Stabilization of nonlinear excitations by disorder

P. L. Christiansen, Yu. B. Gaididei,* M. Johansson, and K. Ø. Rasmussen
Department of Mathematical Modelling, The Technical University of Denmark, DK-2800 Lyngby, Denmark

D. Usero and L. Vázquez
Departamento de Matematica Aplicada, Escuela Superior de Informática, Universidad Complutense, 28040 Madrid, Spain
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Using analytical and numerical techniques we analyze the static and dynamical properties of solitonlike excitations in the presence of parametric disorder in the one-dimensional nonlinear Schrödinger equation with a homogeneous power nonlinearity. Both the continuum and the discrete problem are investigated. We find that otherwise unstable excitations can be stabilized by the presence of disorder in the continuum problem. For the very narrow excitations of the discrete problem we find that the disorder has no effect on the averaged behavior. Finally, we show that the disorder can be applied to induce a high degree of controllability of the spatial extent of the stable excitations in the continuum system. [S0163-1829(97)06346-7]

I. INTRODUCTION

Understanding the interplay between disorder and nonlinearity is of fundamental importance in several physical contexts, and this combination raises a number of unsolved questions.[1] A considerable effort has been invested in understanding the roles of disorder and nonlinearity separately. Each may lead to localizations effects, namely, solitons and collapse effects due to nonlinearity and Anderson localization due to disorder. A natural and important question is therefore how these effects might complement, frustrate, or reinforce each other. It is also an issue of great experimental concern in several fields of modern physics, such as nonlinear optics,[2,3] polaron formation in solid-state materials,[4,5] vibron localization in biomolecules,[6,7] and energy transport in organic thin films.[8]

Investigations of stationary problems in one-dimensional systems have shown that nonlinearity may change the transmission properties of disordered systems, such that the, for a linear system, characteristic exponential decay of the transmission coefficient with systems length changes into a power law in the presence of nonlinearity.[9] This theoretical prediction has recently been confirmed experimentally using nonlinear surface waves on a superfluid helium film.[10] In nonstationary problems nonlinearity creates modulational instabilities, which can be enhanced by disorder.[11] The importance of the modulation instabilities resides in the possibility of formation of nonlinear localized excitations instead of plane waves. It has been demonstrated[12] that due to these nonlinear waves strong nonlinearity may completely inhibit the localization effects stipulated by the disorder. Usually the investigations have been carried out on systems that are integrable — soliton bearing — in the absence of disorder. A common argument is that the equations, despite their exact integrability, provide a sufficient description of the physical systems to display the essential behavior. However, the more common physical situation is that integrability, and thus the exact soliton, is absent. This is the case in discrete systems as well as in continuum systems. In such systems the modulational instabilities are still present but they do not create

solitonic excitations from plane waves. The result is more likely some condensation of the plane waves into intrinsic localized excitations in discrete systems or collapsing excitations in continuum systems.

As an initial step in gaining understanding of the role of disorder in such systems we consider the ubiquitous nonlinear Schrödinger (NLS) equation. This is a natural choice because tight-binding models have proven to be important for gaining insight into the effects of disorder on linear solid-state problems (Anderson model, Hubbard model, etc.) and the discrete NLS model is simply incorporating nonlinear effects into these models. Among these nonlinear models is the model of the so-called self-trapping of electrons in ionic crystals through polaronic lattice distortion[13] (general model for coupled-field systems[7]) where the nonlinearity arises from adiabatic elimination of the lattice distortions. The study of such models with disorder and temporal noise has shown that the ground state is always localized in the presence of disorder[13] while the temporal noise always leads to destruction of the localized states.[14] However, the studied models all have long-lived solitonlike solutions because the continuum limit of these equations is exactly integrable. The situation may change drastically if the continuum limit also is nonintegrable. A relevant example of such an equation is the two-dimensional (or higher-dimensional) NLS equation. The two-dimensional NLS equation is nonintegrable and possesses an unstable ground state solution which in the presence of perturbations either collapses or disperses. In view of the results of Ref. 13 there is a reason to believe that the presence of disorder can stabilize this behavior. This would be an important result in nonlinear optics[15] and in the modeling of organic thin films.[8]

In the present paper we study the effects of disorder on the localized excitations in a one-dimensional NLS equation which is generalized by an arbitrary degree of nonlinearity. It allows us to study the effect of disorder on excitations which are unstable when no disorder is present. This model has also a close relation to higher-dimensional models (for the continuum equation see Ref. 16 and Ref. 17 for the discrete
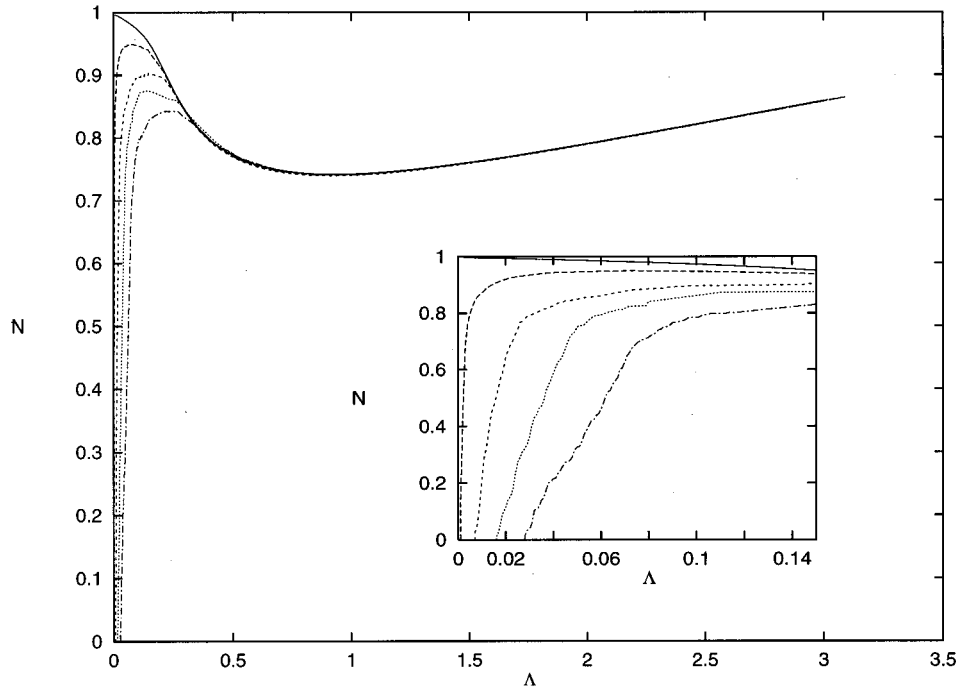
FIG. 1. The norm N vs nonlinear frequency Λ for various disorder strengths η. Homogeneous case $\eta=0$ (solid line), $\eta=0.01$ (long-dashed line), $\eta=0.04$ (short-dashed line), $\eta=0.07$ (dotted line), and $\eta=0.1$ (dash-dotted line). In all cases $\sigma=2$ and $a=3\pi^2/4$.

model) and studying it allows us to predict the behavior of higher-dimensional systems.

The paper is organized as follows. In Sec. II we introduce the model and describe its basic properties in the homogeneous discrete and continuum cases, and we discuss the numerical results obtained when disorder is included in the problem. We present numerical results showing that only the broad excitations are significantly affected by the disorder while the intrinsically localized excitations are rather unaffected. Most importantly, we find that the disorder stabilizes the very broad excitations. In Sec. III we address the problem analytically in the continuum limit. Using a collective coordinate approach we show that the disorder indeed creates a stability window for the localized excitations. Finally, Sec. IV gives a summary.

II. MODEL AND NUMERICAL RESULTS

We shall in the present paper investigate the discrete equation

$$i\dot{\psi}_n+(\psi_{n+1}-2\psi_n+\psi_{n-1})+a|\psi_n|^{2\sigma}\psi_n+\epsilon_n\psi_n=0. \quad (1)$$

Here $\psi_n(t)$ is a complex function of time and of the discrete coordinate n. The disorder ϵ_n is diagonal and assumed to be Gaussian distributed with the probability

$$p(\epsilon_n)=\frac{1}{\eta\sqrt{\pi}}\exp[-(\epsilon_n/\eta)^2] \quad (2)$$

and have the correlation function

$$\langle\epsilon_{n'}\epsilon_n\rangle=\eta^2\delta_{n',n}, \quad (3)$$

where the brackets $\langle\cdots\rangle$ denote averaging over all realizations of the disorder. Equation (1) has the following two conserved quantities: namely, the norm N, defined as

$$N=\sum_n |\psi_n|^2, \quad (4)$$

and the Hamiltonian H, defined as

$$H=\sum_n |\psi_{n+1}-\psi_n|^2-\frac{a}{\sigma+1}\sum_n |\psi_n|^{2(\sigma+1)}-\sum_n \epsilon_n|\psi_n|^2. \quad (5)$$

Equation (1) can be derived from the Hamiltonian (5) using the equation of motion $i\dot{\psi}_n=\partial H/\partial\psi_n^*$.

Considering the stationary solution of Eq. (1) in the form

$$\psi_n(t)=\phi_n\exp(i\Lambda t), \quad (6)$$

the dependence $N(\Lambda)$ can be found numerically. The case $\sigma=2$ [with $a=3\pi^2/4$ yielding $N=1$ in the continuum limit; see Eq. (10) below] is shown with solid line in Fig. 1. It has previously been shown[17] that the stability criterion for the stationary states in the discrete case is $dN/d\Lambda>0$. This together with the solid curve in Fig. 1 shows that an instability region appears in the discrete case, as previously analytically predicted.[17,18] In the discrete case the two-dimensional cubic version of Eq. (1) has similar properties as Eq. (1) with $\sigma=2$. The discrete two-dimensional cubic system has been studied earlier,[19–21] and comparing Fig. 1 to the two-dimensional result (Fig. 1 of Ref. 21) we see that the features are qualitatively the same, although there is a quantitative difference as Λ tends to infinity since $N\propto\Lambda^{1/\sigma}$ independent of the dimension. Evidently the continuum limit can be realized as $\Lambda\to0$ and it means that $N=1$ and $dN/d\Lambda=0$ in this limit.

In the continuum limit Eq. (1) takes the form

$$i\psi_t+\psi_{xx}+a|\psi|^{2\sigma}\psi+\epsilon(x)\psi=0. \quad (7)$$

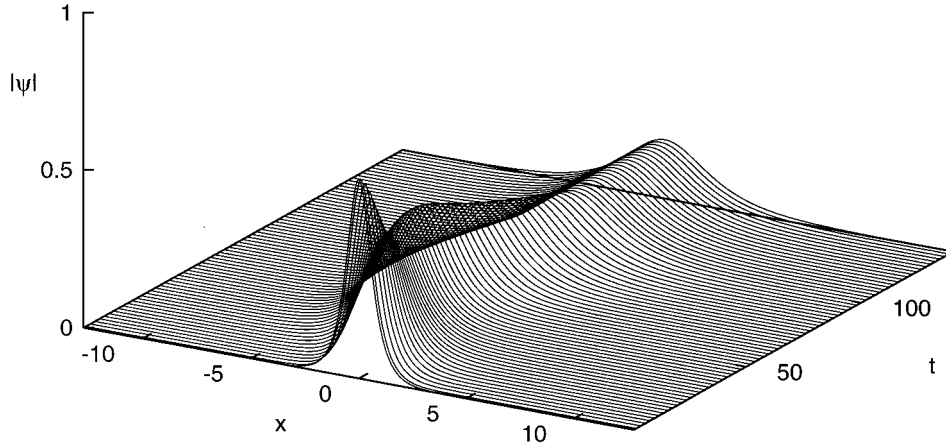Here the random potential is also Gaussian distributed and the correlation function becomes

FIG. 2. Evolution of an initial excitation of the form (9) perturbed such that $N = 0.95$ ($a = 3\pi^2/4$, $\sigma = 2$) in a continuum system with disorder strength $\eta = 0.05$.

$$\langle \epsilon(x)\epsilon(x') \rangle = \eta^2 \delta(x - x'). \qquad (8)$$

In the case of no disorder the continuum, Eq. (7), has the stationary solution

$$\psi(x,t) = \Lambda^{1/(2\sigma)} \left(\frac{\sigma+1}{a} \right)^{1/2\sigma} \text{sech}^{1/\sigma}(\sigma\sqrt{\Lambda}x)e^{i\Lambda t}, \qquad (9)$$

which has the norm

$$N = \left(\frac{\sigma+1}{a} \right)^{1/\sigma} \frac{\Gamma^2(1/\sigma)}{\sigma\Gamma(2/\sigma)} (4\Lambda)^{1/\sigma - 1/2}. \qquad (10)$$

The stability of the stationary solution is determined by the Vakhitov-Kolokolov criterion[22] so that it is stable when $dN/d\Lambda > 0$. Thus from Eq. (10) we have for $\sigma = 2$ $dN/d\Lambda = 0$ which signifies marginal stability of the stationary solutions. This marginal stability also occurs in the cubic two-dimensional NLS equation.

Further we show in Fig. 1 the dependence N on Λ for the stationary solutions of Eq. (1) in the presence of disorder. Results for four values of the variance $\eta = 0.01$ (long-dashed line), 0.04 (short-dashed line), 0.07 (dotted line), and 0.1 (dash-dotted line) are shown. The results have been obtained as averages of 100 realizations of the disorder. Several new features arise as a consequence of the disorder. In the continuum limit we no longer have $N = 1$ and $dN/d\Lambda = 0$. Instead we have $N \to 0$ and $dN/d\Lambda > 0$, signifying that the disorder stabilizes the excitations in the continuum limit. The disorder creates a stability window so that a bistability phenomenon emerges. Consequently there is an interval of excitation norm in which two stable excitations with significantly different widths have the same norm.

Furthermore, we see that the disorder creates a gap at small Λ in which no localized excitation can exist, and that the size of this gap apparently is increased as the variance of the disorder is increased. It is also clearly seen that as Λ increases (decreasing width) the effect of the disorder vanishes so that the very narrow excitations are in average unaffected by the disorder and only the continuum results are affected by disorder. It is important to stress that this is an average effect, because for each realization of the disorder the narrow excitation will be affected. The narrow excitation

will experience a shift in the nonlinear frequency equal to the amplitude of the disorder at the position of the excitation.

The qualitative form of the dependence $N(\Lambda)$ for a particular realization is very similar to the form of the average dependence shown in Fig. 1. It is noteworthy that for all realizations the curve $N(\Lambda)$ is a smooth curve. The basic difference from realization to realization is a displacement of the curve along the Λ axis.

The bistability we observe from Fig. 1 is very similar to the bistability that occurs when long-range effects are included in the NLS framework. An example of this has been studied recently by Gaididei et al.[23] who also showed that the bistability occurred due to competition between two different length scales of the problem, one length scale being defined by the relation between the nonlinearity and the dispersion, while the range of the nonlocal interaction defines the other length scale. The same effect is present in Eq. (1) when σ is in the range $1.4 < \sigma < 2$ (see Ref. 17, e.g.). In our case the bistability arises on similar grounds because of the competition between the length scale defined by the relation between the nonlinearity and the dispersion and the length scale defined by the disorder.

Another interesting numerical experiment is to launch a localized excitation into a disordered chain or a disordered continuum system and then observe the behavior of the excitation. We have done this experiment launching an excitation with a norm below the critical into a continuum system governed by Eq. (7), and observing the behavior of the excitation which in the corresponding homogeneous system would disperse. One example of this experiment is shown in Fig. 2. As is seen the localized excitation initially disperses, but after a short period, during which the center-of-mass motion is clearly seen, this process is arrested by the disorder and the excitation attains some approximately stationary width. Attempting to quantify the observed behavior we have calculated numerically the averaged behavior of the quantity

$$R = \int_{-\infty}^{\infty} |\psi|^6 dx. \qquad (11)$$

This quantity should clearly give a measure of the spatial extension of the excitation. However, we have observed a phenomenon which invalidates the usability of this quantity

for the purpose of quantifying the averaged behavior. The problematic phenomenon is that occasionally the disorder forces the initially localized excitation into two (or more) smaller and spatially separated localized excitations, in which case R is not directly related to the spatial extension of the excitation. Therefore the dynamical simulations can not be averaged in a meaningful way. Despite this problem the example in Fig. 2 clearly shows that the presence of the disorder has a stabilizing effect on the otherwise unstable excitations. This can be taken as a dynamical confirmation of the stationary results, which also showed that a stable stationary state emerged in the continuum limit in the presence of disorder. The conclusion of the dynamical simulations is thus that the disorder allows a stable state to exist even in these systems which have no stable localized excitations when disorder is absent. We merely cannot estimate the averaged behavior of the systems because of the described phenomenon.

III. ANALYTICAL RESULTS

The competition between disorder and nonlinearity has as mentioned above been addressed previously for simple integrable models. In particular Scharf and Bishop[24] have discussed the effects of a periodic potential on the soliton of the cubic NLS equation, and shown on the basis of an averaged NLS equation that the periodic potential leads to a simple renormalization of the solitons and creates a ''dressing'' of the soliton. It was also mentioned that this averaged method in principle could be generalized to account of any potential composed of finitely many short-length-scale (compared to the soliton size) components. Compared to the problem treated by Scharf and Bishop our problem has two distinctly different aspects, namely, that there exists no stable soliton solution and that the perturbing potential is composed of infinitely many short- and long-length-scale components. We have therefore chosen a different approach in order to gain some analytical understanding of how the presence of disorder affects the dynamics of the nonlinear excitations. Apparently, for the average dynamics there are only significant effects of the disorder in the limit of small nonlinear frequencies Λ, and we shall therefore apply the analytical approach in the continuum limit only. Thus it is assumed that the dynamics can be described in terms of collective coordinates using a localized self-similar trial function $\psi(x,t)$ of the form

$$\psi(x,t) = \sqrt{\frac{A(t)}{b(t)}} \operatorname{sech}^{1/\sigma}\left[\frac{x-X(t)}{b(t)}\right]\exp\{i\alpha(t)[x-X(t)]^2$$
$$+ik(t)x+i\phi(t)\}. \tag{12}$$

The real-valued time-dependent parameters $A(t)$ and $\phi(t)$ determine the amplitude and the phase of the excitation, respectively, while $b(t)$ and $\alpha(t)$ determine the width and the chirp of the excitation. Finally, the real-valued parameters $X(t)$ and $k(t)$ determine the excitation center-of-mass motion.

Equation (7) is the Euler-Lagrange equation for the action

$$S = \int_{-\infty}^{\infty} dt \, L, \tag{13}$$

where

$$L = \int_{-\infty}^{\infty} dx \left(\frac{i}{2} (\psi^*\partial_t\psi - \text{c.c.}) - |\psi_x|^2 \right.$$
$$\left. + \frac{a}{\sigma+1}|\psi|^{2(\sigma+1)} + \epsilon(x)|\psi|^2 \right) \tag{14}$$

is the Lagrangian density of the system. Inserting the trial function (12) into Eq. (13) the following equations are derived via the Euler-Lagrange equations:

$$-A\left(\dot{\phi}+\frac{1}{2}\ddot{X}X+\frac{1}{4}(X)^2\right)s\left(0,\frac{2}{\sigma}\right) - \frac{A}{4}\ddot{b}b s\left(2,\frac{2}{\sigma}\right)$$
$$+a\frac{\sigma}{\sigma+1}\frac{A^{\sigma+1}}{b^\sigma}s\left(0,\frac{2}{\sigma}+2\right)=U(b)-F(\{\epsilon\},b,X), \tag{15}$$

$$\frac{As\left(2,\frac{2}{\sigma}\right)}{2}\ddot{b}=-\frac{\partial}{\partial b}[U(b)-F(\{\epsilon\},b,X)], \tag{16}$$

$$\frac{1}{2}\ddot{X}=\frac{\partial F(\{\epsilon\},b,X)}{\partial X}, \tag{17}$$

where

$$U(b)=\frac{A}{\sigma^2}\left[s\left(0,\frac{2}{\sigma}\right)-s\left(0,\frac{2}{\sigma}+2\right)\right]\frac{1}{b^2}$$
$$-a\frac{A^{\sigma+1}}{\sigma+1}s\left(0,\frac{2}{\sigma}+2\right)\frac{1}{b^\sigma} \tag{18}$$

is the effective potential function in the case of no disorder and

$$F(\{\epsilon\},b,X)=\frac{A}{b}\int dx\,\epsilon(x)\operatorname{sech}^{2/\sigma}\left(\frac{x-X}{b}\right) \tag{19}$$

is the additional part of the potential arising from the disorder. The coefficients $s(n,m)$ are defined by

$$s(n,m)=\int_{-\infty}^{\infty} dx\,x^n\operatorname{sech}^m(x). \tag{20}$$

The stationary solutions defined by $\ddot{X}=\dot{X}=\ddot{b}=\dot{b}=0$ and $\phi(t)=\Lambda t$ [see Eq. (9)] are found by solving

$$\frac{\partial}{\partial b}[U(b)-F(\{\epsilon\},b,X)]=0, \tag{21}$$

$$\frac{\partial}{\partial X}F(\{\epsilon\},b,X)=0. \tag{22}$$

Considering the center-of-mass motion described by Eq. (17) we observe that for each realization of the random potential $\epsilon(x)$ the stationary position $X=X_m(\{\epsilon\},b)$ of the excitation is defined by the point where $F(\{\epsilon\},b,X)$ has a maximum with respect to X. Formally we can now insert the value $X=X_m(\{\epsilon\},b)$ into Eq. (21). Solving the resulting equation

the value of the excitation width, $b(\{\epsilon\})$, that minimizes the potential $U(b) - F(\{\epsilon\}, b, X)$ for a given realization $\{\epsilon(x)\}$ can be obtained. Finally, the average value $\langle b(\{\epsilon\}) \rangle$ must be calculated. However, it is difficult to realize the described program simply because we cannot solve Eqs. (21) and (22) for given $\epsilon(x)$. Therefore we will use an approximate approach.

Introducing

$$b = B + \delta, \quad B = \langle b \rangle, \tag{23}$$

and averaging Eq. (21), we get to zeroth order in δ

$$\frac{\partial}{\partial B} U(B) - \left(\frac{\partial}{\partial B} F(\{\epsilon\}, B, X) \right)_{X = X_m(\{\epsilon\}, B)} = 0. \tag{24}$$

Equation (24) shows that the stationary value of the mean excitation width B is determined by the extrema of the function

$$W = U(B) - \langle F(\{\epsilon\}, B, X_m(\{\epsilon\}, B)) \rangle. \tag{25}$$

Clearly the last term in the right-hand side of Eq. (25) can be written as

$$\langle F(\{\epsilon\}, B, X_m(\{\epsilon\}, B)) \rangle = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} df \, f P(f, X), \tag{26}$$

where $P(f,X) df \, dX$ is the probability of the function $F(\{\epsilon\}, B, X)$ having a maximum in the rectangle $(X, X+dX, f, f+df)$.

To calculate W we apply the following theorem by Rice:[25] Let f be a random curve given by

$$f = \Phi(\epsilon_1, \ldots, \epsilon_N; z). \tag{27}$$

The probability that f has a maximum in the rectangle $(z, z+dz, f, f+df)$, dz and df being of the same order of magnitude, is

$$P(f,z) df \, dz = -\frac{1}{M} df \, dz \int_{-\infty}^{0} d\zeta \, \zeta p(f, 0, \zeta). \tag{28}$$

Here $p(\xi, \theta, \zeta)$ is the probability density function for the random variables

$$f(z) = \Phi(\epsilon_1, \ldots, \epsilon_N; z),$$

$$f'(z) = \frac{\partial \Phi}{\partial z}, \quad f''(z) = \frac{\partial^2 \Phi}{\partial z^2}, \tag{29}$$

i.e.,

$$p(\xi, \theta, \zeta) \equiv \langle \delta(\xi - f(z)) \delta(\theta - f'(z)) \delta(\zeta - f''(z)) \rangle, \tag{30}$$

and M is a normalization factor. In the case where $f(z)$ is a stationary centered Gaussian process the probability distribution $p(\xi, \theta, \zeta)$ can be expressed as[26]

$$p(\xi, \theta, \zeta) = \frac{\exp(-{}^T[Z][C]^{-1}[Z]/2)}{(2\pi \, \det C)^{3/2}}, \tag{31}$$

where $Z = {}^T[\xi, \theta, \zeta]$ and

$$C = \begin{pmatrix} M_0 & 0 & -M_2 \\ 0 & M_2 & 0 \\ -M_2 & 0 & M_4 \end{pmatrix} \tag{32}$$

is the covariance of the random vector $(f(z), f'(z), f''(z))$. Here

$$M_0 = \langle [f(z)]^2 \rangle, \quad M_2 = \langle [f'(z)]^2 \rangle,$$

$$M_4 = \langle [f''(z)]^2 \rangle \tag{33}$$

are the spectral moments. From Eqs. (31) and (32) we get

$$p(\xi, 0, \zeta)$$

$$= \frac{\exp[-(M_4 \xi^2 + 2 M_2 \xi \zeta + M_0 \zeta^2)/2(M_0 M_4 - M_2^2)]}{(2\pi)^{3/2} M_2^{1/2} (M_0 M_4 - M_2^2)^{1/2}}. \tag{34}$$

Applying this theorem to our case we see that the random variable

$$f(z) = \frac{A}{B} \int dx \, \epsilon(x) \operatorname{sech}^{2/\sigma} \left(\frac{x-z}{B} \right), \tag{35}$$

where the properties of the random functions $\epsilon(x)$ are given by Eq. (8), is Gaussian and centered. Its spectral moments are

$$M_0 = A^2 s \left(0, \frac{4}{\sigma} \right) \frac{\eta^2}{B},$$

$$M_2 = A^2 \frac{4}{\sigma^2} \left[s \left(0, \frac{4}{\sigma} \right) - s \left(0, 2 + \frac{4}{\sigma} \right) \right] \frac{\eta^2}{B^3},$$

$$M_4 = A^2 \left(\frac{2}{\sigma} \right)^2 \left[\frac{4}{\sigma^2} s \left(0, \frac{4}{\sigma} \right) - \frac{4}{\sigma} \left(1 + \frac{2}{\sigma} \right) s \left(0, 2 + \frac{4}{\sigma} \right) \right.$$

$$\left. + \left(1 + \frac{2}{\sigma} \right)^2 s \left(0, 4 + \frac{4}{\sigma} \right) \right] \frac{\eta^2}{B^5}. \tag{36}$$

The spectral moments do not depend on z and consequently the process $(f(z), f'(z), f''(z))$ is stationary. In this case Eq. (26) can be represented in the form

$$\langle F(\{\epsilon\}, B, X_m(\{\epsilon\}, B)) \rangle = \frac{\int_{-\infty}^{\infty} d\xi \, \xi \int_{-\infty}^{0} d\zeta \, \zeta p(\xi, 0, \zeta)}{\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{0} d\zeta \, \zeta p(\xi, 0, \zeta)}. \tag{37}$$

Inserting Eqs. (34) and (36) into Eq. (37) we get

$$\langle F(\{\epsilon\}, B, X_m(\{\epsilon\}, B)) \rangle = (\pi/2)^{1/2} M_2 M_4^{-1/2} = S(\sigma) A \eta B^{-1/2}, \tag{38}$$
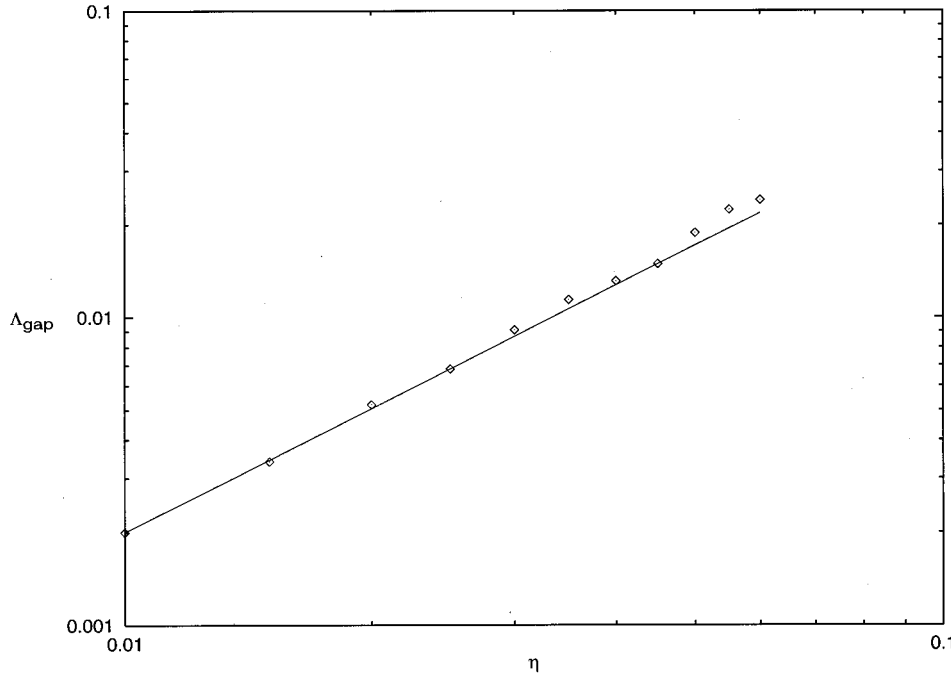
where the abbreviation

FIG. 3. The width of the gap, Λ_{gap} vs the disorder strength η. Results of numerical calculations are indicated by squares while the analytical dependence given by Eq. (46) is shown by the solid line. The gap is measured at $N=0.4$.

$$S(\sigma)=(\pi/2)^{1/2}\frac{2}{\sigma}\frac{s(0,4/\sigma)-s(0,2+4/\sigma)}{\sqrt{(4/\sigma^2)s(0,4/\sigma)-(4/\sigma)(1+2/\sigma)s(0,2+4/\sigma)+(1+2/\sigma)^2s(0,4+4/\sigma)}} \tag{39}$$

was used. Thus the effective averaged potential W takes the form

$$W=\frac{A}{\sigma^2}\left[s\left(0,\frac{2}{\sigma}\right)-s\left(0,\frac{2}{\sigma}+2\right)\right]\frac{1}{B^2}$$

$$-a\frac{A^{\sigma+1}}{\sigma+1}s\left(0,\frac{2}{\sigma}+2\right)\frac{1}{B^\sigma}-S(\sigma)A\,\eta B^{-1/2}. \tag{40}$$

Using this and Eq. (15) the nonlinear frequency $\dot{\phi}=\Lambda$ can be determined:

$$\Lambda=-\frac{1}{N}\left[W-a\frac{\sigma}{\sigma+1}s\left(0,\frac{2}{\sigma}+2\right)\frac{A^{\sigma+1}}{B^\sigma}\right]. \tag{41}$$

In the case of the cubic NLS equation ($\sigma=1$) the potential function (40) can be written as

$$W=\frac{N}{3}B^{-2}-\frac{aN^2}{6}B^{-1}-N\frac{\sqrt{42\pi}}{30}\,\eta B^{-1/2}. \tag{42}$$

Equation (42) has the same form as the effective potential obtained in Ref. 13 where the effects of disorder on the polaron ground state were studied. The authors of Ref. 13 used a quite different approach that combines statistical and scaling analysis.

In the case of the quintic ($\sigma=2$) NLS equation Eqs. (39) and (40) yield

$$W=\frac{N}{8}\left(1-\frac{4\,aN^2}{3\pi^2}\right)B^{-2}-N\sqrt{\frac{10}{42\pi}}\eta B^{-1/2}. \tag{43}$$

Minimizing the potential (43) we obtain for the mean value of the excitation width

$$B\simeq1.49\left(1-\frac{4\,aN^2}{3\pi^2}\right)^{2/3}\eta^{-2/3}. \tag{44}$$

Thus, in the presence of disorder stable excitations exist when $aN^2<3\pi^2/4$. In the opposite case the excitation will collapse.[16]

From Eq. (41) we obtain that the nonlinear frequency Λ in the case of the quintic NLS equation has the form

$$\Lambda=\left(\frac{\eta}{336}\sqrt{\frac{420}{\pi}}\right)^{4/3}\frac{3-4aN^2/3\pi^2}{(1-4aN^2/3\pi^2)^{4/3}}. \tag{45}$$

Since the nonlinear frequency Λ is tightly related to the width of the excitation, this expression clearly shows that the disorder controls a length scale of the problem. As already discussed in Sec. II this gives rise to the bistability phenomenon of the discrete problems.

Equation (45) also shows the appearance of the gap which was seen in the numerical simulations. The width Λ_{gap} of the gap is clearly given by the relation

$$\Lambda_{gap}\propto\eta^{4/3}. \tag{46}$$

In Fig. 3 we have compared this dependence using Eq. (46) [since the numerical coefficient in Eq. (45) is only

within the correct order of magnitude as is common for the collective coordinate method we disregard it in Fig. 3] with the numerically obtained dependence of the gap. Since an essentially equivalent behavior is found, this confirms that the essential features of the problem are captured by the analytical approach.

IV. SUMMARY

In summary we have in this paper shown that the presence of parametric disorder permits the existence of stable localized excitations in the continuum limit. We have shown this via analytical analysis and via numerical simulations of the stationary as well as the dynamical problem. Analyzing the discrete problem the appearance of a bistability phenomenon was observed, and the source of this bistability was identified to be the competition between two length scales. The new length scale was found to be directly related to the strength of the disorder.

Analytically and numerically we have shown the existence of a gap for small nonlinear frequencies in which no stationary state exists. This gap and the appearance of a narrow region where stable excitations exist allows rather accurate controllability of the excitations via the disorder.

In view of the similarity between the dynamics of the two-dimensional cubic ($\sigma = 1$) NLS equation and the one-dimensional quintic ($\sigma = 2$) NLS equation,[16] our results indicate that a two-dimensional optical beam propagating in a Kerr medium can be controlled by disorder effects, at least when the beam power (norm) is below the critical power for collapse. Such stable propagation is not possible in homogeneous two-dimensional Kerr media since the beam will either disperse or collapse. Additionally the beam waist (width) can be controlled by the disorder strength. The same scenario could be important in the modeling of energy transfer and transport in molecular dynamics.

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*Permanent address: Bogolyubov Institute for Theoretical Physics, 252 143 Kiev, Ukraine.

[1] See, e.g, *Fluctuation Phenomena: Disorder and Nonlinearity*, edited by A. R. Bishop, S. Jiménez, and L. Vázquez (World Scientific, Singapore, 1995); *Nonlinearity with Disorder*, edited by F. K. Abdullaev, A. R. Bishop, S. Pnevmatikos, and E. N. Economou (Springer, Berlin, 1992); *Disorder and Nonlinearity*, edited by A. R. Bishop, D. K. Campbell, and S. Pnevmatikos (Springer-Verlag, Berlin, 1989).

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