects present. However, when there are dislocations present Eq. (58) can be shown to be false. It is therefore not at all obvious that the equality is satisfied at this point, because we started out with a description including dislocations. Indeed, the above requirement actually gives a constraint on M, as we will see below. We first calculate $i\rho\partial_\tau\langle(\partial_i u_i)^{\rm phys}\rangle$ and find

$$\begin{aligned}i\rho\partial_\tau\langle(\partial_i u_i)^{\rm phys}\rangle&=-\rho\partial_\tau c_{iikl}^{-1}\langle\sigma_{kl}\rangle\\&=-\rho\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}c_{iikl}^{-1}(\vec{e}_{kl}^{(1)}\cdot\partial_\tau^2\langle\vec{F}\rangle\\&\quad+k^2\vec{e}_{kl}^{(2)}\cdot\partial_\tau\langle\vec{\chi}\rangle)\\&=-\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\{k^2\langle F^{L'}\rangle\\&\quad-\rho(\sqrt{2}a^L\partial_\tau\langle N_\Delta\rangle+kM_{1j}\langle J_j\rangle)\},\end{aligned} \tag{59}$$

where we used the equations of motion in Eq. (43), the completeness relation for the helicity basis in writing

$$\begin{aligned}\vec{e}_{ij}^{(1)}\cdot Q^\dagger+\vec{e}_{ij}^{(2)}\cdot R&=(\vec{e}_{ij}^{(1)}\cdot\vec{e}_{kl}^{(1)}+\vec{e}_{ij}^{(2)}\cdot\vec{e}_{kl}^{(2)})c_{klmn}\vec{e}_{mn}^{(2)}\\&=c_{ijmn}\vec{e}_{mn}^{(2)},\\\vec{e}_{ij}^{(1)}\cdot P+\vec{e}_{ij}^{(2)}\cdot Q&=(\vec{e}_{ij}^{(1)}\cdot\vec{e}_{kl}^{(1)}+\vec{e}_{ij}^{(2)}\cdot\vec{e}_{kl}^{(1)})c_{klmn}\vec{e}_{mn}^{(2)}\\&=c_{ijmn}\vec{e}_{mn}^{(1)},\end{aligned} \tag{60}$$

and the fact that only e_{ij}^L and $e_{ij}^{L'}$ are traceless. If we compare the result in Eq. (59) with the expression for $i\rho\partial_i\langle(\partial_\tau u_i)^{\rm phys}\rangle$, which reads

$$i\rho\partial_i\langle(\partial_\tau u_i)^{\rm phys}\rangle=\partial_i\langle p_i\rangle=-\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}k^2\langle F^{L'}\rangle, \tag{61}$$

we see that $\partial_\tau\langle(\partial_i u_i)^{\rm phys}\rangle=\partial_i\langle(\partial_\tau u_i)^{\rm phys}\rangle$ if

$$M_{1j}J_j=\sqrt{2}a^L kJ^{L'}=-\sqrt{2}a^L\partial_\tau N_\Delta. \tag{62}$$

Furthermore, we take the interaction with the transverse part of the defect current density to be zero. This is justified by noting that the transverse part of the defect density is not a hydrodynamic variable. We will come back to this point in more detail below. The total interaction is now uniquely defined in terms of one parameter a^L and given by

$$S_{\rm int}[\vec{F},\vec{\chi}']=i\int_0^{\hbar\beta}d\tau\int\frac{d\mathbf{k}}{(2\pi)^3}\{a^L(k^2\chi^L+\sqrt{2}\partial_\tau F^{L'})N_\Delta\}. \tag{63}$$

We have now completely specified the interaction of the point defects with the phonon field. However, only the phonon field has dynamics up to now. Because of their interaction with the phonons the point defects would of course effectively acquire dynamics, but one also expects the point defects to behave as dynamical particles if one could freeze out the phonon field. Roughly speaking, they would behave as particles in a periodic potential that could tunnel from one minimum to another. Therefore we have to add a dynamical term for the point defects. The most general dynamical term that describes propagating behavior of the defects is

$$S_0[\{\mathbf{x}^{(n)}\}]=-\frac{1}{2}\int_0^{\hbar\beta}d\tau\sum_n x_i^{(n)}(m_{ij}\partial_\tau^2)x_j^{(n)}. \tag{64}$$

The form of the anisotropic mass m_{ij} is constrained by the symmetry of the crystal. The field equations are now found by varying with respect to $\vec{F}$, $\vec{\chi}$, and the positions $\mathbf{x}^{(n)}$ of the defects, and are given by

$$\begin{aligned}\partial_\tau^2\vec{F}&=\frac{k^2}{2\rho}P\cdot\left(A'\cdot\vec{F}-i\frac{2\rho}{k^2}\sqrt{2}\vec{a}\partial_\tau N_\Delta\right)-iQ^\dagger\cdot\vec{a}\partial_\tau N_\Delta,\\\partial_\tau\vec{\chi}&=\frac{1}{2\rho}Q\cdot\left(A'\vec{F}-i\frac{2\rho}{k^2}\sqrt{2}\vec{a}\partial_\tau N_\Delta\right)-\frac{i}{k^2}R\cdot\vec{a}\partial_\tau N_\Delta,\\\partial_\tau^2 m_{ij}x_j^{(n)}&=-ia^L q^{(n)}(\sqrt{2}\partial_\tau\partial_i F^{L'}|_{\mathbf{x}=\mathbf{x}^{(n)}}-\partial^2\partial_i\chi'^L|_{\mathbf{x}=\mathbf{x}^{(n)}}).\end{aligned} \tag{65}$$

The equations of motion for $\mathbf{x}^{(n)}$ can also be written as

$$\partial_\tau^2 x_i^{(n)}=-iq^{(n)}m_{ij}^{-1}\partial_j B_{kl}\sigma_{kl}|_{\mathbf{x}=\mathbf{x}^{(n)}}, \tag{66}$$

where

$$B_{ij}=\frac{a^L}{\sqrt{2}}\left(\delta_{ij}+\frac{\partial_i\partial_j}{\partial^2}\right). \tag{67}$$

This way of writing it will prove useful when deriving the hydrodynamic equations of motion in Sec. II C.

B. Microscopic picture

An alternative approach to the derivation of an action which describes the coupled dynamics of the point defects and the phonons is to start from a microscopic action. It describes the atoms constituting the crystal by their positions

$$\mathbf{y}^{(i)}=\mathbf{n}^{(i)}+\mathbf{u}^{(i)} \tag{68}$$

relative to the sites $\{\mathbf{n}^{(i)}\}$ of an ideal reference lattice and assumes an isotropic, short-range interaction $V(|\mathbf{y}^{(i)}-\mathbf{y}^{(j)}|)$ between the individual atoms. In this approach, it is clear that the hydrodynamic momentum density is $\mathbf{g}=i\rho\partial_\tau\mathbf{u}$ and, as we will see below, what approximations we implicitly made when we wrote down the free action of a point defect in Eq. (64). In first instance, the microscopic action reads

$$S[\mathbf{u}]=\int_0^{\hbar\beta}d\tau\left\{\frac{1}{2}\sum_i\rho(\partial_\tau\mathbf{u}^{(i)})^2+\sum_{i<j}V(|\mathbf{y}^{(i)}-\mathbf{y}^{(j)}|)\right\}. \tag{69}$$

To explicitly include the point defects, we then decompose the displacement field into a part describing the phonons and a part describing the defects

$$\mathbf{u}^{(i)}=\mathbf{u}^{(i),ph}+\mathbf{u}^{(i),\mathrm{def}}, \tag{70}$$

where the defects are located at the positions $\{\mathbf{x}^{(n)}(\tau)\}$. Inserting this decomposition of the displacement field into the action we get

$$S[\mathbf{u}]=\int_0^{\hbar\beta}d\tau\left\{\frac{1}{2}\sum_i\rho(\partial_\tau\mathbf{u}^{(i),ph}+\partial_\tau\mathbf{u}^{(i),\mathrm{def}})^2+\sum_{i<j}V(|\mathbf{y}^{(i)}-\mathbf{y}^{(j)}|)\right\}. \tag{71}$$

Since the positions $\mathbf{n}^{(i)}$ correspond to the equilibrium positions of the crystal, the total potential $V(\{\mathbf{x}^{(i)}\})\equiv\Sigma_{i<j}V(|\mathbf{x}^{(i)}-\mathbf{x}^{(j)}|)$ satisfies

$$\delta V(\{\mathbf{n}^{(i)}+\mathbf{u}^{(i)}\})=V(\{\mathbf{n}^{(i)}\})+\mathcal{O}[(\mathbf{u}^{(i)})^2]. \tag{72}$$

For slowly varying displacements, the quadratic terms equals $(1/2)\int d\mathbf{x}u_{ij}c_{ijkl}u_{kl}$. However, we cannot use this long wavelength result to find the interaction between the phonons and the defects because defects cause fluctuations in $\mathbf{u}$ on the scale of a few lattice spacings. To proceed we therefore assume that $\mathbf{u}^{\mathrm{def}}$ only changes significantly over a distance which is much smaller than the typical wavelength of a phonon. Moreover, in the continuum limit we can always write for the displacement of a static defect

$$u_i^{\mathrm{def}}(\mathbf{y})=\partial_i f(\mathbf{y}), \tag{73}$$

because a defect is defined by a nonzero value of $v_0=\int d\mathbf{x}u_{ii}$, i.e., the volume that is removed from or added to the crystal due to the presence of a defect. The function f thus satisfies $\int d\mathbf{y}\partial^2 f=qv_0$, where q is either -1 or $+1$ depending on whether we are dealing with an interstitial or a vacancy. Note that in the isotropic case $f(\mathbf{y}-\mathbf{x})\propto|\mathbf{y}-\mathbf{x}|^{-1}$, where $\mathbf{x}$ is again the location of the defect. Furthermore, to first order in the velocity we can write for a moving point defect

$$\partial_\tau u_i^{\mathrm{def}}(\mathbf{y},\tau)\approx\partial_\tau u_i^{\mathrm{def}}[\mathbf{y}-\mathbf{x}(\tau)]=-\partial_\tau x_j\partial_j u_i^{\mathrm{def}}. \tag{74}$$

Expanding the action in Eq. (71) up to second order in the displacements and making use of the above results the effective action describing phonons and point defects is found to be

$$\begin{aligned}S[u_i]=&\int_0^{\hbar\beta}\int d\mathbf{x}\left\{\frac{\rho}{2}(\partial_\tau u_i^{\mathrm{ph}})^2+\frac{1}{2}u_{ij}^{\mathrm{ph}}c_{ijkl}u_{kl}^{\mathrm{ph}}\right\}\\&+\sum_n\int d\tau\left\{-\frac{v_0^2\rho}{2}x_i^{(n)}(\partial_\tau m_{ij}\partial_\tau)x_j^{(n)}+q^{(n)}v_0\widetilde{B}_{ij}u_{ij}^{ph}\right.\\&\times(\mathbf{x}^{(n)},\tau)+q^{(n)}v_0\rho[\partial_\tau x_i^{(n)}]\partial_i\partial_j[\partial_\tau u_j^{\mathrm{ph}}\\&\left.\times(\mathbf{x}^{(n)},\tau)]+E_c\right\},\end{aligned} \tag{75}$$

where we have neglected contributions with $m\neq n$, assuming the defects to be sufficiently far apart to interact only through the phonon field. Furthermore, E_c denotes the energy associated with the creation of a defect. The microscopic action gives certain relations between the coefficients in this action. However, renormalization changes these coefficients and we believe that it does not preserve the relations between them. Therefore they have to be treated as independent. This is important when trying to establish a connection with the action in terms of the stress tensor, as found in the previous section, which can be achieved by means of two Hubbard-Stratonovich transformations and following the same route as before by introducing the gauge fields. The result turns out to be identical and shows in particular that there is indeed only an interaction between the phonons and the longitudinal part of the defect current density.

C. Hydrodynamics

We can now derive the hydrodynamic equations for a crystal with point defects. The number of hydrodynamic modes is fundamentally related to the number of conserved quantities and the number of broken symmetries. The conserved quantities are the total mass, the total momentum, and the netto number of defects N_Δ, the difference between the number of interstitials and vacancies. The associated conservation laws result in five hydrodynamic modes. In principle we also need to take into account energy conservation, which would yield an additional thermal diffusion mode.[18] However, for our purposes it is relatively unimportant and we will not consider it here. Note, however, that we can obtain the Hamiltonian from the action and therefore in principle also include this mode into our considerations. In addition to the conservation laws, translational symmetry is spontaneously broken, which results in three hydrodynamic Goldstone modes. Hence we expect to find a total of 8 hydrodynamic modes. To find the equations of motion describing these modes we first identify the hydrodynamic momentum density g_i with $\langle p_i\rangle=i\rho\langle(\partial_\tau u_i)^{\mathrm{phys}}\rangle$, which is obvious from a microscopic point of view because locally it is just the momentum of the particles of mass m situated on the latice sites. It is important to note that it includes the momentum of the point defects, because we have constructed the physical quantity $(\partial_\tau u_i)^{\mathrm{phys}}$ that way. In the remainder of this article we do not explicitly include the averaging brackets, since it unneccessarily complicates the notation.

We can immediatly write down the equality

$$\partial_\tau g_i=\partial_j\sigma_{ij}. \tag{76}$$

This equation is nothing but the constraints found in Sec. II A in Eq. (9). In a perfect crystal without defects, the hydrodynamic modes are the phonon modes, and their equation of motion is found by taking the time derivative of the above equation

$$\partial_\tau^2 g_i = \partial_j \partial_\tau \sigma_{ij} . \tag{77}$$

Therefore we need to know $\partial_\tau \sigma_{ij}$ which is easily calculated as

$$\begin{aligned}
\partial_\tau \sigma_{ij} &= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (\vec{e}_{ij}^{(1)} \cdot \partial_\tau^2 \vec{F} + k^2 \vec{e}_{ij}^{(2)} \cdot \partial_\tau \vec{\chi}') \\
&= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left\{ \frac{k^2}{2\rho} c_{ijkl} \vec{e}_{kl}^{(1)} \cdot A' \cdot \vec{F} - i c_{ijkl} (\vec{e}_{kl}^{(1)} \cdot \sqrt{2}\vec{a} \right. \\
&\quad \left. + \vec{e}_{kl}^{(2)} \cdot \vec{a}) \partial_\tau N_\Delta \right\} \\
&= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left\{ -\frac{ik_k}{\rho} c_{ijkl} g_l - i c_{ijkl} (\sqrt{2}\vec{e}_{kl}^{(1)} \cdot \vec{a} \right. \\
&\quad \left. + \vec{e}_{kl}^{(2)} \cdot \vec{a}) \partial_\tau N_\Delta \right\} \\
&= -c_{ijkl} \left(\frac{1}{\rho} \partial_k g_l - B_{kl} \partial_i J_i \right),
\end{aligned} \tag{78}$$

where we used the equations of motion for $\vec{F}$ and $\vec{\chi}'$, the continuity equation $\partial_\tau N_\Delta = i \partial_i J_i$, the expression for B_{ij} in Eq. (67) and the equality

$$\begin{aligned}
\frac{k^2}{2} c_{ijkl} \vec{e}_{kl}^{(1)} \cdot A' \cdot \vec{F} &= k^2 c_{ijkl} \hat{k} \left(\frac{e_l^{(1,1)}}{\sqrt{2}} F^{(2,1)} + \frac{e_l^{(1,-1)}}{\sqrt{2}} F^{(2,-1)} \right. \\
&\quad \left. + e_l^{(1,0)} F^{L'} \right) \\
&= k^2 c_{ijkl} \hat{k}_k \hat{k}_m F_{ml} \\
&= -ik_k c_{ijkl} ik_m F_{ml} = -ik_k c_{ijkl} g_l .
\end{aligned} \tag{79}$$

Thus we find for g_i the following equation of motion, describing the phonon modes and their interaction with the point defects:

$$\partial_\tau^2 g_i = -c_{ijkl} \partial_j \left(\partial_k \frac{g_l}{\rho} - B_{kl} \partial_m J_m \right). \tag{80}$$

We can split the above equation into three continuity equations as

$$\partial_\tau \delta\rho = i \partial_i g_i ,$$

$$\partial_\tau \vartheta_i = i \varepsilon_{ijk} \partial_j g_k ,$$

$$\partial_\tau g_i = i c_{ijkl} \partial_j \left\{ \frac{\partial_k}{\rho} \left(\frac{\partial_l \delta\rho}{\partial^2} + \frac{\varepsilon_{lmn} \partial_m \vartheta_n}{\partial^2} \right) - B_{kl} N_\Delta \right\}. \tag{81}$$

Note that it follows from Eq. (81), combined with the continuity equation for g_i, that σ_{ij} can be written as

$$\sigma_{ij} = -i c_{ijkl} \frac{\partial_k}{\rho} \left(\frac{\partial_l \delta\rho}{\partial^2} + \frac{\varepsilon_{lmn} \partial_m \vartheta_n}{\partial^2} \right) - i B_{kl} N_\Delta . \tag{82}$$

To find the total of eight modes, instead of the six given by the equations above, we need to include point defects into our hydrodynamic equations, as was first noted by Martin *et al.*[14] Thus, we have an additional hydrodynamic variable, the netto defect density N_Δ. Note that the transverse part of the defect current density is not a hydrodynamic variable because the momentum of the defects is not conserved. We now want to write down the equation of motion for $N_\Delta = \Sigma_n q^{(n)} \delta(\mathbf{x} - \mathbf{x}^{(n)})$. Using the equations of motion for the point defects, we get

$$\begin{aligned}
\partial_\tau^2 \sum_n \delta(\mathbf{x} - \mathbf{x}^{(n)}) &= \sum_n q^{(n)} \{ \dot{x}_i^{(n)} \dot{x}_j^{(n)} \partial_i \partial_j \delta(\mathbf{x} - \mathbf{x}^{(n)}) \\
&\quad - \ddot{x}_i^{(n)} \partial_i \delta(\mathbf{x} - \mathbf{x}^{(n)}) \} \\
&= \sum_n q^{(n)} \{ \dot{x}_i^{(n)} \dot{x}_j^{(n)} \partial_i \partial_j \delta(\mathbf{x} - \mathbf{x}^{(n)}) \\
&\quad + i m_{ij}^{-1} \partial_j B_{kl} \sigma_{kl} \partial_i \delta(\mathbf{x} - \mathbf{x}^{(n)}) \}.
\end{aligned} \tag{83}$$

To find the hydrodynamics of the netto defect density N_Δ we have to average Eq. (83) over the initial conditions. In an isotropic gas in the absence of external forces, the term proportional to $\Sigma_n \dot{x}_i^{(n)} \dot{x}_j^{(n)} \delta(\mathbf{x} - \mathbf{x}^{(n)})$ would be the only term present, the average of which is just the pressure tensor π_{ij}. A closed set of equations giving the linearized hydrodynamics would then be found by writing down a gradient expansion for π_{ij} in terms of the hydrodynamic variables. In our case to lowest order the pressure tensor can only be a function of $\delta N_\Delta \equiv N_\Delta - \langle N_\Delta \rangle$ because it is the only variable which is even under time reversal, and we get $\pi_{ij} = M_{ij} \delta N_\Delta + \mathcal{O}(\delta N_\Delta^2)$. Neglecting the terms quadratic in the fluctuations we find for δN_Δ the linearized equation of motion

$$\partial_\tau^2 N_\Delta = -M_{ij} \partial_i \partial_j N_\Delta + i m_{ij}^{-1} \partial_i \partial_j B_{kl} \sigma_{kl} , \tag{84}$$

where $B_{kl} \to B_{kl} \langle N_\Delta \rangle$. Note that if we had naively introduced the dynamics of the defects into our theory by adding a Lagrangian density for the defect density instead of the defects $\mathcal{L} = -\frac{1}{2} N_\Delta ([M_{ij} \partial_i \partial_j]^{-1} \partial_\tau^2 + 1) N_\Delta$ as was done by Stoof *et al.*,[12] we would have obtained the same equation by varying the action with respect to N_Δ. However, our approach is more fundamental and illumates clearly the underlying physics of this Lagrangian density. The hydrodynamic equations are usually given as a set of continuity equations, i.e., with first order time derivatives. Therefore we rewrite Eq. (84) as a pair of continuity equations

$$\partial_\tau N_\Delta = i \partial_i J_i ,$$

$$\partial_\tau J_i = i \partial_i \left(\frac{M_{ij} \partial_i \partial_j}{\partial^2} N_\Delta - i \frac{m_{ij}^{-1} \partial_i \partial_j}{\partial^2} B_{kl} \sigma_{kl} \right). \tag{85}$$

We stress that this is actually only an equation for the longitudinal part of the defect current $J_i^L = \partial_i \partial_j / \partial^2 J_j$. The transverse part is not a hydrodynamic variable and is anticipated to relax to zero on a microscopic time scale. This completes

our discussion of the dissipationless hydrodynamic equations. We have obtained a set of hydrodynamic equations describing phonons, point defects, and their interaction for a hcp crystal. They are given by Eqs. (81) and (85).

It is interesting to note that these equations can also be derived from a hydrodynamic action of which the Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2}\sigma_{ij}\left\{\frac{\partial_j\partial_l\delta_{ik}}{\rho\partial_\tau^2}+c^{-1}_{ijkl}\right\}\sigma_{kl}+i\sigma_{kl}B_{kl}N_\Delta - \frac{1}{2}N_\Delta\left\{\frac{\partial_\tau^2}{m_{ij}\partial_i\partial_j}+E_c\right\}N_\Delta, \tag{86}$$

using $M_{ij}\partial_i\partial_j=E_c m^{-1}_{ij}\partial_i\partial_j$. From the associated action, the hydrodynamic equations describing the phonon modes and their coupling to N_Δ are found by writing down the field equations for σ_{ij} and defining g_i by the constraint $\partial_\tau g_i=\partial_j\sigma_{ij}$. Note that in Eq. (86) the term quadratic in σ_{ij} is just $p_i^2/2\rho+\sigma_{ij}c^{-1}_{ijkl}\sigma_{kl}$, which is the free part of the action in Eq. (8) with the substition $p_i\rightarrow\partial_j\sigma_{ij}/\partial_\tau$. Therefore, Eq. (86) is the analog of the hydrodynamic action describing density fluctuations in a normal fluid.[23,24]

For completeness we write down the total set of hydrodynamic equations which as expected amount to a total of eight continuity equations:

$$\partial_\tau\delta\rho=i\partial_i g_i,$$

$$\partial_\tau\vartheta_i=i\varepsilon_{ijk}\partial_j g_k,$$

$$\partial_\tau g_i=ic_{ijkl}\partial_j\left\{\frac{\partial_k}{\rho}\left(\frac{\partial_l\delta\rho}{\partial^2}+\frac{\varepsilon_{lmn}\partial_m\vartheta_n}{\partial^2}\right)-B_{kl}N_\Delta\right\},$$

$$\partial_\tau N_\Delta=i\partial_i J_i,$$

$$\partial_\tau J_i=i\partial_i\left(\frac{M_{ij}\partial_i\partial_j}{\partial^2}N_\Delta-\frac{im^{-1}_{ij}\partial_i\partial_j}{\partial^2}B_{kl}\sigma_{kl}\right). \tag{87}$$

To check heuristically if we ended up with the right equations we write down the hydrodynamic equations in the case of a isotropic crystal and compare these to the ones found for a two-dimensional isotropic crystal by Stoof *et al.*[12] In the isotropic case c_{ijkl}, M_{ij}, and m_{ij} are given by

$$c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

$$M_{ij}=M\delta_{ij},$$

$$m_{ij}=m\delta_{ij}. \tag{88}$$

This implies the equalities

$$c_{ijkl}\partial_j\partial_k\partial_l\frac{\delta\rho}{\rho\partial^2}=\frac{(\lambda+2\mu)}{\rho}\partial_i\delta\rho,$$

$$c_{ijkl}\partial_j\partial_k\varepsilon_{lmn}\partial_m\frac{\vartheta_n}{\rho\partial^2}=\frac{\mu}{\rho}\varepsilon_{imn}\partial_m\vartheta_n,$$

$$c_{ijkl}B_{kl}\partial_j N_\Delta=\frac{4}{\sqrt{2}}(\mu+\lambda)a^L\partial_i N_\Delta. \tag{89}$$

Furthermore, we write the hydrodynamic equations in real time, which amount to the substitution $\partial_\tau\rightarrow-i\partial_t$. As a result the hydrodynamic equations for an isotropic three-dimensional crystal with point defects are given by

$$\partial_t\delta\rho=-\partial_i g_i,$$

$$\partial_t\vartheta_i=-\varepsilon_{ijk}\partial_j g_k,$$

$$\partial_t g_i=-\left\{\frac{\lambda+2\mu}{\rho}\partial_i\delta\rho+\frac{\mu}{\rho}\varepsilon_{ikl}\partial_k\vartheta_l-\frac{4}{\sqrt{2}}(\mu+\lambda)a^L\partial_i N_\Delta\right\},$$

$$\partial_t N_\Delta=-\partial_i J_i,$$

$$\partial_t J_i=-c_\rho\partial_i\delta\rho-c_\Delta\partial_i N_\Delta, \tag{90}$$

where $c_\rho=[ma^L/\sqrt{2}](4\lambda+4\mu)$ and $c_\Delta=M-2m(a^L)^2(4\lambda+3\mu)$. These are indeed the three-dimensional generalizations of the equations found by Stoof *et al.* for the two-dimensional isotropic cristal without dissipation.

In order to give a realistic description of the system, we need to include dissipational effects into our hydrodynamic equations. Although there is a coupling between the phonon field and the defect density and thus a ''shake up'' of the phonon field if a defect moves, up to this point there is no real dissipation because the bilineair coupling between the phonon and the defect modes causes mixing of these modes, but no dissipation. Therefore we include dissipation into our hydrodynamic equations in the standard way by first expanding the dissipative part of the stress tensor to linear order in the conjugate forces and requiring the coefficients to be compatible with the symmetry of the system under consideration and then in turn expanding the conjugate forces in terms of the currents.[13,14,25] A particularly clear treatment of this standard method is given by Jähnig and Schmidt.[26] Quite generally, our hydrodynamic equations have the form

$$\partial_t\delta\rho=-\partial_i g_i,$$

$$\partial_t\vartheta_i=-\varepsilon_{ijk}\partial_j g_k,$$

$$\partial_t g_i=-\partial_j(\sigma_{ij}+\sigma^D_{ij}),$$

$$\partial_t N_\Delta=-\partial_i J_i,$$

$$\partial_t J_i=-\partial_j(\pi_{ij}+\pi^D_{ij}), \tag{91}$$

where the superscript D denotes the dissipative part of the ''stress'' tensors, and the nondissipative part has already been determined. Roughly speaking the variables ϑ_i and J_i are associated with the three broken symmetries, whereas ρ, g_i, N_Δ account for the conservation of mass, momentum, and defects. The most general dissipative terms allowed by the requirement that the time reversal symmetry of the dissipative currents is opposite to the associated hydrodynamic variable are given by

$$\sigma^D_{ij}=\frac{1}{\rho}\eta^{(1)}_{ijkl}\partial_k g_l+\zeta^{(1)}_{ij}\partial_k J_k,$$

$$\pi^D_{ij}=\frac{1}{\rho}\delta_{ij}\eta^{(2)}_{kl}\partial_k g_l+\delta_{ij}\zeta^{(4)}\partial_k J_k, \tag{92}$$

where we used that J_i contained only a longitudinal degree of freedom. The specific form of the parameters is determined by the discrete symmetries of the system, which in the case of the hcp crystal ^{4}He form the group $\mathcal{C}_{6h}$. It should be noted that in Stoof *et al.* it was incorrectly assumed that the transverse part of the defect current behaves as in a gas and diffuses to zero.[12] As we have seen, the correct behavior of the transverse part of the defect current is a relaxation to zero on a microscopic time scale.

III. SUPERSOLID HYDRODYNAMICS

In view of the exciting experiments by Lengua and Goodkind, our aim in writing this paper was also to formulate the hydrodynamic equations of supersolid ^{4}He. Hence we have come to the point where we have to include into our hydrodynamic equations the superfluid degree of freedom. From microscopic theories developed for superfluid liquids and gases it is well known how we should proceed to include these additional degrees of freedom into the hydrodynamic equations for the normal phase.[4,27] First, the density ρ is split into a normal part ρ^n_{ij} and a superfluid part ρ^s_{ij}, satisfying

$$\rho\,\delta_{ij}=\rho^s_{ij}+\rho^n_{ij}. \tag{93}$$

Note that the tensorial nature of the densities is of importance in the case of an anisotropic hcp crystalline structure. Second, we split the total momentum density of the system into a normal part $\rho^n_{ij}v^n_j$ and a superfluid part $\rho^s_{ij}v^s_j$ according to

$$g_i=\rho^s_{ij}v^s_j+\rho^n_{ij}v^n_j=\rho v^n_i+\rho^s_{ij}(v^s_j-v^n_j), \tag{94}$$

where the superfluid velocity is purely longitudinal, i.e., $\varepsilon_{ijk}\partial_j v^s_k=0$, because it is proportional to the gradient of the superfluid phase ϕ^s.

Furthermore, the dissipative terms have to be generalized for an anisotropic superfluid, and the dynamics of the superfluid velocity has to be determined. Following the standard treatment, the dissipative part of the stress tensor σ^D_{ij} becomes

$$\sigma^D_{ij}=\eta^{(1)}_{ijkl}\partial_k v^n_l+\zeta^{(1)}_{ij}\partial_k J_k+\frac{1}{\rho}\zeta^{(3)}_{ij}\partial_k\rho^s_{kl}(v^s_l-v^n_l). \tag{95}$$

The last term on the right-hand side is the most general term containing the conjugate variable of the phase field, i.e., $\partial_i g^s_i\equiv\partial_i\rho^s_{ij}(v^n_j-v^s_j)$.[13,14,25] Furthermore, the dynamics of the superfluid phase field is basically determined by the Josephson relation and is given by

$$\partial_t v^s_i=-\frac{B_\rho}{\rho^2}\partial_i\delta\rho+\beta_\Delta\partial_i N_\Delta+\partial_i\zeta^{(7)}_{jk}\partial_j v^n_k+\zeta^{(8)}\partial_i\partial_j J_j+\frac{\zeta^{(10)}}{\rho}\partial_i\partial_j\rho^s_{jk}(v^s_k-v^n_k), \tag{96}$$

where $B_\rho=\rho^2(\partial\mu/\partial\rho)|_{T,n_\Delta}$ is the isothermal bulk modulus, μ is the chemical potential per unit mass, and $\beta_\Delta=-\partial\mu/\partial n_\Delta|_{\rho,T}$. By adding the last three terms in the right-hand side of Eq. (96) we have also included dissipation. However, at this point we have to realize that we were already dealing with a two fluid hydrodynamics in the normal solid phase, due to the presence of defects. This means that we also have to split the defect current density J_i into a normal and a superfluid part, i.e., $J_i=J^n_i+J^s_i$. Physically, this means that the superfluid current density can be caused both by the motion of defects and by lattice vibrations.[12] As a result we end up with the following hydrodynamic equations describing supersolid ^{4}He:

$$\begin{aligned}
\partial_t\delta\rho&=-\partial_i g_i,\\
\partial_t\vartheta_i&=-\varepsilon_{ijk}\partial_j g_k,\\
\partial_t g_i&=-c_{ijkl}\partial_j\left\{\frac{\partial_k}{\rho}\left(\frac{\partial_l\delta\rho}{\partial^2}+\frac{\varepsilon_{lmn}\partial_m\vartheta_n}{\partial^2}\right)-B_{kl}N_\Delta\right\}\\
&\quad+\eta^{(1)}_{ijkl}\partial_j\partial_k v^n_l+\zeta^{(1)}_{ij}\partial_j\partial_k J^n_k+\zeta^{(2)}_{ij}\partial_j\partial_k(J^s_k-J^n_k)\\
&\quad+\frac{1}{\rho}\zeta^{(3)}_{ij}\partial_k\rho^s_{kl}(v^s_l-v^n_l),\\
\partial_t N_\Delta&=-\partial_i J_i,
\end{aligned} \tag{97}$$

$$\begin{aligned}
\partial_t J_i&=-\partial_i\left(\frac{\partial_i\partial_j M_{ij}}{\partial^2}N_\Delta-\frac{\partial_i\partial_j m^{-1}_{ij}}{\partial^2}\partial_j B_{kl}i\sigma_{kl}\right)+\eta^{(2)}_{jk}\partial_i\partial_j v^n_k\\
&\quad+\zeta^{(4)}\partial_i\partial_j J^n_j+\zeta^{(5)}\partial_i\partial_j(J^s_j-J^n_j)\\
&\quad+\frac{1}{\rho}\zeta^{(6)}\partial_i\partial_j\rho^s_{jk}(v^s_k-v^n_k),\\
\partial_t v^s_i&=-\frac{B_\rho}{\rho^2}\partial_i\delta\rho+\beta_\Delta\partial_i N_\Delta+\partial_i\zeta^{(7)}_{jk}\partial_j v^n_k+\zeta^{(8)}\partial_i\partial_j J^n_j\\
&\quad+\zeta^{(9)}\partial_i\partial_j(J^s_j-J^n_j)+\frac{\zeta^{(10)}}{\rho}\partial_i\partial_j\rho^s_{jk}(v^s_k-v^n_k),\\
\partial_t J^s_i&=-\frac{B_\Delta}{\rho^2}\partial_i N_\Delta+\beta_\rho\partial_i\delta\rho+\partial_i\zeta^{(11)}_{jk}\partial_j v^n_k+\zeta^{(12)}\partial_i\partial_j J^n_j\\
&\quad+\zeta^{(13)}\partial_i\partial_j(J^s_j-J^n_j)+\frac{1}{\rho}\zeta^{(14)}\partial_i\partial_j\rho^s_{jk}(v^s_k-v^n_k).
\end{aligned}$$

The large number of dissipative terms makes these equations look rather intricate, but in the limit $k\to 0$ only the nondissipative terms remain and a considerable simplification occurs, as we will see below. They are easily seen to represent ten equations for the ten unknown quantities $\delta\rho,\vartheta_i,v^n_i,v^s_i,N_\Delta,J_i$, and J^S_i, realizing that v^s_i, J^s_i, and J^n_i have only one degree of freedom and θ_i has only two degrees of freedom.

IV. COMPARISON WITH EXPERIMENT

We now want to compare our results with the equations used by Lengua and Goodkind to fit the data of their experiment in which they may have observed the supersolid phase of ^{4}He.[11] Their phenomenological equations describe a set of two coupled harmonic oscillators. We show below that these equations essentially follow from our hydrodynamic equations describing a normal crystal with defects.

To find the mode structure present in our dissipationless hydrodynamic equations, it is convenient to rewrite Eq. (84)

in terms of the longitudinal part of the defect current density J_i. After taking the time derivative of the second equation of Eq. (85) and inserting the first equation we get

$$\partial_t^2 J_i = \partial_i \left(\frac{m_{ij}^{-1} \partial_i \partial_j}{\partial^2} B_{kl} i \partial_t \sigma_{kl} + \frac{M_{ij} \partial_i \partial_j}{\partial^2} \partial_k J_k \right). \quad (98)$$

To obtain a closed set of equations we then use the real time version of Eq. (78), which expresses σ_{ij} in terms of g_i and N_Δ. We find

$$\partial_t^2 J_i = \partial_i \left[\frac{m_{ij}^{-1} \partial_i \partial_j}{\partial^2} B_{mn} c_{mnkl} \left(\frac{1}{\rho} \partial_k g_l - B_{kl} \partial_i J_i \right) + \frac{M_{ij} \partial_i \partial_j}{\partial^2} \partial_k J_k \right]. \quad (99)$$

We now turn to Eq. (80) which describes the phonon modes. First we define the eigenvectors $A_i^{(n)}(\hat{\mathbf{k}})$, $n=\{1,2,3\}$, of the matrix $c_{ijkl} k_j k_k$ as

$$c_{ijkl} k_j k_k A_l^{(n)}(\hat{\mathbf{k}}) = k^2 \lambda_n^2(\hat{\mathbf{k}}) A_i^{(n)}(\hat{\mathbf{k}}). \quad (100)$$

The six phonon modes of the ideal crystal are thus given by $A_i^{(n)}(\hat{\mathbf{k}}) e^{i(\omega(\mathbf{k}) t \pm i \mathbf{k}\cdot\mathbf{x})}$, with $\omega^2(\mathbf{k}) = k^2 \lambda_n^2(\hat{\mathbf{k}})$. In order to find the equations used by Lengua and Goodkind we first expand g_i in terms of the eigenvectors $A_i^{(n)}(\hat{\mathbf{k}})$, i.e., $g_i = \Sigma_n g^{(n)}(\mathbf{k}) A_i^{(n)}(\hat{\mathbf{k}})$. We then write $J_i = J^L(\mathbf{k},\tau)\hat{k}_i$ and insert these expressions into Eq. (80) and Eq. (99). After contracting the first equation with the eigenvectors A_i and the second with $\hat{k}_i$, this leads to the following equations in Fourier space:

$$\partial_t^2 g^{(n)} = -\frac{1}{\rho} k^2 \lambda_n^2(\hat{\mathbf{k}}) g^{(n)} + k^2 \alpha^{(n)}(\hat{\mathbf{k}}) J^L,$$

$$\partial_t^2 J^L = -m_{ij} k_i k_j \left(\beta(\hat{\mathbf{k}}) J^L + \frac{1}{\rho} \sum_n \alpha^{(n)}(\hat{\mathbf{k}}) g^{(n)} \right), \quad (101)$$

where we defined $\alpha^{(n)}(\hat{\mathbf{k}}) \equiv B_{ij} c_{ijkl} \hat{k}_j A_l^{(n)}$ and $\beta(\hat{\mathbf{k}}) \equiv (M_{ij} k_i k_j)/(m_{ij} k_i k_j) - B_{ij} c_{ijkl} B_{kl}$. Finally we consider one particular mode, say m, and eliminate the two modes with $n \neq m$. After Fourier transforming also the time variable the equations for $g^{(n)}$ with $n \neq m$ are solved by

$$g^{(n)} = \frac{k^2 \alpha^{(n)}(\hat{\mathbf{k}}) J^L}{\omega^2 - k^2 \lambda_n^2/\rho}. \quad (102)$$

Inserting this into Eq. (101) we find

$$-\omega^2 g^{(m)} = -\frac{1}{\rho} k^2 \lambda_m^2 g^{(m)} + k^2 \alpha^{(m)}(\hat{\mathbf{k}}) J^L,$$

$$-\omega^2 J^L = -m_{ij} k_i k_j \left\{ \beta(\hat{\mathbf{k}}) J^L + \frac{1}{\rho} \alpha^{(m)}(\hat{\mathbf{k}}) g^{(m)} + \frac{1}{\rho} \sum_{n \neq m} \frac{\alpha^{(n)}(\hat{\mathbf{k}}) \alpha^{(n)}(\hat{\mathbf{k}}) k^2 J^L}{\omega^2 - k^2 \lambda_n^2/\rho} \right\}. \quad (103)$$

These equations still contain four separate modes. However, solutions to these equations have $\omega^2 \propto k^2$. Therefore we essentially find the equations

$$\partial_t^2 g^{(m)} = -\frac{1}{\rho} k^2 \lambda_n^2 g^{(m)} + k^2 \alpha^{(m)}(\hat{\mathbf{k}}) J^L,$$

$$\partial_t^2 J^L = k_i m_{ij} k_j \left(\beta'(\hat{\mathbf{k}}) J^L + \frac{\alpha^{(m)}(\hat{\mathbf{k}})}{\rho} g^{(m)} \right). \quad (104)$$

These indeed describe a set of coupled harmonic oscillators and agree with the dissipationless limit of the equations used by Lengua and Goodkind to interpret their data.

If we now add dissipation, the modes $A^{(n)}$ no longer diagonalize Eq. (80). However, there will be a new set of damped phonon modes with imaginary eigenvalues. Proceeding as before, we can again eliminate two modes. We then find a coupled set of damped harmonic oscillators that now precisely agree with the equations used by Lengua and Goodkind.

To conclude this section, let us consider the dissipationless hydrodynamic equations describing an isotropic supersolid. The transverse phonon modes then decouple, and for the longitudinal part we find schematically the equations

$$\partial_t^2 \begin{pmatrix} \delta\rho \\ N_\Delta \end{pmatrix} = \partial^2 \begin{pmatrix} \frac{\lambda + 2\mu}{\rho} & -\frac{4}{\sqrt{2}}(\mu + \lambda) a^L \\ c_\rho & c_\Delta \end{pmatrix} \begin{pmatrix} \delta\rho \\ N_\Delta \end{pmatrix},$$

$$\partial_\tau \begin{pmatrix} v_i^s \\ J_i^s \end{pmatrix} = \partial_i \begin{pmatrix} -\frac{B_\rho}{\rho^2} & \beta_\Delta \\ -\frac{B_\Delta}{\rho^2} & \beta_\rho \end{pmatrix} \begin{pmatrix} \delta\rho \\ N_\Delta \end{pmatrix}. \quad (105)$$

The hydrodynamic modes can in principle be found by diagonalizing the two matrices. If we are in the normal phase, the first equation remains unchanged, whereas the second is absent. Clearly we then have four propagating sound modes. In the supersolid phase the second equation is also present, and we find two second sound modes in addition to the four first sound modes. These are, however, not accurately described by Eq. (105), because for that it is essential to include temperature fluctuations, which we have neglected throughout this article. Nevertheless, it is clear from the above that to show experimentally the existence of a supersolid, it would be very convincing if one observes an additional resonance due to one of the modes associated with the superfluid degrees of freedom.

V. CONCLUSION

We have derived the hydrodynamic equations for the solid and supersolid phases of ^{4}He. It is well known that to describe the normal solid phase, it is essential to include defects into the hydrodynamic equations to find the right number of modes predicted by the conservation laws and broken symmetries. Because we know that there are six phonon modes, the defects are usually assumed to have diffusive dynamics, giving a total of $6+1=7$ hydrodynamic modes. This is then in agreement with the $8-1$ modes one expects

from the usual counting argument, excluding a thermal diffusion mode. However, Lengua and Goodkind in their experiment observe instead propagating behavior of the defect mode. This brings the total number of hydrodynamic modes to $6+2=8$. Therefore we introduced another hydrodynamic variable, the longitudinal part of the defect momentum. We believe that this is justified by noting that, when counting the number of conserved quantities, we should also include the conservation of defects. Hence the continuity equations for N_Δ and J_i are roughly speaking associated with respectively a conservation law and a broken symmetry. Indeed, our equations reproduce the set of coupled wave equations which were used by Lengua and Goodkind to interpret their data, and lead them to the identification of the observed collective mode as a propagating defect mode.

Furthermore, we have considered the hydrodynamic equations of supersolid ^{4}He by allowing both fluctuations in the defects density and lattice vibrations to lead to superfluid motion.[12] If we include these superfluid degrees of freedom into our hydrodynamic equations in the standard way, we end up with what one might call a four fluid hydrodynamics instead of the usual two fluid hydrodynamics. As a result we end up with two second sound modes instead of one. We expect on general grounds that including temperature fluctuations leads to one of these modes becoming propagating whereas the other will remain diffusive. Given these results it should then be possible in principle to identify experimentally an additional resonance in the attenuation and velocity of sound due to the coupling of these modes to the phonons. In our opinion this would be a more convincing experimental proof for the existence of a supersolid phase than the analysis made by Lengua and Goodkind.

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[1] W. E. Keller, *Helium-3 and Helium-4* (Plenum, New York, 1969).
[2] L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).
[3] R. P. Feynman, Phys. Rev. **91**, 1301 (1953); **94**, 262 (1954).
[4] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).
[5] I. Rudnick, Phys. Rev. Lett. **40**, 1454 (1978); D. J. Bishop and J. D. Reppy, *ibid.* **40**, 1727 (1978).
[6] C. M. Varma and N. R. Werthamer, in *The Physics of Liquid and Solid Helium*, edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1976).
[7] M. W. Meisel, Physica B **178**, 121 (1992), and references therein.
[8] A. F. Andreev and I. M. Lifshitz, Sov. Phys. JETP **29**, 1107 (1969).
[9] G. V. Chester, Phys. Rev. A **2**, 256 (1970).
[10] A. J. Leggett, Phys. Rev. Lett. **25**, 1543 (1970).
[11] G. A. Lengua and J. M. Goodkind, J. Low Temp. Phys. **79**, 251 (1990).
[12] H. T. C. Stoof, K. Mullen, M. Wallin, and S. M. Girvin, Phys. Rev. B **53**, 5670 (1996).
[13] M. Liu, Phys. Rev. B **18**, 1165 (1978).
[14] P. C. Martin, O. Parodi, and P. S. Pershan, Phys. Rev. A **6**, 2401 (1972).
[15] M. T. Chen, J. Roesler, and J. M. Mochel, J. Low Temp. Phys. **89**, 125 (1992); J. M. Mochel and M. T. Chen, Physica B **197**, 278 (1994).
[16] H. Kleinert, *Gauge Fields in Condensed Matter Physics* (World Scientific, Singapore, 1989).
[17] H. Kleinert, J. Phys. A **19**, 1855 (1986).
[18] A. Zippelius, B. I. Halperin, and D. R. Nelson, Phys. Rev. B **22**, 2514 (1980).
[19] J. P. Hirth and J. Lothe, *Theory of Dislocations* (McGraw-Hill, New York, 1968).
[20] L. D. Landau and E. M. Lifschitz, *Theory of Elasticity* (Pergamon, Oxford, 1970).
[21] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).
[22] M. Hamermesh, *Group Theory* (Addison-Wesley, London, 1962).
[23] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (W. A. Benjamin, Reading, MA, 1975).
[24] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), Chap. 6.
[25] S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland, Amsterdam, 1974).
[26] F. Jähnig and H. Schmidt, Ann. Phys. (N.Y.) **71**, 129 (1972).
[27] T. R. Kirkpatrick and J. R. Dorfman, J. Low Temp. Phys. **58**, 301 (1985); **58**, 399 (1985).