Theoretical study of Landau-Zener tunneling at the $M+N$ level crossing

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Two rules have been analytically obtained for the well-known problem of Landau-Zener tunneling at the $M+N$ level crossing. Both rules are valid for the arbitrary crossing rate, level splitting vectors, and mixing matrix. The first rule is given by a simple expression for an arbitrary diagonal element of the S matrix. The second rule is that the S matrix has $(M-1)\times(M-1)$ and $(N-1)\times(N-1)$ triangles of zeros in its $M\times M$ and $N\times N$ submatrices, i.e., some interlevel transitions are strictly forbidden. Some other features of the off-diagonal elements are also studied, using a $2+2$ level crossing as an example. Numerical and analytical results of the $2+2$ crossing show difference from a $1+2$ level, and we conclude that the $M+N$ level crossing cannot be expressed as a mere composition of $1+1$ level crossings except for the above two rules. Finally we discuss the possibility of a single-electron operation using these results. [S0163-1829(97)00844-8]

I. INTRODUCTION

Landau-Zener tunneling has been widely studied in the context of inelastic atomic collisions, and others. It is also important for the field of solid-state physics and electronics, commonly called ''single electronics'' (for general reviews, see Refs. 1–3). Single electronics is based on the tunnel transfer of single electrons from one small conducting region (''island'') to another under ''Coulomb blockade'' conditions when a single tunneling event leads to a considerable change in the voltage between the islands. When the electron energy spectra of the islands are quasicontinuous, the single-electron transfer may be analyzed using the so-called ''orthodox'' theory with simple master equations for the tunneling.[2] However, if the islands are formed in two-dimensional (2D) electron gas in semiconductor heterostructures, the discreteness of the electron energy levels may considerably affect the tunnel transfer.[4–6] Moreover, maturing nanofabrication techniques make smaller and smaller islands possible, so that the energy quantization effects become noticeable even in 3D semiconductor[7] and metallic[8] structures. This development makes a return to the old problem of Landau-Zener tunneling at the $M+N$ level crossing of great interest.

Let us consider the basic process of single-electron tunneling between two conducting islands A and B, each with a set of discrete energy levels for the conductivity electrons [Fig. 1(a)]. An external electric field applied to this system (created either by external signals or by similar single-electron devices[1,2]) changes the mutual position of the level sets, while leaving each level set intact. Assuming that near the transition point the energy shift is linear in time, $\Delta E=(2\alpha^2/\hbar)t$, we arrive at the well-known problem of Landau-Zener transitions[9,10] between the sets of M and N levels, described by the Hamiltonian

$$\mathbf{H}=\begin{pmatrix}\mathbf{A} & \mathbf{T}^{\dagger}\\ \mathbf{T} & \mathbf{B}\end{pmatrix}, \tag{1}$$

$$\mathbf{A}=\mathrm{diag}\left[A_1+\frac{\alpha^2 t}{\hbar},\ldots,A_M+\frac{\alpha^2 t}{\hbar}\right] \quad \text{as } A_i<A_j \text{ for } i<j, \tag{2}$$

$$\mathbf{B}=\mathrm{diag}\left[B_1-\frac{\alpha^2 t}{\hbar},\ldots,B_N-\frac{\alpha^2 t}{\hbar}\right] \quad \text{as } B_i<B_j \text{ for } i<j, \tag{3}$$

as $A_i<A_j$, $B_i<B_j$ for $i<j$. Here, the α is the crossing rate parameter with the dimension of energy. The coupling $N\times M$ matrix $\mathbf{T}$ is in general arbitrary, because the overlap of the corresponding eigenfunctions may be a very intricate function of their energies, especially in islands with complex geometries. The general solution of the Landau-Zener problem may be expressed in the form of an S matrix relating $(M+N)$-long vectors of the initial and finite eigenstates of the system.

Landau[9] and Zener[10] obtained simple S-matrix elements for the $1+1$ level crossing:

$$|S_{11}|^2=|S_{22}|^2=p_1 \quad \text{and} \quad |S_{11}|^2=|S_{22}|^2=q_1, \tag{4}$$

where $p_1=\exp(-\pi|T_{11}|^2/\alpha^2)$ and $q_1=1-p_1$. Demkov and Osherov[11] generalized this solution to the $1+N$ case. Their result may be expressed as follows:

$$\begin{gathered}|S_{11}|^2=p_1\ldots p_N, \quad |S_{1n+1}|^2=p_1\ldots p_{n-1}q_n,\\ |S_{n+11}|^2=q_n p_{n+1}\ldots p_N,\\ |S_{n+1n+1}|^2=p_n, \quad |S_{m+1n+1}|^2=0, \quad \text{and } |S_{n+1m+1}|^2\\ =q_m p_{m+1}\ldots p_{n-1}q_n,\end{gathered} \tag{5}$$

where $m<n$, $p_n=\exp(-\pi|T_{n1}|^2/\alpha^2)$ and $q_n=1-p_n$. This solution shows that regardless of the coupling strength and interlevel spacing, the S matrix has the same form as if the spacing is very large, i.e., the process may be considered as a set of sequential Landau-Zener tunneling events between individual crossing levels.

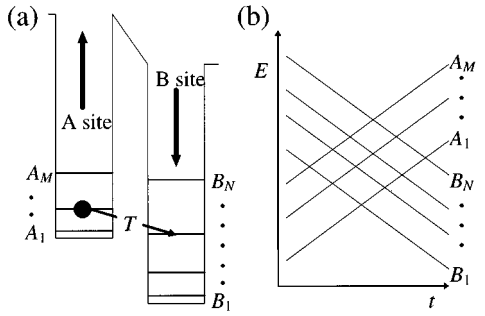

FIG. 1. $M+N$ level crossing: (a) Two conducting islands with discrete energy levels. (b) Diabatic energy diagram.

For the $M+N$ level crossing with $M>1$ and $N>1$, the picture becomes much more complex, because the energy diagram [Fig. 1(b)] acquires closed loops. This means that different paths of the system evolution may interfere destructively or constructively, depending on the difference of quantum phase accumulation along the paths.[12,13] This interference results in the so-called Stückelberg oscillations[14] of some elements of the S matrix as functions of the interlevel spacing (see, e.g., numerical plots in Ref. 15).

It may seem that these fast oscillations strip us of any hope of obtaining simple analytical results for the $M+N$ level crossing problem with $M>1$ and $N>1$. The goal of our work is to show that even under these conditions some important components of the S matrix behave very simply and in particular do not depend on the interlevel spacing. In Sec. II, we will derive these simple expressions. In Sec. III, we will discuss numerical results for other matrix elements in the case of $2+2$ level crossing. A summary and discussion will be given in Sec. IV.

II. ANALYSIS OF $M+N$ LEVEL CROSSING

We start with using a simple unitary transformation to remove diagonal elements from $\mathbf{H}$ of Eq. (1):

$$\tilde{\mathbf{H}}=\begin{pmatrix}\mathbf{0} & \tilde{\mathbf{T}}^{\dagger}\\ \tilde{\mathbf{T}} & \mathbf{0}\end{pmatrix},\quad \tilde{T}_{nm}=T_{nm}\exp[-i(u^2-2u\,\eta_{nm})], \tag{6}$$

where, $u=\alpha t/\hbar$, $\eta_{nm}=(B_n-A_m)/(2\alpha)$. By using Eq. (6), we can write

$$\begin{pmatrix}\mathbf{S}_{AA} & \mathbf{S}_{AB}\\ \mathbf{S}_{BA} & \mathbf{S}_{BB}\end{pmatrix}=\mathbf{1}+\sum_{k=1}^{\infty}(i\alpha)^{-k}\prod_{l=1}^{k}\int_{-\infty}^{u_{l+1}}\tilde{\mathbf{H}}(u_l)du_l \tag{7}$$

at $u_{k+1}=\infty$, where the S matrix is expressed as a composite of four partial matrices. [In this work, the product sign is defined as $\Pi_{l=1}^{k}\tilde{\mathbf{H}}(u_l)\equiv\tilde{\mathbf{H}}(u_k)\ldots\tilde{\mathbf{H}}(u_1)$.] In particular, the $M\times M$ matrix $\mathbf{S}_{AA}$ in Eq. (7) is given by

$$\mathbf{S}_{AA}=\mathbf{1}+\sum_{k=1}^{\infty}\mathbf{S}_{AA}^{(k)}(-\infty),$$

$$\mathbf{S}_{AA}^{(k)}(u_0)=\left(\frac{-1}{\alpha^2}\right)^k\prod_{l=1}^{k}\int_{u_0}^{u_{2l+1}}du_{2l}\int_{u_0}^{u_{2l}}du_{2l-1}\times\tilde{\mathbf{T}}^{\dagger}(u_{2l})\tilde{\mathbf{T}}(u_{2l-1})\exp[\tilde{o}(u_{2l}+u_{2l-1})] \tag{8}$$

at $u_{2k+1}=|u_0|$. At this point, we assume that the elements of $\mathbf{S}_{AA}$ are the analytic function of α^{-2}. To avoid logarithmic divergence of the following calculation, we can formally change the mixing matrix $\mathbf{T}$ to $\mathbf{T}\exp(\tilde{o}u)$, where the small damping factor $\tilde{o}$ satisfies the conditions $\lim_{|u_0|\to\infty}\tilde{o}u_0=0$ and $\lim_{|u_0|\to\infty}(-\ln|\tilde{o}|)/\ln|u_0|<\infty$. The variables u_l in Eq. (8) can be transformed to variables x_l and y_l, where

$$x_{k+1}=|u_0|,\quad x_l=u_{2l}+\frac{A_{m_l}}{\alpha}-\sum_{j=l+1}^{k}y_j,$$

$$y_l=u_{2l-1}-u_{2l}+a_l,$$

and

$$a_l=\eta_{n_lm_l}-\eta_{n_lm_{l-1}}=-\frac{1}{2\alpha}(A_{m_l}-A_{m_{l-1}}).$$

By using the above variables, an element of the matrix $\mathbf{S}_{AA}^{(k)}$ is expressed as follows:

$$\begin{aligned}[\mathbf{S}_{AA}^{(k)}(u_0)]_{m_km_0}&=\left(\frac{-1}{\alpha^2}\right)^k\sum_{n_1,\cdots,n_k}^{N}\sum_{m_1,\cdots,m_{k-1}}^{M}\\&\times\left(\prod_{l=1}^{k}T_{n_lm_l}^{*}T_{n_lm_{l-1}}\exp[-i(\eta_{n_lm_l}^2-\eta_{n_lm_{l-1}}^2)]\right)\\&\times\left(\prod_{l=1}^{k}\int_{-R}^{a_l}dy_l\int_{u_0}^{x_{l+1}}dx_l\right)\\&\times\exp\left\{-i\left(\sum_{j=1}^{k}y_j\right)^2+i\left[\sum_{j=1}^{k}\left(\frac{B_{n_j}}{\alpha}-2x_j\right)y_j'\right]\right\}\\&\times\exp\left[\tilde{o}\sum_{l=1}^{k}\left(y_l+\eta_{n_lm_l}+\eta_{n_lm_{l-1}}+2\sum_{j=l+1}^{k}y_j\right)\right]\\&+O(R^{-1}),\end{aligned} \tag{9}$$

where $y_l'=y_l+i\tilde{o}$. (The lower bound $-R<0$ of the multiple integral by y_l satisfies the conditions $\lim_{|u_0|\to\infty}R^{-1}=0$, and $\lim_{|u_0|\to\infty}R/u_0=0$.) For a large $|u_0|$, we can ignore the exponential factor of $\tilde{o}$ in the end of the first term, and the

second term $O(R^{-1})$ in the right-hand side of Eq. (9). By calculating the multi-integral of variables x_l, Eq. (9) is reduced to

$$[\mathbf{S}_{AA}^{(k)}(u_0)]_{m_k m_0} = \left(\frac{-1}{2i\alpha^2}\right)^k \sum_{n_1,\cdots,n_k=1}^{N} \sum_{m_1,\cdots,m_{k-1}=1}^{M} \times \left(\prod_{l=1}^{k} T^*_{n_l m_l} T_{n_l m_{l-1}} \exp[-i(\eta^2_{n_l m_l} - \eta^2_{n_l m_{l-1}})]\right) \times \int_{-R}^{a_k} dy_k \cdots \int_{-R}^{a_1} dy_1 \exp\left[-i(y_k + \cdots + y_1)^2 + i\left(\frac{B_{n_k}}{\alpha} y'_k + \cdots + \frac{B_{n_1}}{\alpha} y'_1\right)\right] f(y_k, \ldots, y_1) \tag{10}$$

and

$$f(y_k, \ldots, y_1) = \sum_{j=1}^{k} \sum_{\mathbf{L}^{(j)}} \frac{2i \sin[2|u_0|(y'_k + \cdots + y'_{L_j})]}{(y'_k + \cdots + y'_{L_j}) \cdots (y'_{L_j+1} + y'_{L_j}) y'_{L_j}} \prod_{h=1}^{j-1} \frac{\exp[2i|u_0|(y'_{L_{h+1}-1} + \cdots + y'_{L_h})]}{(y'_{L_{h+1}-1} + \cdots + y'_{L_h}) \cdots (y'_{L_h+1} + y'_{L_h}) y'_{L_h}}, \tag{11}$$

where $\mathbf{L}^{(j)} = (L_1, \ldots, L_j)$ for $1 = L_1 < \cdots < L_j \leqslant k$. Note that Eq. (11) oscillates with high-frequency at $|u_0| \gg 1$. The multiple integral with this high frequency factor strongly depends on the upper bound parameter a_l of the integral, e.g., when all parameters $a_l < 0$, the multiple integral becomes $O(|u_0|^{-1})$ at $|u_0| \gg 1$. In general, the multiple integral does not contribute to the element of $\mathbf{S}_{AA}^{(k)}(-\infty)$, when even an integer h as $1 \leqslant h \leqslant j$ satisfies that $\Sigma_{l=L_h}^{L_{h+1}-1} a_l \leqslant 0$ and $\Sigma_{l=L_h}^{L_{h+1}-1} |a_l| \neq 0$ as $L_{j+1} = k + 1$ (see the Appendix).

When $m_k = m_0$, only the terms with $m_k = m_{k-1} = \cdots = m_1$ (i.e.. all upper bound parameters $a_l = 0$) contribute to Eq. (10), and other terms are $O[|u_0|^{-1}(i\ln|\tilde{o}|)^{k-2}] \to 0$ as $|u_0| \to \infty$. The significant terms reduces the Eq. (9) as $u_0 \to -\infty$ to

$$[\mathbf{S}_{AA}^{(k)}(-\infty)]_{m_0 m_0} = \left(\frac{-1}{\alpha^2}\right)^k \lim_{u_0 \to -\infty} \sum_{\{I_j | I_1 + \cdots + I_N = k, I_j \geqslant 0\}} |T_{1m_0}|^{2I_1} |T_{2m_0}|^{2I_2} \cdots |T_{Nm_0}|^{2I_N} \prod_{j=1}^{N} \frac{1}{I_j!} \left(\prod_{l=1}^{I_j} \int_{-R}^{0} dy_l \frac{2\sin(y'_l |u_0|)}{y'_l}\right) \times \exp\left[-i\left(\sum_{i=1}^{I_j} y_i\right)^2 + i\left(\frac{B_j}{2\alpha} \sum_{i=1}^{I_j} y'_i\right)\right] = \frac{1}{k!}\left(\frac{-\pi}{2\alpha^2}\right)^k (|T_{1m_0}|^2 + |T_{2m_0}|^2 + \cdots + |T_{Nm_0}|^2)^k. \tag{12}$$

Immediately, we obtain the diagonal elements of $\mathbf{S}_{AA}$:

$$(\mathbf{S}_{AA})_{mm} = e^{-\gamma_{Am}}, \quad \gamma_{Am} = \frac{\pi}{2\alpha^2} \sum_{n=1}^{N} |T_{nm}|^2. \tag{13}$$

For $\mathbf{S}_{BB}$, similarly,

$$(\mathbf{S}_{BB})_{nn} = e^{-\gamma_{Bn}}, \quad \gamma_{Bn} = \frac{\pi}{2\alpha^2} \sum_{m=1}^{M} |T_{nm}|^2. \tag{14}$$

When $m_k > m_0$, i.e., $\Sigma_{l=1}^{k} a_l < 0$, all terms in Eq. (10) are $O(|u_0|^{-1}(i\ln|\tilde{o}|)^{k-1}) \to 0$ as $|u_0| \to \infty$. We thus obtain a ''forbidden transition'' rule for the off-diagonal elements:

$$(\mathbf{S}_{AA})_{mm'} = 0, \quad m > m'. \tag{15}$$

For $\mathbf{S}_{BB}$,

$$(\mathbf{S}_{BB})_{nn'} = 0, \quad n < n'. \tag{16}$$

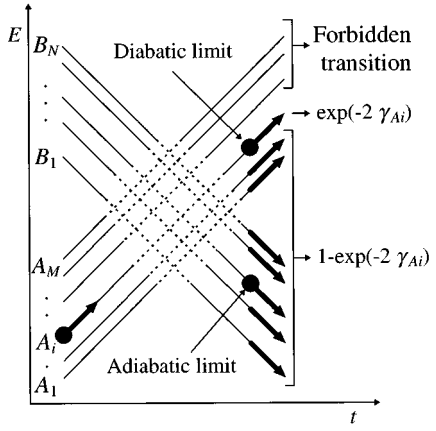

FIG. 2. Schematic presentation of the analytical results. If only level A_i is initially occupied, the transitions to upper levels of the same group are completely forbidden. The diagonal probability can be expressed as products of successive $1+1$ level transitions. In the diabatic limit ($\alpha^2 \to \infty$), only level A_i is finally occupied, while, in the adiabatic limit ($\alpha^2 \to 0$), only level B_i is finally occupied.

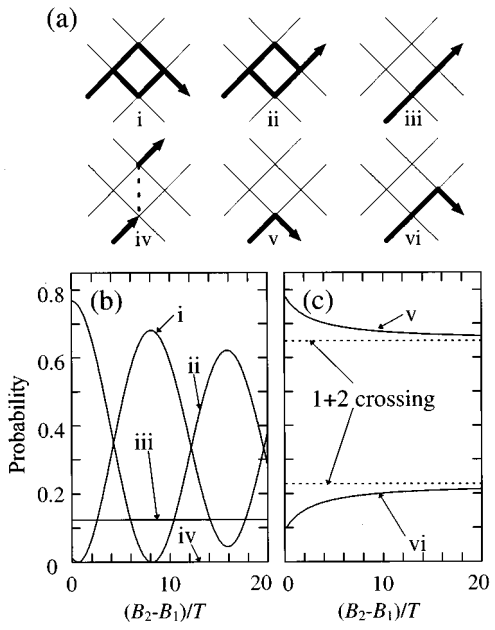

FIG. 3. Numerical results of the $2+2$ crossing. (a) Six different processes corresponding to 16 elements of the S matrix. (b) and (c) Nonvanishing probabilities as functions of the normalized level splitting $(B_2-B_1)/T$ at $\alpha^2/T^2=3$ and $(A_2-A_1)/T=2$. The dashed lines in (c) show probabilities of transitions at the $1+2$ crossing, topologically similar to processes v and vi.

Equations (13)–(16) are our most important results. Equations (13) and (14) show that, exactly as in the $1+N$ level case,[11] the diagonal probabilities can be expressed as products of successive $1+1$ level transitions, and thus are independent of level splittings A_j and B_j. Equations (15) and (16) show that the transitions between energy levels of the same set are forbidden[16] if they would go in the energy direction opposite to that of the energy evolution of the second set (Fig. 2).

III. NUMERICAL AND ANALYTICAL RESULTS FOR 2+2 LEVEL CROSSING

Other S-matrix elements do depend on level splittings if $M>1$ and $N>1$. We investigated their behavior numerically for the simplest nontrivial $2+2$ level case with $T=T_{nm}$: $m,n=1,2$. In this case, 16 elements of the S matrix are reduced to only six different elements characterizing the processes shown in Fig. 3(a). The Schrödinger equation was reduced to a system of difference equations by the midpoint method, and they were numerically solved (similarly to our former calculations of the lattice model[17]) under an additional technical modification that the coupling parameter T slowly increases (decreases) for $u\ll -1$ $(u\gg 1)$.

Figures 3(b) and 3(c) show the transition probabilities as functions of the B-site splitting $(B_2-B_1)/T$ ratio at fixed $\alpha^2/T^2=3$ and $(A_2-A_1)/T=2$. Each of the processes denoted as i and ii in Fig. 3(a) has two paths and thus shows clear Stückelberg oscillations. The period of the oscillations is very close to the area of the loop at the energy diagram even for small level splitting. The probability of process iii is in perfect agreement with the analytical results given by Eqs. (13) and (14): the numerical error is less than 10^{-7}. The calculated probability of process iv, which is analytically forbidden [see Eqs. (15) and (16)] has been found to be extremely low (less than 10^{-19}).

Processes v and vi in Fig. 3(a) are apparently similar to those in the $1+2$ level crossing problem, for which the probabilities are given by Eq. (5):

$$P_v^{(1+2)}=1-p, \quad P_{vi}^{(1+2)}=p(1-p), \quad p\equiv\exp(-\pi T^2/\alpha^2). \tag{17}$$

However, the actual probabilities in the $2+2$ crossing have a substantial dependence on the level splitting, though they do not oscillate—see Fig. 3(c). The difference between the $2+2$ and $1+2$ crossings (dashed lines) is largest at $(B_2-B_1)/T\to 0$. This result obviously shows that an ambiguous picture as ''independent crossing'' is *not* correct in general. This statement may be proven as follows. For large splitting, i.e., $(B_2-B_1)/T\gg 1$, we can separate the crossings for B_1 and B_2. In this limit, the ''independent crossing'' is certainly correct, and the probabilities of v and vi converge with those of Eq. (17). On the other hand, for small splitting we cannot neglect the transitions between B_1 and B_2 even far from the crossings, because a mixing between the B levels through the A levels A_2 is large even if $\alpha^2 t/\hbar\gg T$. Note that the mixing effects have the order of $\hbar T^2/[\alpha^2 t(B_2-B_1)]$. We can fortunately study the details in the vicinity of $B_2=B_1$, because the Hamiltonian in this case can be reduced to the $1+2$ crossing problem:

$$\mathbf{U}_1^{\dagger}\tilde{\mathbf{H}}\mathbf{U}_1=\begin{pmatrix} & & \sqrt{2}\tilde{T}^*_{11} & 0 \\ & \mathbf{0} & & \\ & & \sqrt{2}\tilde{T}^*_{12} & 0 \\ \sqrt{2}\tilde{T}_{11} & \sqrt{2}\tilde{T}_{12} & & \\ & & \mathbf{0} & \\ 0 & 0 & & \end{pmatrix}, \tag{18}$$

as

$$\mathbf{U}_1=\begin{pmatrix} \mathbf{1} & \mathbf{0} & \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \mathbf{0} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{19}$$

Accordingly, we can use the results of the $1+2$ crossing, namely, the S matrix for Eq. (18), given by

$$\mathbf{S}^{(1+2)}=\begin{pmatrix} p & (1-p^2)e^{i\theta_1} & p\sqrt{1-p^2}e^{i\theta_2} & 0 \\ 0 & p & \sqrt{1-p^2}e^{i\theta_3} & 0 \\ \sqrt{1-p^2}e^{i\theta_4} & p\sqrt{1-p^2}e^{i\theta_5} & p^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

where $p=\exp(-\pi T^2/\alpha^2)$, and θ_j is phase factor. Furthermore, the S matrix is transformed to

$$\mathbf{U}_2\mathbf{S}^{(1+2)}\mathbf{U}_2^{\dagger}=\begin{pmatrix} p & (1-p^2)e^{i\theta_1} & c^*p\sqrt{1-p^2}e^{i\theta_2} & s^*p\sqrt{1-p^2}e^{i\theta_2} \\ 0 & p & c^*\sqrt{1-p^2}e^{i\theta_3} & s^*\sqrt{1-p^2}e^{i\theta_3} \\ c\sqrt{1-p^2}e^{i\theta_4} & cp\sqrt{1-p^2}e^{i\theta_5} & |c|^2p^2+|s|^2 & -cs^*(1-p^2) \\ s\sqrt{1-p^2}e^{i\theta_4} & sp\sqrt{1-p^2}e^{i\theta_5} & -c^*s(1-p^2) & |s|^2p^2+|c|^2 \end{pmatrix}, \tag{21}$$

where

$$\mathbf{U}_2=\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ & c \quad s^* \\ \mathbf{0} & \\ & s \quad -c^* \end{pmatrix} \quad \text{as} \quad |c|^2+|s|^2=1. \tag{22}$$

If $\mathbf{U}_2=\mathbf{U}_1$, the transformed S matrix of Eq. (21) expresses the scattering just at $B_2=B_1$. The transformed S matrix also shows that the four matrix elements within the island B always break the rules given by Eqs. (14) and (16), as is evident from the condition in Eq. (3). To estimate the probabilities approaching $B_2=B_1$, we try to determine the parameter $|c|^2$, so that the other 12 elements keep the properties for $B_2>B_1$. One of the properties is that the sum of two probabilities, i.e., $P_{31}+P_{32}$, is equal to $1-p^2$ when the subscript corresponds to the elements of Eq. (21). Accordingly, we can chose $|c|^2=1/(1+p^2)$, and obtain the probabilities for the processes i–$v\,i$ in $(B_2-B_1)/T\to 0$ as follows:

$$P_i^{(2+2)}=\frac{p^4(1-p^2)}{1+p^2}, \quad P_{ii}^{(2+2)}=(1-p^2)^2, \quad P_{iii}^{(2+2)}=p^2,$$

$$P_{iv}^{(2+2)}=0, \quad P_v^{(2+2)}=\frac{1-p^2}{1+p^2}, \quad P_{vi}^{(2+2)}=\frac{p^2(1-p^2)}{1+p^2}. \tag{23}$$

These are consistent with the numerical results in the vicinity of $B_2=B_1$ (see Fig. 3). As $(B_2-B_1)/T\to 0$, we can analytically obtain a difference between the cases of $2+2$ and $1+2$ crossings:

$$P_v^{(2+2)}-P_v^{(1+2)}=-(P_{vi}^{(2+2)}-P_{vi}^{(1+2)})=\frac{p(1-p)^2}{1+p^2}, \tag{24}$$

where $p=\exp(-\pi T^2/\alpha^2)$. This difference is independent of the level splitting for the island A, and so the difference always exists in the processes v and $v\,i$. Furthermore, the numerical results in Fig. 3(b) clearly show that the mixing between B levels also affects the amplitude of oscillations in the processes i and ii. Consequently, processes iii and iv expressed by Eqs. (13)–(16) are a unique exception when the ''independent crossing'' picture may be used.

IV. SUMMARY AND DISCUSSION

We have investigated generalized Landau-Zener tunneling at the $M+N$ level crossing with arbitrary parameters for mixing matrix, crossing rate, and level splitting, and have found two exact results. First, diagonal elements of the S matrix are independent of energy splitting, and can be expressed by successive application of the Landau-Zener formula for each individual level crossing unexpectedly. Second, some transitions, namely, the transitions within the same site in the direction opposite to that of the crossing are completely forbidden. Note that the other elements are not mere compositions of the two-level crossing, as mentioned in Sec. III.

These analytical results provides an ideal relation between an operation speed and an operation error of the simplified single-electron device, and suggest a possibility about single-electron operation. In quite low temperatures, we can consider that the electron of the initial state localizes only at the lowest level of the island A if the single-electron system is sufficiently relaxed before the operation. A diagonal element of Eq. (13) for the lowest level can be applied to a survival probability at the island A after crossing the energy levels, since transitions to the upper levels of the island A are forbidden by Eq. (15). The single-electron device may have an exponentially small error-rate derived from Eq. (13) when the operation speed is slow, although we should rigorously discuss the electron transfer within the finite swing on the picture of the adiabatic potentials.[16]

Accordingly this work brightens up the operation of the single-electron devices, and may be also useful for the physics of Rydberg atoms[12] and optical physics.[18] However, for tunneling in real single-electron systems and others, a coupling to the environment (neglected in this work) may be important, especially in the adiabatic limit.[19]

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APPENDIX: ESTIMATION OF MULTI-INTEGRAL

We estimate the multi-integral in Eq. (10), and focus on the following integral:

$$\int_{-R}^{a_k}dy_k\cdots\int_{-R}^{a_2}dy_2\int_{-R}^{a_1}dy_1\exp\{-i[(y_1+\cdots+y_k)^2+C_1y_1+\cdots+C_ky_k]\}\frac{\exp[-i(y_1+\cdots+y_k)\zeta_0]}{(y_1'+\cdots+y_k')(y_1'+\cdots+y_{k-1}')\cdots y_1'}$$

$$= \int_{-R}^{0} dw_k \exp[-i(w_k^2 + c_k w_k)] \frac{\exp[-i(w_k+\sigma_k)|u_0|]}{w_k'+\sigma_k} \int_{w_k}^{0} dw_{k-1} \cdots \int_{w_2}^{0} dw_1$$

$$\times \frac{\exp[-ic_{k-1}(w_{k-1}+\sigma_{k-1})]}{w_{k-1}'+\sigma_{k-1}} \cdots \frac{\exp[-ic_1(w_1+\sigma_1)]}{w_1'+\sigma_1} + O(R^{-1}), \tag{A1}$$

where

$$y' = y + i\tilde{o}, \quad w_k = y_k - a_k, \quad w_j = y_j - a_j + w_{j+1}, \quad w_j' = w_j + (i\tilde{o})j, \quad c_j = \sum_{l=j}^{k} C_l, \quad \sigma_j = \sum_{l=1}^{j} a_l .$$

The following integral can be estimated directly:

$$\int_{w}^{0} dw \frac{\exp[-ic(w+\sigma)]}{w'+\sigma} = O(w) \quad \text{for} \quad \sigma \neq 0, \tag{A2}$$

and

$$\left| \int_{w}^{0} dw \frac{\exp(-icw)}{w'} \right| \leqslant \left| \int_{w}^{0} dw \frac{1}{w'} \right| \leqslant \left[\frac{1}{4} \left[\ln\left(\frac{\tilde{o}^2}{w^2+\tilde{o}^2} \right) \right]^2 + \left(\frac{\pi}{2} \right)^2 \right\}^{1/2} \leqslant O(\ln|\tilde{o}|). \tag{A3}$$

Furthermore, we summarize two hyperfunctions ($|u_0| \to \infty$) in the following table:

Integral/Hyperfunction	$\Psi_{\lvert u_0\rvert}(y) = \dfrac{\cos(y\lvert u_0\rvert)}{i(y+i\tilde{o})}$	$\Psi_{\lvert u_0\rvert}(y) = \dfrac{\sin(y\lvert u_0\rvert)}{y+i\tilde{o}}$
$\int_{-\lvert y_0\rvert}^{\lvert y_0\rvert} f(y)\Psi_{\lvert u_0\rvert}(y)dy$	$-\pi \mathrm{sign}(\tilde{o}) f(0) + O(\lvert u_0\rvert^{-1})$	$\pi f(0) + O(\lvert u_0\rvert^{-1})$
$\int_{-\lvert y_0\rvert}^{0} f(y)\Psi_{\lvert u_0\rvert}(y)dy$	$f(0)O(\ln\lvert\tilde{o}\rvert) + O(\lvert u_0\rvert^{-1})$	$\frac{\pi}{2} f(0) + O(\lvert u_0\rvert^{-1})$
$\int_{\lvert y_1\rvert}^{\lvert y_2\rvert} f(y)\Psi_{\lvert u_0\rvert}(y)dy$	$O(\lvert u_0\rvert^{-1})$	$O(\lvert u_0\rvert^{-1})$

From Eqs. (A2) and (A3), and results of the above hyperfunctions, we can estimate the multi-integral Eq. (A1) for $\sigma_k \leqslant 0$.

For $\sigma_k = 0$, $|\sigma_{k-1}| + \cdots + |\sigma_1| \neq 0$,

$$\text{Eq. (A1)} \leqslant O\left(\frac{(\ln|\tilde{o}|)^{k-2}}{|u_0|} \right) \to 0 \quad \text{as} \quad |u_0| \to \infty.$$

For $\sigma_k < 0$, $\sigma_{k-1} = \cdots = \sigma_1 = 0$,

$$\text{Eq. (A1)} \leqslant O\left(\frac{(\ln|\tilde{o}|)^{k-1}}{|u_0|} \right) \to 0 \quad \text{as} \quad |u_0| \to \infty.$$

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