Delocalization transition via supersymmetry in one dimension

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(Received 16 July 1997)

We use supersymmetric (SUSY) methods to study the delocalization transition at zero energy in a one-dimensional tight-binding model of spinless fermions with particle-hole symmetric disorder. Like the McCoy-Wu random transverse-field Ising model to which it is related, the fermionic problem displays two different correlation lengths for typical and mean correlations. Using the SUSY technique, mean correlators are obtained as quantum-mechanical expectation values for a U(2|1,1) ''superspin.'' In the scaling limit, this quantum mechanics is closely related to a $0+1$-dimensional Liouville theory, allowing an interpretation of the results in terms of simple properties of the zero-energy wave functions. Our primary results are the exact two-parameter scaling functions for the mean single-particle Green's functions. We also show how the Liouville quantum-mechanics approach can be extended to obtain the full set of multifractal scaling exponents $\tau(q)$, $y(q)$ at criticality. A thorough understanding of the unusual features of the present theory may be useful in applying SUSY to other delocalization transitions. [S0163-1829(97)03143-3]

I. INTRODUCTION

Delocalization transitions control the physical behavior of a number of electronic systems, including dirty semiconductors, metals, and two-dimensional electron gases in the quantum Hall regime.[1,2] In three dimensions, such critical points occur at the boundary between a diffusive metal and a localized insulating phase. In two or fewer dimensions, however, the metallic state is generally unstable (''weak localization''), so that delocalization transitions are typically isolated conducting *points* separating two localized phases. The prototypical example is the transition between plateaus in the integer quantum Hall effect (IQHE): Within a model of noninteracting electrons, the localization length in each disorder-broadened Landau band only diverges at one isolated energy.[2] Intense experimental effort has focused on the quantum Hall plateau transitions, and has led to an unprecedented characterization of the universal scaling behavior. Indeed, from the experimental point of view, this system probably provides the best example of random quantum critical behavior.

Theoretically, however, such systems still present a grand puzzle, in which but a few pieces are in place. Some analytical progress has been made for metal-insulator transitions in $d = 2 + \epsilon$ dimensions.[3–5] But in two dimensions, despite a set of simple noninteracting Fermion models that describe the IQHE plateau transition, a controlled analytic treatment is sorely missing. Efficient numerical methods have been developed to investigate these models, and provide a number of significant empirical observations.[6,2] In both these cases, scaling is manifest in the vicinity of the critical point, with a diverging localization length $\xi \sim |M|^{-\nu}$, where M measures the deviation from criticality. For the IQHE, $\nu \approx 7/3$. For distances shorter than ξ, the single-particle electronic wave functions $[\psi(x)]$ are extended, but exhibit complex *multifractal* scaling.[2] In particular, each disorder-averaged moment scales with an independent pair of critical exponents, which we denote as $\tau(q)$ and $y(q)$ for the qth moment:

$$[|\psi(x)\psi(0)|^q]_{\text{ens}} \sim L^{-d-\tau(q)}|x|^{-y(q)}. \quad (1.1)$$

Here the square brackets denote an ensemble average over disorder configurations, L is the linear extent of the system, and $|x| \ll L$ is assumed. The scaling for essentially all physical quantities can be formulated in terms of the set of exponents ν, $\tau(q)$, and $y(q)$ (a simple example is given in Ref. 7).

Supersymmetric (SUSY) techniques offer the tantalizing possibility of a field-theoretic treatment of such delocalization transitions. SUSY has a long history in disordered electronic systems, where it was first introduced by Efetov to describe diffusive metals.[8] The SUSY nonlinear σ model, when linearized, provides a *Gaussian or free* field description of a diffusive metal. Expansions about the metallic phase in $d = 2 + \epsilon$ dimensions give a fixed point that can be extrapolated to describe a three-dimensional (3D) metal-insulator transition. However, isolated delocalization transitions in $d \leq 2$, such as in the IQHE, do not afford the luxury of expanding about a diffusive metallic phase. Recently, Zirnbauer and others have used SUSY to map the Chalker-Coddington network model for the IQHE transition into an *interacting* $1+1$ dimensional field theory—a supersymmetric antiferromagnetic spin chain.[9–11] Unfortunately, this model has resisted all attempts at an analytic treatment, despite the essentially complete solution of a related supersymmetric *ferromagnetic* chain that describes transport in a dirty 2D *chiral* metal.[12] Falco and Efetov have recently applied the SUSY nonlinear σ model in two dimensions to extract multifractal wave function correlations, but in a crossover regime rather than at an isolated delocalization transition.[13]

Some analytic progress has been made using a toy model of Dirac fermions in a random vector potential (RVP), which exhibits a 2D delocalization transition.[14–17] This model has the simplifying feature that a zero-energy (critical) wave function can be found exactly for any realization of the disorder, which enables analytic study of wave-function multifractality. A number of different formulations are possible, but a particularly intriguing approach involves mapping to

2D Liouville field theory,[17] which has been extensively studied in string theory. Away from criticality, however, far fewer results are known, and at present SUSY techniques have not been successful in this regard.

In this paper, we study a one-dimensional tight-binding system of spinless fermions with random hopping matrix elements,

$$\mathcal{H} = -\sum_n t_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n). \qquad (1.2)$$

Here the $c's$ are canonical Fermion operators satisfying $\{c_m, c_n^\dagger\} = \delta_{mn}$, and the random hopping strengths t_n can be taken as positive without loss of generality. The continuum limit of this model is in fact a one-dimensional analog of the 2D RVP theory, and many of the same properties obtain. An exact zero-energy wave function is known for each realization of the disorder. Critical singularities are present in the single-particle density of states as in the 2D RVP model (but in contrast to the IQHE transition). Indeed, for the 1D random hopping model the density of states *diverges* at the band center.

There has been considerable prior work on the 1D transition in the random hopping model, primarily focussing on properties derivable from the mean local Green's function: the mean density of states and the typical localization length. This work was recently summarized in Ref. 18. Employing a real-space renormalization group (RG) method, D. S. Fisher (DSF) has obtained the spatial dependence of mean spin-correlation functions in several closely related 1D models: the McCoy-Wu random transvere field Ising model and random Heisenberg and *XX* spin chains.[19,20] In this paper, we extend the above analyses using SUSY methods to obtain the spatial dependence of the *exact* critical and off-critical scaling functions for the mean fermion Green's function, summarized in Eqs. (5.31)–(5.40) (unpublished work by DSF using real-space methods[21] corroborates our results). Our results are also reminiscent of forms obtained by Bouchaud *et al.*[22] for two-time correlation functions in the mathematically similar problem of a 1D random walker in a random potential, though the precise relation to our calculation has not been investigated. An important feature not present in the local properties is the emergence of a *mean* localization length that controls the spatial decay of the average Green's function,

$$\xi_\epsilon \sim |\ln\epsilon|^2, \qquad (1.3)$$

with ϵ the energy from the center of the band. This length is much *longer* than the *typical* localization length $\tilde{\xi}_\epsilon \sim |\ln\epsilon|$, found previously by many authors. This important distinction between typical and mean correlation lengths has been emphasized by Fisher in his analysis of 1D random spin chains.

In contrast to the IQHE transition that is mapped into a SUSY spin chain, the SUSY formulation of the 1D random hopping model is equivalent to the quantum mechanics of a *single* superspin, with ''Hamiltonian''

$$H = 2\omega\mathcal{J}^z + 2m_0\mathcal{J}^x - 4g(\mathcal{J}^x)^2. \qquad (1.4)$$

The superspin operators $\mathcal{J}$ are defined in Sec. III. This simplification enables us to systematically carry through the analysis from start to finish to obtain the *exact* critical and off-critical scaling functions. A key motivation for doing this was to investigate in detail the different elements that arise in a SUSY formulation of a delocalization transition. Indeed, the ''spin'' Hamiltonian in Eq. (1.4) has many unconventional properties. It is non-Hermitian, requiring a distinction between left and right eigenstates. The spin operators themselves are in fact elements of the noncompact superalgebra U(2|1,1). The noncompactness of the SU(1,1) subalgebra is manifest in the representations of the spin operators which are infinite dimensional (i.e., the ladder of discrete $\mathcal{J}^z$ eigenstates is infinite). As demonstrated in Secs. IV-V, near criticality the system explores the far reaches of this manifold of spin states, in a manner that can be described by *Liouville quantum mechanics*, which was recently introduced in studies of the zero-energy eigenstates in Ref. 23. Furthermore, the SUSY Hamiltonian is *defective*, i.e., the (right) eigenstates do not span the Hilbert space. To surmount this difficulty requires the definition of ''pseudoeigenstates'' to complete the eigenbasis. A knowledge and familiarity of these features will likely be crucial to the success of future work applying SUSY to other (e.g., 2D) critical points.

As a major function of this paper is pedagogy, we have attempted to present the material in enough detail to allow the reader to appreciate the technical elements of the calculations. Section II discusses the model, its relations to various random spin chains, the continuum limit, and the relevant single-particle Green's functions. In Sec. III, we describe the mapping to quantum mechanics, derive the SUSY Hamiltonian and its (super)symmetries, and detail the organization of states into superspin ladders and supermultiplets. The exact ground state and a class of excited states needed to compute the desired correlators are found in Secs. IV and V, leading to the final results in Eqs. (5.31)–(5.40). Last, in Sec. VI we pursue the Liouville quantum-mechanics formulation, extending the treatment of Ref. 23 to determine the full set of multifractal exponents,

$$\tau(q) = 0, \quad y(q) = 3/2. \qquad (1.5)$$

II. MODEL AND SYMMETRIES

A. Lattice model and continuum limit

We begin with the free-fermion model, Eq. (1.2). We assume that t_n consists of a large uniform part t and a small random piece δt_n. Under a Jordan-Wigner transformation, this model is equivalent to a random exchange spin-1/2 *XX* chain,

$$\mathcal{H}_{XX} = -\sum_n 2t_n(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y), \qquad (2.1)$$

where $S_n = \sigma_n/2$, with σ the usual vector of Pauli matrices.

For uniform hopping the single-particle states are plane waves, and $\mathcal{H}$ describes a band at half filling, with zero Fermi energy, and two Fermi points at $k_{F\pm} = \pm \pi/2$. With a small random component in the hopping strengths, the single-particle states will be localized away from the band center, but due to a special particle-hole symmetry (see below) the localization length diverges upon approaching zero energy. The density of states is also singular at zero energy. To study this delocalization transition, it suffices to focus on states near zero energy. Provided δt_n is small compared to

the band width t, it is legitimate to take a continuum limit, retaining a narrow shell of pure energy levels near the two Fermi points.

To this end, we decompose the Fermi fields as

$$c_n = (i)^n \psi_R(n) + (-i)^n \psi_L(n), \tag{2.2}$$

and assume that ψ varies slowly with n. To take the continuum limit we replace n by a continuous coordinate x, and approximate discrete differences with x derivatives. For the fermion hopping term this gives

$$c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n \sim i[\psi_R^\dagger \partial_x \psi_R - (\partial_x \psi_R^\dagger)\psi_R - (R \leftrightarrow L)]$$
$$- 2i(-1)^n[\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R]. \tag{2.3}$$

For uniform hopping, $t_n = t \rightarrow dx/2$, the second term is rapidly varying and can be ignored, giving the expected (pure) Hamiltonian,

$$\mathcal{H}_0 = -\int dx [\psi_R^\dagger i \partial_x \psi_R - \psi_L^\dagger i \partial_x \psi_L], \tag{2.4}$$

which describes right and left moving modes at the two Fermi points.

A small random hopping δt_n causes scattering between the plane-wave states. The important Fourier components of δt_n are at π, since these cause backscattering between the right and left movers. We thus decompose $\delta t_n \rightarrow (-1)^n m(x) dx/2$, where $m(x)$ is assumed slowly varying. From the second term in Eq. (2.3), this leads to a (random) backscattering term in the continuum limit:

$$\mathcal{H}_1 = -i \int dx\, m(x)(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R). \tag{2.5}$$

Employing a spinor notation, $\psi = (\psi_R, \psi_L)$, the full continuum Hamiltonian, $\mathcal{H}_c = \mathcal{H}_0 + \mathcal{H}_1$ takes the form

$$\mathcal{H}_c = \int dx\, \psi^\dagger h \psi, \tag{2.6}$$

with a single-particle Hamiltonian,

$$h = -i\sigma^z \partial_x + m(x)\sigma^y. \tag{2.7}$$

It is convenient to decompose the function $m(x)$ into a uniform and random piece as

$$m(x) = m_0 + \tilde{m}(x), \tag{2.8}$$

where $[\tilde{m}]_{\text{ens}} = 0$, with the square brackets denoting an ensemble average. Nonzero m_0 corresponds to a (uniform) staggering in the hopping, $\delta t_n \sim (-1)^n m_0$, which opens a gap in the pure spectrum about the band center. In the XX spin chain, nonzero m_0 corresponds to a dimerization in the bond strengths, and the gap is a spin gap due to singlet formation across the stronger bonds. Both m_0 and the energy ϵ are tuning parameters that take one away from the delocalized critical point.

B. Symmetries and delocalization

The lattice free fermion Hamiltonian, Eq. (1.2), is invariant under the canonical transformation,

$$c_n \rightarrow (-1)^n c_n^\dagger, \tag{2.9}$$

due to time-reversal and particle-hole symmetries, present even with random hopping strengths. As a consequence of this symmetry, the single-particle wave functions can be chosen to be real (time-reversal invariance) and come in conjugate pairs with energy $\pm \epsilon$. Specifically, for a given eigenfunction, $\phi_\epsilon(n)$ at energy ϵ, there is a partner eigenstate with energy $-\epsilon$, given by $\phi_{-\epsilon} = (-1)^n \phi_\epsilon(n)$. At zero energy one thus anticipates special properties, as discussed below.

In the continuum, the symmetry Eq. (2.9) becomes an invariance of $\mathcal{H}_c$ under the canonical transformation,

$$\psi_\alpha \rightarrow \psi_\alpha^\dagger, \tag{2.10}$$

for $\alpha = R, L$. This symmetry restricts the allowed form of the single-particle Hamiltonian h. Specifically, h cannot have terms (with no gradients) proportional to σ^x, σ^z or the identity. A generic random 1D tight-binding fermion model, in which the density of states is regular and all the eigenstates are localized, would *not* be particle-hole symmetric, and additional terms, such as a spatially random σ^x term, would be present in the continuum Hamiltonian. The above symmetry is clearly crucial for the existence of delocalization at the band center, $\epsilon = 0$.

An (un-normalized) extended state at zero energy can in fact be directly extracted from the continuum wave equation: $h\Phi(x) = 0$, where Φ is a two-component wave function. Writing

$$\Phi(x) = \phi_\pm(x)\binom{1}{\pm 1}, \tag{2.11}$$

the scalar function $\phi(x)$ satisifies

$$[\partial_x \pm m(x)]\phi_\pm = 0. \tag{2.12}$$

This can be integrated to give

$$\phi_\pm(x) \propto e^{\pm \int^x dx'\, m(x')}. \tag{2.13}$$

For random $m(x)$, with mean zero, this wave function is clearly *not* exponentially localized. If the random function $m(x)$ has short-ranged spatial correlations, the logarithm of the wave function undergoes a 1D random walk. For a Gaussian distribution of $m(x)$ the (un-normalized) wave function is log-normally distributed. This wave function is a one-dimensional analog of the exact zero-energy wave functions written down for 2D free fermions described by a Dirac theory with random vector potential. As in the 2D case, the wave function is very broadly distributed, and its correlations can be characterized by a multifractal scaling description. We return to a discussion of the multifractal characteristics of this wave function in Sec. VI, where we compute the multifractal spectrum explicitly, following recent work by Shelton and Tsvelik.[23]

Away from criticality, for nonzero m_0, the zero-energy wave functions in Eq. (2.13) are exponentially growing and decaying functions, $\phi_\pm(x) \sim e^{\pm m_0 x}$. While they are non-normalizable in infinite space, for a finite system they describe solutions that decay exponentially into the system, with an associated localization (or correlation) length, $\tilde{\xi} = 1/m_0$. The critical exponent, defined via

$$\widetilde{\xi} \sim m_0^{-\widetilde{\nu}}, \tag{2.14}$$

is $\widetilde{\nu} = 1$. As emphasized by DSF, in addition to the length $\widetilde{\xi}$ that describes the decay of a typical (unaveraged) correlation function, there is another divergent length ξ that describes the decay of ensemble-averaged correlation functions. Consistent with arguments by DSF, we find below that this latter length diverges more rapidly with an exponent $\nu = 2$.

Two similar lengths may be defined by approaching the critical point at finite energy ϵ, but with zero mass $m_0 = 0$. Using the Thouless construction from the local Green's function, previous authors have found a (typical) localization length $\widetilde{\xi}_\epsilon \sim |\ln\epsilon|$. In contrast, employing a real-space RG approach, DSF has shown that mean correlation functions decay with a longer length, which varies as $\xi_\epsilon \sim |\ln\epsilon|^2$.

Another important characteristic of the above exact zero-energy wave function is that it is *nodeless*, for each and every realization of the random potential $m(x)$. Because of this, critical properties of the 1D localization transition at $\epsilon = 0$ are contained in the ensemble averaged single-particle Green's function, in contrast to the conventional Anderson transition in higher dimensions.

Below we briefly consider symmetry properties of the single Fermion Green's function, and obtain expressions in the continuum limit. The next sections are devoted to evaluating the ensemble averaged Green's function using super-symmetry methods.

C. Green's functions

Consider the single fermion Green's function at energy ϵ, defined as

$$\mathcal{G}(n,n';\epsilon+i\omega) = i\int_0^\infty dt\, e^{i(\epsilon+i\omega)t}\langle v|c_n(t)c_{n'}^\dagger(0)|v\rangle, \tag{2.15}$$

where $|v\rangle$ denotes the fermion vacuum, and $c(t) = e^{i\mathcal{H}t}ce^{-i\mathcal{H}t}$ with $\mathcal{H}$ the lattice Hamiltonian. Here ω is a small imaginary part to the energy. In practice below, we will take the real part of the energy to be zero, $\epsilon = 0$, calculate $\mathcal{G}(i\omega)$ for real ω, and then extract the energy dependence via an analytic continuation.

The spectral decomposition of $\mathcal{G}$ in terms of the exact eigenstates ϕ_ϵ takes the form

$$\mathcal{G}(n,n';i\omega) = \sum_\epsilon \frac{\phi_\epsilon(n)\phi_\epsilon(n')}{\epsilon-i\omega}. \tag{2.16}$$

Using the symmetry property $\phi_{-\epsilon}(n) = (-1)^n\phi_\epsilon(n)$, one can readily show that $\mathcal{G}(n,n',i\omega)$ is *real* and even in ω for $n-n'$ odd, and purely *imaginary* and odd in ω for $n-n'$ even. For example, with $n-n'$ even, Eq. (2.16) can be rewritten as

$$\mathcal{G}(n,n';i\omega) = \sum_{\epsilon>0}\phi_\epsilon(n)\phi_\epsilon(n')\frac{-2i\omega}{\epsilon^2+\omega^2}. \tag{2.17}$$

The Green's function, as defined in Eq. (2.15), can be reexpressed in terms of the continuum fermion fields $\psi(x)$ by employing the decomposition, Eq. (2.2). For $n-n' \gg 1$,

the discrete separation can be replaced by a continuous distance $x-x'$ (lattice constant equal to one). One finds

$$\mathcal{G}(n,n';i\omega) = i^{n-n'}\sum_{\alpha,\beta}(-1)^{\alpha n+\beta n'}G_{\alpha\beta}(x,x';i\omega), \tag{2.18}$$

with α and β running over the two spinor components, denoted as either (R,L), (0,1), or ($\uparrow,\downarrow$). Here $G_{\alpha\beta}$ is defined in terms of the continuum fermion fields ψ as

$$G_{\alpha\beta}(x,x';i\omega) = i\int_0^\infty dt\, e^{-\omega t}\langle v|\psi_\alpha(x,t)\psi_\beta^\dagger(x',0)|v\rangle, \tag{2.19}$$

with $\psi(t) = e^{i\mathcal{H}_c t}\psi e^{-i\mathcal{H}_c t}$ and $\mathcal{H}_c$ the *continuum* Hamiltonian. This continuum Green's function can alternatively be expressed in terms of the single-particle Hamiltonian, h in Eq. (2.7), as

$$G_{\alpha\beta}(x,x';i\omega) = \langle x,\alpha|\frac{1}{h-i\omega}|x',\beta\rangle, \tag{2.20}$$

where $|x,\alpha\rangle$ denotes a fermion at position x with "spin" component α.

Of interest is the behavior of the single-particle Green's function upon ensemble averaging over disorder realizations. To be concrete, we take the random function $\widetilde{m}(x)$ to be Gaussian with

$$[\widetilde{m}(x)\widetilde{m}(x')]_{\text{ens}} = 2g\,\delta(x-x'). \tag{2.21}$$

Ensemble-averaged Green's functions, which we denote with an overbar, become translationally invariant:

$$\overline{\mathcal{G}}(x;i\omega) = \frac{1}{N}\sum_{n=1}^N [\mathcal{G}(n+x,n;i\omega)]_{\text{ens}}, \tag{2.22}$$

with $N\to\infty$ the number of sites in the tight-binding lattice. From Eq. (2.18), this is related to the averaged continuum Green's functions,

$$\overline{G}_{\alpha\beta}(x;i\omega) = [G_{\alpha\beta}(x,0;i\omega)]_{\text{ens}}, \tag{2.23}$$

via

$$\overline{\mathcal{G}}(x;i\omega) = i^x\sum_\alpha(-1)^{\alpha x}\overline{G}_{\alpha\alpha}(x;i\omega), \tag{2.24}$$

with the separation x either even or odd.

The mean density of states for the original lattice fermions can be written

$$\rho(\epsilon) = \lim_{\omega\to 0}\frac{1}{\pi}\text{Im}\overline{\mathcal{G}}(x=0;\epsilon+i\omega). \tag{2.25}$$

We shall also be interested in the spatial dependence of the correlation function,

$$C(x,\epsilon) = \frac{1}{\pi}\text{Im}\overline{\mathcal{G}}(x;\epsilon+i0^+). \tag{2.26}$$

This function is expected to decay exponentially with a mean correlation length ξ. Due to the delocalized zero-energy

wave function (at $m_0 = 0$), $\xi(\epsilon)$ should diverge upon approaching the band center, $\epsilon \to 0$.

In Sec. III we will construct a generating functional that can be used to extract $\bar{G}$. Our strategy will be to calculate $\bar{G}(x, i\omega)$ for *real* ω, and then perform an analytic continuation to extract the density of states and $C(x, \epsilon)$.

D. Related random models

As noted above, under a Jordan-Wigner transformation the lattice free-Fermion Hamiltonian $\mathcal{H}$ in Eq. (1.2) is identical to a random exchange spin-1/2 XX chain. Some properties of the spin chain can be extracted from the fermion density of states, specifically the specific heat and the z-component magnetization $\langle S^z \rangle$ in response to a magnetic field along the z axis. Unfortunately, spin correlation functions are notoriously difficult to extract from the free-fermion representation, due to the nonlocal relation between spin and fermion operators (Jordan-Wigner string). Nevertheless, one expects that the correlation decay length of the fermion Green's functions will also control the decay of spin correlations.

As shown originally by Shankar and Murthy, a second spin model that is equivalent upon fermionization to the free Fermion model $\mathcal{H}$, is the 1D random quantum Ising chain in transverse field, with Hamiltonian

$$\mathcal{H}_I = \sum_n [2K_{1,n} S_n^x + 4K_{2,n} S_n^z S_{n+1}^z]. \tag{2.27}$$

Here K_1 and K_2 are spatially random field and Ising exchange constants, respectively. This model is an anisotropic (''time-continuum'') version of a 2D classical Ising model with random exchange interactions perfectly correlated in one of the two directions, a model first studied by McCoy and Wu. For completeness, we sketch this fermionization procedure in Appendix A, where we show that the low-energy properties of $\mathcal{H}_I$ follow from the properties of the single-particle Hamiltonian h in Eq. (2.7).

Finally, we should mention the equivalence between the free-fermion Hamiltonian $\mathcal{H}$ and a 1D model of quantum particles connected by random strength harmonic springs, a model first introduced and analyzed by Dyson almost 50 years ago.[24]

III. QUANTUM MECHANICS

As discussed above, much of the information of interest is contained in the mean single-particle Green's function $\bar{G}_{\alpha\beta}(x, \epsilon + i\omega)$. In the following we will construct a generating functional that can be used to extract $\bar{G}$. Our strategy will be to calculate $\bar{G}(x, i\omega)$ for *real* ω, and then perform an analytic continuation to extract the density of states and $C(x, \epsilon)$.

A. SUSY generating functional

Our analysis is based on employing the well-known field-theoretic representation of an operator inverse to express the ensemble averaged Green's function,

$$\bar{G}_{\alpha\beta}(x, i\omega) = \left[\langle x, \alpha | \frac{1}{h - i\omega} | 0, \beta \rangle \right]_{\text{ens}}, \tag{3.1}$$

as a functional integral,

$$\bar{G}_{\alpha\beta}(x, i\omega) = i \langle \psi_\alpha(x) \bar{\psi}_\beta(0) \rangle_S, \tag{3.2}$$

with

$$\langle O \rangle_S = \left[\int D\psi D\bar{\psi} D\xi D\xi^* O e^{-S} \right]_{\text{ens}}. \tag{3.3}$$

Here the functional integration is over Grassmann fields $\psi(x)$ and complex fields $\xi(x)$, with the action

$$S = \int dx [\bar{\psi}(ih + \omega)\psi + \xi^*(ih + \omega)\xi]. \tag{3.4}$$

The most noteworthy point is that in Eq. (3.3), we have included the complex scalar field ξ in order to cancel the fermionic determinant that naturally occurs due to the ψ integration. In doing so, we obtain the supersymmetric form in Eq. (3.4). This enables an ensemble average over the Gaussian disorder $\tilde{m}(x)$ to be readily performed.

For ease in presentation, it is convenient to speak of a slightly simpler object, the partition function

$$\mathcal{Z} = \int D\psi D\bar{\psi} D\xi D\xi^* e^{-S}, \tag{3.5}$$

where the functional integration is, as before, over Grassmann fields $\psi(x)$ and complex fields $\xi(x)$, with the action of Eq. (3.4). Correlation functions are obtained by simply inserting the appropriate fields after the integration measure and ensemble averaging. The crucial cancellation of the fermionic and Bosonic determinants then gives the trivial identity $\langle 1 \rangle = 1$, or $\mathcal{Z} = 1$. From the functional integral formulation, the reason for generating Green's functions for real ω becomes clear: while the fermionic functional integral is always well defined, the bosonic one is only convergent provided that the action is bounded below. This is the case here provided $\omega > 0$, since h is Hermitian.

B. Tranformation to quantum formulation

The above action corresponds to a random one-dimensional statistical mechanics problem. After ensemble averaging over the random function $m(x)$, the model is translationally invariant. Our approach is to extract the transfer matrix, $\hat{T} = e^{-H}$, which can be used to reconstruct the averaged generating function:

$$[\mathcal{Z}]_{\text{ens}} = \lim_{L \to \infty} \text{STr} e^{-LH}, \tag{3.6}$$

where L is the length of the system. The symbol STr indicates the supertrace, defined by

$$\text{STr} O = \text{Tr}[(-1)^{N_f} O], \tag{3.7}$$

where Tr is the conventional trace, and N_f is a fermion number operator defined below. Although H is an operator, easily expressed in terms of Fermi and Bose operators (see below), it is ''zero dimensional,'' being independent of the spatial

coordinate x. The problem is thus reduced to studying the quantum mechanics of the ''Hamiltonian'' H. As usual, the spectrum of H contains information about the correlation length of the 1D system—in this case the localization length.

To extract the operator H, we massage the action Eq. (3.4) into the form of a coherent state path integral with x playing the role of imaginary time. To this end we let $\xi_\downarrow \rightarrow -\xi_\downarrow^*$, leaving $\xi_\uparrow$ unchanged. Similarly we transform the independent Grassmann fields as $\psi_\downarrow \rightarrow -\bar\psi_\downarrow$ and $\bar\psi_\downarrow \rightarrow \psi_\downarrow$, leaving the spin-up Grassmann fields unchanged. The action can then be written

$$S = \int dx (\mathcal{L}_0 + \mathcal{L}_\omega + \mathcal{L}_m), \tag{3.8}$$

with

$$\mathcal{L}_0 = \bar\psi \partial_x \psi + \xi^* \partial_x \xi, \tag{3.9}$$

$$\mathcal{L}_\omega = \omega(\bar\psi \psi + \xi^* \xi), \tag{3.10}$$

and $\mathcal{L}_m = m(x)A$ with

$$A = \bar\psi_\uparrow \bar\psi_\downarrow - \psi_\uparrow \psi_\downarrow + (\psi \rightarrow \xi). \tag{3.11}$$

Notice that $\mathcal{L}_0$ is now in the standard form for a coherent state path integral if x is reinterpreted as an imaginary time coordinate.

Before extracting H, we perform an ensemble average over the disorder. Since we have assumed a Gaussian distribution, this is readily performed to extract $[\mathcal{Z}]_{\mathrm{ens}}$. The only term in the action that is modified is $\mathcal{L}_m$, which now becomes

$$\mathcal{L}_m^{\mathrm{ens}} = m_0 A - g A^2. \tag{3.12}$$

The transfer ''Hamiltonian'' H can now be read off, since the full action takes the form $S = \int_x [\mathcal{L}_0 + H(\psi, \xi)]$. In passing to the Hamiltonian framework, the Grassmann fields are replaced by Fermion operators, $\psi \rightarrow f$, $\bar\psi \rightarrow f^\dagger$, and the complex fields by Bose operators, $\xi \rightarrow b$, $\xi^* \rightarrow b^\dagger$, where f and b satisfy canonical commutation relations:

$$[f_\alpha, f_\beta^\dagger]_- = [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}. \tag{3.13}$$

The resulting ''Hamiltonian'' is

$$H = \omega[f^\dagger f + b^\dagger b] + m_0 A - g A^2, \tag{3.14}$$

with

$$A = f_\uparrow^\dagger f_\downarrow^\dagger - f_\uparrow f_\downarrow + (f \rightarrow b). \tag{3.15}$$

Although we will hereafter refer to H as a Hamiltonian, it is important to keep in mind that this operator is *not* Hermitian.

Since H does not conserve the fermion number, $N_f = f^\dagger f$, it is convenient to perform a particle-hole transformation, defining new fermion fields via a canonical transformation,

$$F_\uparrow = f_\uparrow, \quad F_\downarrow = f_\downarrow^\dagger, \tag{3.16}$$

where $F^\dagger F$ commutes with H. To preserve the Bose-Fermi supersymmetry one can also define,

$$B_\uparrow = b_\uparrow, \quad B_\downarrow = b_\downarrow^\dagger. \tag{3.17}$$

However, $B_\downarrow$ does *not* satisfy the canonical Boson commutator, but rather, $[B_\downarrow, B_\downarrow^\dagger] = -1$. To restore the canonical form we define

$$\bar B = B^\dagger \sigma^z, \tag{3.18}$$

so that B and $\bar B$ satisfy,

$$[B_\alpha, \bar B_\beta] = \delta_{\alpha\beta}. \tag{3.19}$$

However, it must be kept in mind that $\bar B \neq B^\dagger$.

In term of these new operators, the Hamiltonian H becomes

$$H = \omega[F^\dagger \sigma^z F + \bar B \sigma^z B] + m_0 A - g A^2, \tag{3.20}$$

with

$$A = F^\dagger \sigma^x F + \bar B \sigma^x B. \tag{3.21}$$

At this stage it is convenient to express the Green's function, Eq. (3.1), as a supertrace over quantum states. Consider the Eq. (3.2) representation in terms of fermions. After transforming to the coherent state path integral form, the Grassmann fields can be replaced by fermion operators: $\psi(x) \rightarrow e^{xH} f e^{-xH}$. The average in Eq. (3.2) is replaced by a supertrace over quantum states:

$$\langle O(\psi, \xi)\rangle_S \rightarrow \langle O(f,b)\rangle \equiv \mathrm{STr}[O e^{-LH}]. \tag{3.22}$$

This gives

$$\bar G_{\alpha\beta}(x; i\omega) = i(-1)^\alpha \, \mathrm{STr}[F_\alpha(x) F_\beta^\dagger e^{-LH}], \tag{3.23}$$

with $F_\alpha(x) = e^{xH} F_\alpha e^{-xH}$.

C. Supersymmetry

In order to disucss the symmetries of the Hamiltonian H it is useful to introduce a four-component superfield,

$$\Psi = (F, B), \quad \bar\Psi = (F^\dagger, \bar B). \tag{3.24}$$

We will use latin indices $(a, b, \dots)$ to denote the fermion/boson label, i.e., $a = B, F \leftrightarrow 0,1$. From this superfield, one may build a three-component superspin,

$$\mathcal{J}_{ab} = \frac{1}{2}\bar\Psi_{a\alpha} \boldsymbol{\sigma}_{\alpha\beta} \Psi_{b\beta}, \tag{3.25}$$

where sums on the greek indices are implied. One can also define a set of ''charges,''

$$Q_{ab} = \bar\Psi_{a\alpha} \Psi_{b\alpha}, \tag{3.26}$$

(sum on α).

The diagonal components of $\mathcal{J}$ have special significance. In the fermionic sector,

$$\mathcal{J}_{11} \equiv S = F^\dagger \frac{\boldsymbol{\sigma}}{2} F \tag{3.27}$$

forms a set of ordinary Hermitian SU(2) spin operators, satisfying $[S^i, S^j] = i\epsilon^{ijk} S^k$. In the bosonic sector, we may similarly define three other currents,

$$\mathcal{J}_{00} \equiv J = \bar{B}\frac{\boldsymbol{\sigma}}{2}B. \tag{3.28}$$

These also satisfy the SU(2) algebra, $[J^i, J^j] = i\epsilon^{ijk}J^k$. However, although J^z is Hermitian, J^x and J^y are *anti*-Hermitian. One could define a Hermitian set of operators, multiplying the x and y components of J by a factor of i. These would then satisfy SU(1,1) commutation relations, instead of SU(2).

A useful object is the total spin current,

$$\mathcal{J} = \mathcal{J}_{aa}, \tag{3.29}$$

where again the sum on repeated indices is implied. This spin current commutes with the charges:

$$[\mathcal{J}, Q_{ab}] = 0. \tag{3.30}$$

Since the Hamiltonian can be expressed in terms of this total spin current,

$$H = 2\omega\mathcal{J}^z + 2m_0\mathcal{J}^x - 4g(\mathcal{J}^x)^2, \tag{3.31}$$

the charges Q_{ab} also commute with H. Thus Q_{ab} generate a set of (super)symmetries of H. Because of their importance, it is convenient to name them individually:

$$N_B \equiv Q_{00} = \bar{B}B, \quad N_F \equiv Q_{11} = F^\dagger F, \tag{3.32}$$

$$Q \equiv Q_{01} = \bar{B}F, \quad \bar{Q} \equiv Q_{10} = F^\dagger B. \tag{3.33}$$

These latter two operators are fermionic ''charges,'' which will be extremely useful in determining the ground state of H. They obey

$$Q^2 = \bar{Q}^2 = 0, \tag{3.34}$$

$$\{Q, \bar{Q}\} = N, \tag{3.35}$$

where the total charge is defined as

$$N = N_B + N_F, \tag{3.36}$$

which commutes with *all* sixteen of the U(2|1,1) currents.

D. Hilbert space and representations

Finding the ground-state and low-energy excitations of H is complicated by the enormity of the Hilbert space. Indeed, since the number of bosons with spin α, $n_\alpha = b_\alpha^\dagger b_\alpha$, is unbounded, the Hilbert space is actually infinite. One basis of states spanning the Hilbert space may be written as a direct product of bosonic and fermionic states:

$$|n_\uparrow n_\downarrow \alpha\rangle = |n_\uparrow n_\downarrow\rangle \otimes |\alpha_F\rangle, \tag{3.37}$$

where $n_\uparrow, n_\downarrow = 0 \cdots \infty$ are the number of up- and down-spin bosons, respectively. The parameter α_F labels the fermionic sector, which is spanned by the Fermionic vacuum $|vac\rangle$, which is anihillated by F, and three other states

$$|\uparrow\rangle = F_\uparrow^\dagger|vac\rangle, \tag{3.38}$$

$$|\downarrow\rangle = F_\downarrow^\dagger|vac\rangle, \tag{3.39}$$

$$|\downarrow\uparrow\rangle = F_\uparrow^\dagger F_\downarrow^\dagger|vac\rangle. \tag{3.40}$$

As is usual in a quantum-mechanics problem, we may simplify matters by choosing a maximal set of commuting variables, whose eigenvalues are good quantum numbers. In this case, N_B and N_F are obvious choices. The fermionic charges Q and $\bar{Q}$ cannot, of course, be diagonalized. They can, however, be combined to form two additional bosonic charges

$$\Gamma = \bar{Q}Q, \quad \bar{\Gamma} = Q\bar{Q}. \tag{3.41}$$

It is straightforward to show that

$$\Gamma^2 = N\Gamma, \quad \bar{\Gamma}^2 = N\bar{\Gamma}, \tag{3.42}$$

$$\Gamma + \bar{\Gamma} = N, \quad \bar{\Gamma}\Gamma = 0. \tag{3.43}$$

These relations imply that the eigenvalues of $(\Gamma, \bar{\Gamma})$ are either $(0,N)$ or $(N,0)$. The four operators $N_B, N_F, \Gamma, \bar{\Gamma}$ form the desired set of good quantum numbers, and it is desirable to reorganize the states given above into a basis diagonal in these variables.

1. SU(2)

Before proceeding to determine this basis, consider first the fermionic sector of the theory. The fermion number is in fact related to the total spin via

$$S^2 = s(s+1), \tag{3.44}$$

with

$$s = N_F(2 - N_F)/2. \tag{3.45}$$

So we can think of N_F as determining the representation of SU(2). Note that the singlet ($s=0$) representation occurs twice—for $N_F = 0, 2$.

2. SU(1,1) and bosonic ladders

Similarly, the bosonic states may be separated into multiplets with fixed $N_B = n_\uparrow - n_\downarrow - 1$. Each such multiplet is actually a distinct irreducible representation of SU(1,1). To see this, consider the Casimir operator,

$$J^2 = (J^x)^2 + (J^y)^2 + (J^z)^2 = (N_B^2 + 2N_B)/4. \tag{3.46}$$

Fixing N_B thus fixes the ''total spin'' of the SU(1,1) representation. Following the analogy with SU(2), we may label the states by their total spin and, e.g., the spin J^z along the z axis,

$$J^2|jn\rangle = j(j+1)|jn\rangle, \tag{3.47}$$

$$N_B|jn\rangle = 2j|jn\rangle, \tag{3.48}$$

$$J^z|jn\rangle = \left[\frac{1 + |2j+1|}{2} + n\right]|jn\rangle. \tag{3.49}$$

Note that in the last equation we have departed from the usual convention for denoting J^z eigenvalues. This is convenient because the quantum number n as we have defined it takes integer values $n = 0, \ldots \infty$. The total spin can take half-integer values $j = 0, \pm 1/2, \pm 1, \ldots$.

It is also helpful to have explicit expressions in terms of the previous basis. Since the jth block of states corresponds to a ladder satisfying $n_\uparrow - n_\downarrow = 2j+1$, it can be conveniently generated using the raising and lowering operators: $J^\pm = J^x \pm iJ^y$. As usual, the lowest-weight state in each ladder, denoted $|j0\rangle$, is constructed to be annihilated by $J_- = -b_\uparrow b_\downarrow$:

$$|j0\rangle = \begin{cases} \dfrac{1}{\sqrt{(2j+1)!}}(b_\uparrow^\dagger)^{2j+1}|v\rangle, & j \geq -1/2 \\[3mm] \dfrac{1}{\sqrt{|2j+1|!}}(b_\downarrow^\dagger)^{-(2j+1)}|v\rangle, & j < -1/2, \end{cases} \quad (3.50)$$

where $|v\rangle$ denotes the bosonic vacuum: $b_\alpha|v\rangle = 0$. Each ladder is constructed by acting on the associated lowest weight state with powers of $J^+ = b_\uparrow^\dagger b_\downarrow^\dagger$;

$$|jn\rangle = \left(\frac{|2j+1|!}{n!(n+|2j+1|)!}\right)^{1/2}(J^+)^n|j0\rangle, \quad (3.51)$$

where n runs from zero to ∞.

3. SUSY ladders

Clearly the set of eigenvalues of $N_B, N_F, \Gamma, \overline{\Gamma}$ is insufficient to distinguish all the states in the Hilbert space. To provide a unique labeling, we will choose to diagonalize the additional operator $\mathcal{J}^z = J^z + S^z$. This choice is natural in that $\mathcal{J}^z$ commutes with the other four diagonal charges. Moreover, the other current present in the Hamiltonian, $\mathcal{J}^x$, leaves the original four quantum numbers unchanged, mixing only different values of $\mathcal{J}^z$. The collection of states with different $\mathcal{J}^z$ but with the other four charges fixed may be viewed as the basis for a representation of the algebra of the $\mathcal{J}$ operators, i.e., a peculiar (nonunitary) representation of SU(2).

Two sets of such states are easily constructed. These are

$$|2j,0,0,N,(|N+1|+1)/2+n\rangle = |jn\rangle \otimes |\text{vac}\rangle, \quad (3.52)$$

$$|2j,2,N,0,(|N-1|+1)/2+n\rangle = |jn\rangle \otimes |\downarrow\uparrow\rangle, \quad (3.53)$$

where the quantum numbers inside the bras on the left-hand side denote eigenvalues of $(N_B, N_F, \Gamma, \overline{\Gamma}, \mathcal{J}^z)$, in that order.

States with $N_F = 1$ are slightly more complicated, since they can involve linear combinations of up or down fermions. For $N \neq 0$, these are

$$|2j,1,(N-|N|)/2,(N+|N|)/2,|N|/2+n\rangle$$
$$= \sqrt{\frac{n+|N|}{2n+|N|}}|jn\rangle \otimes |\downarrow\rangle + \sqrt{\frac{n}{2n+|N|}}|jn-1\rangle \otimes |\uparrow\rangle,$$
$$(3.54)$$

$$|2j,1,(N+|N|)/2,(N-|N|)/2,|N|/2+n\rangle$$
$$= \sqrt{\frac{n}{2n+|N|}}|jn\rangle \otimes |\downarrow\rangle + \sqrt{\frac{n+|N|}{2n+|N|}}|jn-1\rangle \otimes |\uparrow\rangle,$$
$$(3.55)$$

where in the first set above $n = 0,1,2,\ldots\infty$, while in the second set $n = 1,2,\ldots\infty$.

Apparently these two ladders of states become identical for $N = 0$. What happens in that case? There are two states which are eigenstates of $N_B = -1$, $N_F = 1$, and $\mathcal{J}^z = n$:

$$|-1/2,n\rangle \otimes |\downarrow\rangle, |-1/2,n-1\rangle \otimes |\uparrow\rangle. \quad (3.56)$$

If, however, we attempt to diagonalize, e.g., Γ, in this two-dimensional space, we find that it is impossible! In fact, there is only a single eigenstate for each n,

$$|-1,1,0,0,n\rangle = \begin{cases} \dfrac{1}{\sqrt{2}}[|-1/2,n\rangle \otimes |\downarrow\rangle + |-1/2,n-1\rangle \otimes |\uparrow\rangle], & n > 0 \\[3mm] |-1/2,0\rangle \otimes |\downarrow\rangle, & n = 0, \end{cases} \quad (3.57)$$

which are annihilated by both Q and $\overline{Q}$. Clearly this set of states does not span the full subspace. To complete it, we may define an orthogonal ladder of states,

$$|-1,1,*,*,n\rangle = \frac{1}{\sqrt{2}}[|-1/2,n\rangle \otimes |\downarrow\rangle - |-1/2,n-1\rangle \otimes |\uparrow\rangle], \quad (3.58)$$

for $n = 1,2,\ldots\infty$. It is important to note that $|-1,1,*,*,n\rangle$ *is not an eigenstate of* Γ *and* $\overline{\Gamma}$. Instead, acting with these operators on $|-1,1,*,*,n\rangle$ gives back $|-1,1,0,0,n\rangle$, i.e., Γ and $\overline{\Gamma}$ act like projection operators in this subspace.

E. Eigenstates

1. Supermultiplets

In a conventional quantum-mechanics problem, we can look for eigenstates of the Hamiltonian separately for each distinct set of eigenvalues of the chosen commuting operators. As we have seen above, most of the states in the Hilbert space can be specified in this way. To avoid the exceptions for the moment, consider first the sectors with $N \neq 0$. Then we may find *right* eigenstates of the Hamiltonian,

$$H|N_B,N_F,\Gamma,\ \overline{\Gamma},E\rangle = E|N_B,N_F,\Gamma,\ \overline{\Gamma},E\rangle, \quad (3.59)$$
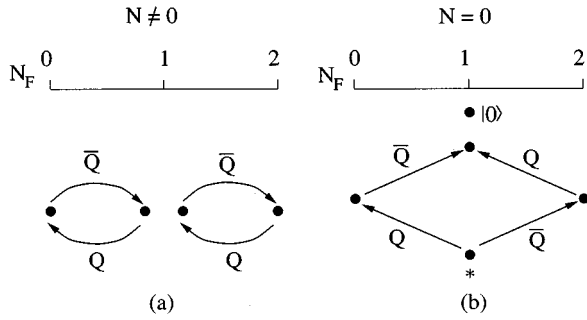
FIG. 1. Organization of multiplets in the SUSY Hamiltonian. (a) Doublets, involving one bosonic and one fermionic state for total number $N \neq 0$. (b) Singlet ground state $|0\rangle$ and quadruplets, with two fermionic and two bosonic states for $N=0$. See text for details.

where each such state can be expanded in a basis of appropriate $\mathcal{J}^z$ eigenstates, i.e.,

$$|N_B,N_F,\Gamma,\bar{\Gamma},E\rangle = \sum_{\mathcal{J}^z} \chi_E^{N_B,N_F,\Gamma,\bar{\Gamma}}(\mathcal{J}^z)|N_B,N_F,\Gamma,\bar{\Gamma},\mathcal{J}^z\rangle.$$
(3.60)

In a theory with ordinary bosonic symmetries, completely specifying the quantum numbers usually determines a unique set of energies—accidental degeneracies are rare. In a SUSY theory, however, the additional fermionic generators Q and $\bar{Q}$ lead to extra relations between states.

To see this, note that since Q and $\bar{Q}$ commute with H, acting with them upon an eigenstate must either produce another eigenstate with the same energy or annihilate it. Because Q and $\bar{Q}$ do not commute with the four diagonal operators, however, they *must* change these eigenvalues, and hence connect distinct states. Because the total charge N *does* commute with Q and $\bar{Q}$, it (along with the energy E) can be used to characterize a multiplet of states connected in this way.

For $N \neq 0$, the states are in fact organized into doublets, as indicated graphically in Fig. 1(a). One class of doublets includes the ladder of states with $N_F=0$,

$$\bar{Q}|N,0,0,N,E\rangle = c|N-1,1,N,0,E\rangle,$$
(3.61)

$$Q|N-1,1,N,0,E\rangle = (N/c)|N,0,0,N,E\rangle,$$
(3.62)

$$Q|N,0,0,N,E\rangle = \bar{Q}|N-1,1,N,0,E\rangle = 0,$$
(3.63)

while another includes the states with $N_F=2$,

$$Q|N-2,2,N,0,E\rangle = c|N-1,1,0,N,E\rangle,$$
(3.64)

$$\bar{Q}|N-1,1,0,N,E\rangle = (N/c)|N-2,2,N,0,E\rangle,$$
(3.65)

$$\bar{Q}|N-2,2,N,0,E\rangle = Q|N-1,1,0,N,E\rangle = 0,$$
(3.66)

where different constants c can appear in different expressions.

For $N=0$, things are more complicated. Distinct eigenstates can be found in the sectors with $N_F=-N_B=0,2$, and for the ladder of states with $N_F=-N_B=1$ and $\Gamma=\bar{\Gamma}=0$. The fourth subspace, spanned by the $|-1,1,*,*,n\rangle$ basis,

however, *is not closed under the action of H*. The best that one can hope to achieve in this sector, therefore, is to find an eigenstate of the Hamiltonian projected back onto the same sector. Such a state $|-1,1,*,*,E\rangle$ satisfies

$$H|-1,1,*,*,E\rangle = E|-1,1,*,*,E\rangle + |\psi\rangle,$$
(3.67)

where

$$|\psi\rangle = \sum_n \widetilde{\chi}_n|-1,1,0,0,n\rangle,$$
(3.68)

and is hence orthogonal to $|-1,1,*,*,n\rangle$.

Taken together with the other three states of energy E, $|-1,1,*,*,E\rangle$ forms the fourth member of a superquadruplet [see Fig. 1(b)]. Using Eq. (3.67), it is straightforward to see that the three true eigenstates may be obtained through the action of Q and $\bar{Q}$ on $|-1,1,*,*,E\rangle$. In particular,

$$Q|-1,1,*,*,E\rangle = c_1|0,0,0,0,E\rangle,$$
(3.69)

$$\bar{Q}|-1,1,*,*,E\rangle = c_2|-2,2,0,0,E\rangle,$$
(3.70)

$$\bar{Q}|0,0,0,0,E\rangle = c_3|-1,1,0,0,E\rangle,$$
(3.71)

$$Q|-2,2,0,0,E\rangle = c_4|-1,1,0,0,E\rangle,$$
(3.72)

$$\bar{Q}|-2,2,0,0,E\rangle = Q|0,0,0,0,E\rangle = \bar{Q}|-1,1,0,0,E\rangle$$
$$= Q|-1,1,0,0,E\rangle = 0.$$
(3.73)

We expect the exception to this rule to be a unique ground state with zero energy, forming a supersinglet. We can essentially pin down the sector in which such a singlet can occur. It cannot occur for $N \neq 0$, since $\Gamma+\bar{\Gamma}=N \neq 0$, and hence either Q or $\bar{Q}$ would create a new degenerate state. For $N=0$, it is possible only for a state in the $N_F=-N_B=1$, $\Gamma=\bar{\Gamma}=0$ ladder. This is consistent with the existence of one additional state $|-1,1,0,0,n=0\rangle$ in this basis.

2. Partition function and non-Hermiticity

Thus far, we have described the organization of eigenstates. To calculate physical quantities, we will need to perform a trace over the Hilbert space of e^{-LH}. Once the Hamiltonian is diagonalized, this trace can be performed separately in each multiplet. Some care must be taken to account, however, for the non-Hermiticity of H.

To do so, let us consider the behavior of the currents under Hermitian conjugation. The usual SU(2) spin generators S are of course Hermitian. The bosonic currents J are, however, mixed. In particular J^z is hermitian while J^x and J^y are anti-Hermitian. Since the Hamiltonian involves only $\mathcal{J}^z$ and $\mathcal{J}^x$, the net effect of conjugation is to change the sign of J^x. This can in fact be accomplished by the unitary transformation ($U=U^\dagger$),

$$U = e^{i\pi J^z},$$
(3.74)

which rotates J^x,

$$U^\dagger J^x U = -J^x,$$
(3.75)

and therefore conjugates the Hamiltonian,

$$U^{\dagger}HU = H^{\dagger}. \qquad (3.76)$$

It therefore follows that if $|E\rangle_R$ is a right eigenstate of H with energy E,

$$|E\rangle_L = cU|E\rangle_R \qquad (3.77)$$

is a left eigenstate of H, i.e.,

$$H^{\dagger}|E\rangle_L = E|E\rangle_L \Leftrightarrow {}_L\langle E|H = {}_L\langle E|E. \qquad (3.78)$$

Within each of the doublets, it is advantageous to normalize the states such that

$$_L\langle E|E\rangle_R = 1. \qquad (3.79)$$

Left and right eigenstates with different quantum numbers are automatically orthogonal, so this normalization guarantees the resolutions of the identity

$$1_{N;N_F=0,1} = \sum_E \ [|N,0,0,N,E\rangle_{RL}\langle N,0,0,N,E|$$
$$+ |N-1,1,N,0,E\rangle_{RL}\langle N-1,1,N,0,E|],$$
$$(3.80)$$

$$1_{N;N_F=1,2} = \sum_E \ [|N-2,2,N,0,E\rangle_{RL}\langle N-2,2,N,0,E|$$
$$+ |N-1,1,0,N,E\rangle_{RL}\langle N-1,1,0,N,E|],$$
$$(3.81)$$

for $N \neq 0$. Inserting these into the usual expression for the trace in terms of an orthonormal basis gives

$$\mathrm{STr}_{N;N_F=0,1}O = \sum_E \ [-{}_L\langle N,0,0,N,E|O|N,0,0,N,E\rangle_R$$
$$+ {}_L\langle N-1,1,N,0,E|O|N-1,1,N,0,E\rangle_R],$$
$$(3.82)$$

$$\mathrm{STr}_{N;N_F=1,2}O = \sum_E \ [-{}_L\langle N-2,2,N,0,E|O|N-2,2,N,0,E\rangle_R$$
$$+ {}_L\langle N-1,1,0,N,E|O|N-1,1,0,N,E\rangle_R],$$
$$(3.83)$$

where the minus signs arise from the $(-1)^{N_f}$ in the definition of the supertrace, and again this holds only for $N \neq 0$. Similarly, the ground state gives a unique contribution,

$$\mathrm{STr}_0 O = {}_L\langle 0|O|0\rangle_R, \qquad (3.84)$$

where $|0\rangle \equiv |-1,1,0,0,E=0\rangle$ is the ground state.

The quadruplets are somewhat trickier. The two states with $s=0$ (i.e., $N_F=0,2$) in each quadruplet can be separately orthogonalized, i.e.,

$$_L\langle 0,0,0,0,E|0,0,0,0,E\rangle_R = 1, \qquad (3.85)$$

$$_L\langle -2,2,0,0,E|-2,2,0,0,E\rangle_R = 1, \qquad (3.86)$$

since $U|-2,2,0,0,E\rangle_R$ remains in the subspace with $N_F=-N_B=2$, and similarly for the $N_F=N_B=0$ state. The same is not true for the two states with $N_F=-N_B=1$. In

fact, it is straightforward to show that corresponding left and right eigenstates are actually orthogonal! That is,

$$_L\langle -1,1,0,0,E|-1,1,0,0,E\rangle_R = 0, \qquad (3.87)$$

$$_L\langle -1,1,*,*,E|-1,1,*,*,E\rangle_R = 0, \qquad (3.88)$$

for $E \neq 0$. This is because the U operator takes each basis state from one subspace into the other, as can be seen directly from Eqs. (3.57,3.58). Instead of the usual normalization condition, therefore, we must require

$$_L\langle -1,1,0,0,E|-1,1,*,*,E\rangle_R = 1, \qquad (3.89)$$

$$_L\langle -1,1,*,*,E|-1,1,0,0,E\rangle_R = 1. \qquad (3.90)$$

The corresponding resolution of the identity is

$$1_{N=0} = \sum_E \ [|2,-2,0,0,E\rangle_{RL}\langle 2,0,0,0,E| + |0,0,0,0,E\rangle_{RL}$$
$$\times \langle 0,0,0,0,E| + |-1,1,0,0,E\rangle_{RL}\langle -1,1,*,*,E|$$
$$+ |-1,1,*,*,E\rangle_{RL}\langle -1,1,0,0,E|]. \qquad (3.91)$$

From this identity follows the expression for the trace,

$$\mathrm{STr}_{N=0}O = \sum_E \ [-{}_L\langle 2,-2,0,0,E|O|2,-2,0,0,E\rangle_R$$
$$- {}_L\langle 0,0,0,0,E|O|0,0,0,0,E\rangle_R$$
$$+ {}_L\langle -1,1,0,0,E|O|-1,1,*,*,E\rangle_R$$
$$+ {}_L\langle -1,1,*,*,E|O|-1,1,0,0,E\rangle_R]. \qquad (3.92)$$

In this paper we are studying primarily properties of the system in the thermodynamic limit $L \to \infty$. In this case the (super)trace of e^{-LH} reduces to an expectation value in the singlet ground state, i.e.,

$$\langle O \rangle = \mathrm{STr} O e^{-LH} \to {}_L\langle 0|O|0\rangle_R. \qquad (3.93)$$

It is worth noting, however, that SUSY indeed implies that the partition function itself is exactly one, even in the finite system. This is because each multiplet other than the singlet contains equal numbers of fermionic (N_f odd) and bosonic (N_f even) states, which thereby cancel in the supertrace. All that remains is the expectation value in the zero-energy ground state, which is unity even for L finite.

IV. GROUND-STATE PROPERTIES

A. Schrödinger equation

As we saw in the previous section, the ground state resides in a unique (nonunitary) singlet representation of SU(2) algebra of the total current $\mathcal{J}$. As such, it must be annihilated by both Q and $\bar{Q}$, and hence fits into the subspace with $N_F = -N_B = 1$, $\Gamma = \bar{\Gamma} = 0$, i.e.,

$$|0\rangle_R = \sum_{n=0}^{\infty} \phi_n |-1,1,0,0,n\rangle. \qquad (4.1)$$

In this basis, $\mathcal{J}^x$ and $\mathcal{J}^z$ have the very simple matrix elements

$$\mathcal{J}^z|n\rangle_0 = n|n\rangle_0, \qquad (4.2)$$

$$\mathcal{J}^x|n\rangle_0 = \tfrac{1}{2}[(n+1)|n+1\rangle_0 - (n-1)|n-1\rangle_0], \quad (4.3)$$

where we have introduced the more compact notation $|n\rangle_0 \equiv |-1,1,0,0,n\rangle$. Using these results, and the requirement of a zero-energy ground state,

$$H|0\rangle_R = E|0\rangle_R = 0, \qquad (4.4)$$

we obtain straightforwardly the Schrödinger equation,

$$n[-(n+1)\phi_{n+2} + 2n\phi_n - (n-1)\phi_{n-2} - m_0(\phi_{n+1} - \phi_{n-1})$$
$$+ 2\omega\phi_n] = 0, \qquad (4.5)$$

which implies

$$-(n+1)\phi_{n+2} + 2n\phi_n - (n-1)\phi_{n-2} - 2M(\phi_{n+1} - \phi_{n-1})$$
$$+ 2\omega\phi_n = 0 \quad \text{for } n \neq 0. \qquad (4.6)$$

Here we have defined $M = m_0/2$ to simplify some expressions in what follows.

B. Normalization and density of states

From the previous section, given the right eigenstate $|0\rangle_R$, a corresponding left eigenstate is obtained by

$$|0\rangle_L = cU|0\rangle_R = (-1)^{J^z - 1/2}|0\rangle_R, \qquad (4.7)$$

with an appropriate choice for the constant c. Applying this to the basis of $\mathcal{J}^z$ eigenstates gives

$$(-1)^{J^z - 1/2}|n\rangle_0 = \begin{cases} (-1)^n|-1,1,*,*,n\rangle, & n>0 \\ |0\rangle_0, & n=0. \end{cases} \qquad (4.8)$$

Since the $|n\rangle_0$ and $|-1,1,*,*,n\rangle$ states are orthogonal, the normalization requirement reduces to

$$_L\langle 0|0\rangle_R = |\phi_0|^2 = 1. \qquad (4.9)$$

The density of states is obtained in a similar way. From Eq. (2.25) the density of states is obtained by analytically continuing,

$$\overline{\mathcal{G}}(x=0;i\omega) = \sum_\alpha \overline{G}_{\alpha\alpha}(0,i\omega) = -2i_L\langle 0|S^z|0\rangle_R, \qquad (4.10)$$

where the second equality follows from Eq. (3.23). Simple manipulations then give

$$\overline{\mathcal{G}}(0;i\omega) = i\sum_{n=0}^{\infty} (-1)^n|\phi_n|^2. \qquad (4.11)$$

C. Continuum limit and boundary conditions

In order to solve for the ground-state wave function, first note the following simple property: if $\phi_n^+(M)$ is a solution of Eq. (4.6), then so is

$$\phi_n^-(M) = (-1)^n\phi_n^+(-M). \qquad (4.12)$$

Since this is a fourth-order linear difference equation, we would expect four linearly independent solutions. The above result reduces these to two trivially related pairs. To understand the nature of these two solutions, it is instructive to consider the limit $M = \omega = 0$. In this case there are two obvious solutions: $\phi_n = 1$ and $\phi_n = (-1)^n$ satisfy the $E = 0$ Schrödinger equation (4.6). However, these wave functions are *not* normalizable, a feature due to a pathology of the theory *at* $\omega = 0$. Indeed, nonzero ω is essential in guaranteeing convergence of the bosonic functional integral in the generating function. Moreover, we are interested in behavior off criticality with $M \neq 0$ and at finite energy $i\omega \to \epsilon$.

A general solution of the Schrödinger equation, Eq. (4.6), is daunting (although possible for $\omega \neq 0$—see Appendix B), so for simplicity we focus our attention on the critical regime very close to the localization transition where both $\omega, M \ll g = 1$. In this scaling regime one expects the wave function to remain close to a superposition of the two trivial [constant and $(-1)^n$] solutions, i.e.,

$$\phi_n = c_1\phi(n,M) + c_2(-1)^n\phi(n,-M), \qquad (4.13)$$

where $\phi(n,M)$ is slowly varying with $|\phi(n+1,M) - \phi(n,M)| \ll |\phi(n,M)|$. This suggests a ''continuum'' limit, in which $\phi(n,M)$ may be regarded as a continuous function of n, and discrete differences in the Schrödinger equation are replaced with derivatives. In this continuum approximation the Schrödinger equation, Eq. (4.6), becomes

$$n\frac{d^2\phi}{dn^2} + (1+M)\frac{d\phi}{dn} = \frac{\omega}{2}\phi. \qquad (4.14)$$

For $\omega, M \ll 1$ the solution of this continuum equation is expected to coincide with the exact eigenfunction of Eq. (4.6) for $n \gg 1$. In Appendix B, we verify this for the special case $M = 0$, where it is possible to solve directly the discrete Schrödinger equation.

The continuum differential equation (4.14) must be supplemented by an appropriate boundary condition. A natural physical requirement is that $\phi(n,M) \to 0$ for $n \to \infty$. Because we have still have the freedom to choose c_1 and c_2 in Eq. (4.13), the normalization at the origin can remain arbitrary at this stage.

The two constants in Eq. (4.13) then require two additional constraints. The first comes from Eq. (4.9), $\phi_0 = 1$, giving

$$c_1\phi(1,M) + c_2\phi(1,-M) = 1. \qquad (4.15)$$

The second constraint is obtained from Eq. (4.6) for $n = 1$:

$$\phi_3 - \phi_1 = -M(\phi_2 - \phi_0) + \omega\phi_1. \qquad (4.16)$$

In the limit $M, \omega \ll 1$, and using Eq. (4.13), this becomes

$$c_1\phi'(1,M) = c_2\phi'(1,-M). \qquad (4.17)$$

D. Solution

We are now in a position to obtain the solution to the continuum equation. Under an exponential change of variables, $n = e^z$, with $\Phi(z,M) = \phi(e^z,M)$, Eq. (4.14) takes a more illuminating form,

$$\left[-\frac{d^2}{dz^2} - M\frac{d}{dz} + \frac{\omega}{2}e^z\right]\Phi_\pm = 0. \qquad (4.18)$$

Indeed, for $M=0$ this is equivalent to a Schrödinger equation for a particle moving in an exponential potential. Since $z=\ln n$, the domain of the equation is $0 \leq z < \infty$. For small ω, however, the potential is negligible for small z. It rises very abruptly and becomes of order 1 for $z \sim z_w$, where

$$z_w = |\ln \omega|. \tag{4.19}$$

In the small ω limit of interest, then, there is a region of divergent width over which Eq. (4.18) effectively describes a *free* particle. The width of the potential is, however, only of order 1. Thus we expect that on the scale of z_w, the exponential potential can be well approximated by a *hard wall*.

In the following we make this hard-wall approximation, replacing the continuum equation (4.18) by

$$\left[-\frac{d^2}{dz^2} - M \frac{d}{dz} \right] \Phi = 0, \tag{4.20}$$

with $0 \leq z \leq z_w$. and the boundary condition

$$\Phi(z_w) = 0. \tag{4.21}$$

This simple equation can be readily solved (below) and the density of states extracted.

As shown in Appendix C, the hard-wall approximation is not necessary, since the exact continuum equations can be solved explicitly. Although the hard-wall wave function differs from the exact solution, the resulting density of states coincides, up to an overall (nonuniversal) multiplicative constant.

The solution of Eq. (4.20) consistent with the hard-wall boundary condition [Eq. (4.21)] is

$$\Phi(z, M) = \frac{1}{M z_w} [e^{-Mz} - e^{-M z_w}], \tag{4.22}$$

where we have assumed that $\omega, M \ll 1$, and chosen a convenient (but arbitrary) normalization. As $M \to 0$ this reduces to

$$\Phi(z) = 1 - \frac{z}{z_w}. \tag{4.23}$$

Applying the constraints in Eqs. (4.15,4.17), determines the constants as

$$c_1 = c_2 = \frac{z_w M \omega^M}{1 - \omega^{2M}}. \tag{4.24}$$

To extract the density of states, we use Eq. (4.11) to write

$$\bar{\mathcal{G}}(0, i\omega) = 2i c_1 c_2 \int_1^\infty dn \, \phi(n, M) \phi(n, -M)$$

$$= 2i c_1 c_2 \int_0^{z_w} dz \, e^z \Phi(z, M) \Phi(z, -M). \tag{4.25}$$

The integral can be readily performed giving

$$\bar{\mathcal{G}}(0, i\omega) = i \frac{M^2}{\omega} f(\omega^M), \tag{4.26}$$

with the *exact* scaling function

$$f(Y) = c \left(\frac{Y}{1 - Y^2} \right)^2. \tag{4.27}$$

Our present implementation of the hard-wall approximation reproduces the exact value of the nonuniversal constant, $c=4$, obtained in Appendix C, but this result depends upon the precise position of the wall.

An exact expression for the density of states in the critical regime can now be obtained from Eq. (2.25) by performing an analytic continuation, $\rho(\epsilon) = (1/\pi) \text{Im} \bar{\mathcal{G}}(0; \epsilon + i0^+)$. Noting that $\bar{\mathcal{G}}(0, i\omega)$ is purely imaginary and odd in ω, one readily obtains (for $\epsilon > 0$),

$$\rho(\epsilon) = \frac{M^3}{\epsilon} g(\epsilon^M), \tag{4.28}$$

with

$$g(Y) = \tfrac{1}{2} Y f'(Y) = c \frac{Y^2(1 + Y^2)}{(1 - Y^2)^3}. \tag{4.29}$$

In the $M \to 0$ limit, this reduces to

$$\rho(\epsilon) \sim \frac{1}{\epsilon |\ln \epsilon|^3}, \tag{4.30}$$

a result obtained previously by other methods.

A special feature of one dimension, is that the *typical* localization length $\tilde{\xi}$ can be extracted from the real part of the Green's function *at $x=0$*. As derived in Ref. 25, $\tilde{\xi}(\epsilon)$ satisfies

$$\frac{\partial \tilde{\xi}^{-1}}{\partial \epsilon} = P \int d\epsilon' \frac{\rho(\epsilon')}{\epsilon - \epsilon'} = -\frac{1}{\pi} \text{Re} \bar{\mathcal{G}}(x=0; \epsilon). \tag{4.31}$$

Performing an analytic continuation to real energy using Eq. (4.26) gives $\text{Re} \bar{\mathcal{G}}(0; \epsilon) \sim -1/\epsilon |\ln \epsilon|^2$. Integration from $\epsilon=0$ using the fact that $\tilde{\xi}^{-1}(0) = 0$ gives the result $\tilde{\xi} \sim |\ln \epsilon|$.

V. FERMION GREEN'S FUNCTION

To determine the mean correlation length, we need to calculate the Green's function, $\bar{G}(x, i\omega)$, at *nonzero x*. From Eq. (3.23) this takes the form

$$\bar{G}_{\alpha\beta}(x, i\omega) = i(-1)_L^\alpha \langle 0 | F_\alpha e^{-xH} F_\beta^\dagger | 0 \rangle_R. \tag{5.1}$$

Given the quantum numbers of the ground state, the state $F_\beta^\dagger |0\rangle_R$ clearly has $N_B = -1$, $N_F = 2$, $\Gamma = N = 1$, $\bar{\Gamma} = 0$. We can, therefore, insert the resolution of the identity from Eq. (3.81) to give

$$\bar{G}_{\alpha\beta}(x, i\omega) = i(-1)^\alpha \sum_E \,\, L\langle 0 | F_\alpha | -1, 2, 1, 0, E \rangle_{RL}$$

$$\langle -1, 2, 1, 0, E | F_\beta^\dagger | 0 \rangle_R e^{-Ex}. \tag{5.2}$$

Our task is thus to determine the matrix elements and eigenvalues needed to carry out this sum. To do so, we expand the eigenstate in the appropriate basis,

$$|-1,2,1,0,E\rangle_R=\sum_{n=0}^{\infty}\chi_n|n\rangle_1, \qquad (5.3)$$

where we have abbreviated

$$|n\rangle_1\equiv|-1,2,1,0,1/2+n\rangle. \qquad (5.4)$$

The wave function χ_n then satisfies the Schrödinger wave equation,

$$2\omega[n+\tfrac{1}{2}]\chi_n+2M[n\chi_{n-1}-(n+1)\chi_{n+1}]-(n+2)(n+1)$$
$$\times\chi_{n+2}+(2n^2+2n+1)\chi_n-n(n-1)\chi_{n-2}=E\chi_n, \qquad (5.5)$$

where we have again set $g=1$. Again, this equation has the property that multiplication of a solution by $(-1)^n$ yields a solution for $M\to-M$. We therefore expect

$$\chi_n(M,E)=c_3\chi(n,M,E)+c_4(-1)^n\chi(n,-M,E). \qquad (5.6)$$

Naively, $\chi(n,M,E)$ can be obtained in the continuum limit. Converting finite differences to derivatives, we obtain

$$\left[-\left(2n\frac{d}{dn}+1\right)^2-2M\left(2n\frac{d}{dn}+1\right)+2\omega n\right]\chi(n,M,E)$$
$$=E\chi(n,M,E). \qquad (5.7)$$

Upon tranforming to logarithmic variables,

$$\chi(n,M)=\Phi_E(z=\ln n,M), \qquad (5.8)$$

one has

$$[-(2\partial_z+1)^2-2M(2\partial_z+1)+2\omega e^z]\Phi_E(z,M)$$
$$=E\Phi_E(z,M). \qquad (5.9)$$

Based on previous experience, we expect that the hard-wall approximation gives exact results in the scaling limit. In Appendix D we verify this explicitly, by constructing exact solutions of the continuum equation (5.7). Within the hard-wall approximation the potential $2\omega e^z\to0$ in Eq. (5.9) is dropped, and replaced by a boundary condition at $z_w=|\ln\omega|$,

$$\Phi_E(z_w,M)=0. \qquad (5.10)$$

The general solution of Eq. (5.9) with the hard-wall boundary condition is

$$\Phi_E(z)=e^{-(1+M)z/2}\sin[\beta(z-z_w)/2], \qquad (5.11)$$

where $\beta=\sqrt{E-M^2}$.

So far, we have not determined the spectrum, or allowed values of β. In an ordinary quantum problem, these eigenvalues would be fixed by a boundary condition at $z=0$. In this case, however, such a simple treatment is problematic. The difficulty arises because, unlike in the ground-state sector, neither $\chi_n=1$ nor $\chi_n=(-1)^n$ are solutions in the limit $\omega=M=0$. We therefore expect a nontrivial solution for $n=O(1)$, in that the discreteness of n is important and the continuum limit is *not* valid.

Fortunately, for $1\lesssim n\ll1/\omega$, we can obtain an asymptotic approximation which does not rely upon the continuum limit.

This is possible because for $n\ll1/\omega$, the $\omega\mathcal{J}^z$ term in the Hamiltonian, Eq. (3.31), can be regarded as a small perturbation. Neglecting this term, H is a function only of $\mathcal{J}^x$, so that the eigenfunctions of H are simply eigenfunctions of $\mathcal{J}^x$. As a first step, consider the state $|\alpha\rangle$ with

$$\mathcal{J}^x|\alpha\rangle=i\alpha|\alpha\rangle. \qquad (5.12)$$

Expanding $|\alpha\rangle$ in the $\mathcal{J}^z$ basis,

$$|\alpha\rangle=\sum_n\psi_n(\alpha)|n\rangle_1, \qquad (5.13)$$

one finds the simpler Schrödinger equation,

$$(n+1)\psi_{n+1}(\alpha)-n\psi_{n-1}(\alpha)=-2i\alpha\psi_n(\alpha). \qquad (5.14)$$

This equation can be solved exactly (see Appendix E). For large n, the solution, which is well behaved at the origin behaves asyptotically as

$$\psi_n(\alpha)\sim n^{-1/2}[(2n)^{-i\alpha}\Gamma(1/2+i\alpha)$$
$$+(-1)^n(2n)^{i\alpha}\Gamma(1/2-i\alpha)]. \qquad (5.15)$$

Eigenstates of H therefore take the form

$$\chi_n\sim c_+\psi_n(\alpha_+)+c_-\psi_n(\alpha_-), \quad \text{for } 1\ll n\ll1/\omega, \qquad (5.16)$$

where $\alpha_{\pm}$ are the two roots of the equation $4\alpha^2+4iM\alpha=E$, i.e.,

$$\alpha_{\pm}=\frac{-iM\pm\beta}{2}, \qquad (5.17)$$

and $\beta=\sqrt{E-M^2}$.

Mathematically, Eq. (5.16) is an *outer* solution, valid outside a boundary layer that occurs for large n. To obtain a complete solution, it must be matched to the *inner* solution, valid ''inside'' the boundary layer, which is just the continuum regime of large n. Within the hard-wall approximation, this is just the standing wave in Eq. (5.11).

To match the two solutions, we let $n=e^z$ in Eqs. (5.15,5.16), which gives

$$\chi_n\sim e^{-z/2}\{e^{-Mz/2}[c_+e^{-i\beta z/2}+c_-e^{i\beta z/2}]$$
$$+(-1)^ne^{Mz/2}[c_+e^{i\beta z/2}+c_-e^{-i\beta z/2}]\}. \qquad (5.18)$$

Similarly, using Eqs. (5.6,5.11), the continuum solution gives

$$\chi_n\sim e^{-z/2}\{c_3e^{-Mz/2}+(-1)^nc_4e^{Mz/2}\}\sin[\beta(z-z_w)/2]. \qquad (5.19)$$

These expressions are equal in two situations. One can take $c_+=c_-=c_3/2=c_4/2$ if βz_w is an odd multiple of π. Alternatively, $c_+=-c_-=c_3/2=-c_4/2$ if βz_w is an even multiple of π. The final, matched solution for both cases can thus be written

$$\Phi^{(k)}(z,M)\equiv\Phi_{E_k}(z,M)=e^{-(1+M)z/2}\sin(\beta_kz/2+\theta_k), \qquad (5.20)$$

where

$$\beta_k=\pi k/z_w, \qquad (5.21)$$

$$E_k = M^2 + \left(\frac{\pi k}{z_w}\right)^2, \tag{5.22}$$

$$\theta_k = \begin{cases} 0, & k \text{ even} \\ \pi/2, & k \text{ odd,} \end{cases} \tag{5.23}$$

$$c_3 = (-1)^{k+1}c_4 \equiv c. \tag{5.24}$$

The discrete quantum number $k = 1,2,\ldots \infty$.

The corresponding left eigenstate is obtain by acting with U from Eq. (3.74). Defining

$$|k\rangle_{R/L} = |-1,2,1,0,E_k\rangle_{R/L}, \tag{5.25}$$

we choose

$$|k\rangle_L = (-1)^{k+1} e^{i\pi(J^z - 1/2)}|k\rangle_R. \tag{5.26}$$

With this choice, normalization implies that the constant $c = z_w^{-1/2}$. Thus the final expressions for the (normalized) eigenstates are

$$|k\rangle_R = z_w^{-1/2}\sum_n \left[\chi(n,M) + (-1)^{n+k+1}\chi(n,-M)\right]|n\rangle_1, \tag{5.27}$$

$$|k\rangle_L = z_w^{-1/2}\sum_n \left[(-1)^{n+k+1}\chi(n,M) + \chi(n,-M)\right]|n\rangle_1. \tag{5.28}$$

These actually form an orthonormal set

$$_L\langle k|k'\rangle_R = \delta_{kk'}, \tag{5.29}$$

as can be verified by direct computation.

Having obtained the full set of eigenvalues and left and right eigenfunctions of H in the appropriate sector, it is a simple matter to evaluate the Green's function using Eq. (5.2). Using the hard-wall eigenfunctions, we find

$$\overline{G}_{\alpha\beta}(x,i\omega) = i\frac{32\pi^2}{\omega|\ln\omega|^3}\frac{M^2\omega^{2M}}{(1-\omega^{2M})^2}(-1)^\alpha$$
$$\times \sum_{k=1}^{\infty}(-1)^{k+1}k^2[(-1)^\alpha\omega^{M/2}$$
$$+(-1)^{k+1}\omega^{-M/2}][(-1)^\beta\omega^{M/2}$$
$$+(-1)^{k+1}\omega^{-M/2}]e^{-\pi^2 k^2 x/|\ln\omega|^2}. \tag{5.30}$$

One thereby obtains the final form for the Green's function, exact in the scaling limit:

$$\overline{G}(x,i\omega) = \begin{cases} \dfrac{i^{x+1}}{\omega|\ln\omega|^5}f_\omega^e(\omega^M)F_\omega^e(x/\xi_\omega)e^{-x/\xi_M}, & x \text{ even} \\[2ex] \dfrac{i^{x+1}}{\omega|\ln\omega|^5}f_\omega^o(\omega^M)F_\omega^o(x/\xi_\omega)e^{-x/\xi_M}, & x \text{ odd,} \end{cases} \tag{5.31}$$

where the universal scaling functions are given by

$$f_\omega^e(Y) = Yf_\omega^o(Y) = \left(\frac{Y\ln Y}{1-Y^2}\right)^2, \tag{5.32}$$

$$F_\omega^e(Y) = A\sum_{k=1}^{\infty}k^2 e^{-k^2 Y}, \tag{5.33}$$

$$F_\omega^o(Y) = A\sum_{k=1}^{\infty}(-1)^{k+1}k^2 e^{-k^2 Y}. \tag{5.34}$$

Here the nonuniversal amplitude $A = 32\pi^2$ within the hard-wall approximation, and we have defined two correlation lengths,

$$\xi_M = 1/M^2, \quad \xi_\omega = \left(\frac{\ln\omega}{\pi}\right)^2. \tag{5.35}$$

Notice that $\overline{\mathcal{G}}(x;i\omega)$ is pure imaginary for x even and pure real for x odd, as dictated by particle-hole symmetry. Equation (5.31) is valid for $\omega > 0$. Particle-hole symmetry then determines the Green's function for $\omega < 0$, since $\overline{\mathcal{G}}(x;i\omega)$ is odd in ω for even x and even in ω for odd x.

Equation (5.35) gives us the correlation length exponent, $\nu = 2$, defined by $\xi \sim M^{-\nu}$, and the dynamical exponent, $z = \infty$, defined by $\omega \sim \xi^z$. Equation (5.31) is actually simpler than one would generally expect on the basis of scaling — the most general scaling form would not factorize as it does here.

The correlator $C(\epsilon) = (1/\pi)\text{Im}\overline{\mathcal{G}}(\epsilon + i0^+)$ can be obtained via analytic continuation, using the symmetry properties of the Green's function under $\omega \to -\omega$. One finds

$$C(x=2n,\epsilon,M) = \frac{(-1)^n}{\epsilon|\ln\epsilon|^6}F_\epsilon^e\left(\frac{x}{\xi_\epsilon},\epsilon^M\right)e^{-x/\xi_M}, \tag{5.36}$$

$$C(x=2n+1,\epsilon,M) = \frac{(-1)^{n+1}}{\pi\epsilon|\ln\epsilon|^5}F_\epsilon^o\left(\frac{x}{\xi_\epsilon},\epsilon^M\right)e^{-x/\xi_M}, \tag{5.37}$$

where the energy dependent ''localization'' length is

$$\xi_\epsilon = \left(\frac{\ln\epsilon}{\pi}\right)^2. \tag{5.38}$$

The even sublattice scaling function no longer factors,

$$F_\epsilon^e(Y,Z) = f_\omega^e(Z)\left[\tfrac{5}{2}F_\omega^e(Y) + YF_\omega^{e\prime}(Y)\right]$$
$$+\tfrac{1}{2}Z\ln Zf_\omega^{e\prime}(Z)F_\omega^e(Y), \tag{5.39}$$

while the odd sublattice scaling function remains simple,

$$F_\epsilon^o(Y,Z) = F_\omega^o(Y)f_\omega^o(Z). \tag{5.40}$$

The scaling forms, Eqs. (5.36)–(5.40), encode several significant physical properties. First consider the same sublattice correlation $(x=2n)$, for simplicity at zero staggering $(M=0)$. For distances shorter than the correlation length, this has a slow power-law decay, since $F_\epsilon^e(Y,1) \sim Y^{-3/2}$, for $Y \ll 1$. In particular,

$$C(x=2n,\epsilon,M=0) \sim \frac{(-1)^n}{\epsilon|\ln\epsilon|^3}\frac{1}{|x|^{3/2}}, \tag{5.41}$$

for $|x| \ll \xi_\epsilon$. This can be understood as the product of the density of states and a two-point ''wave-function correla-
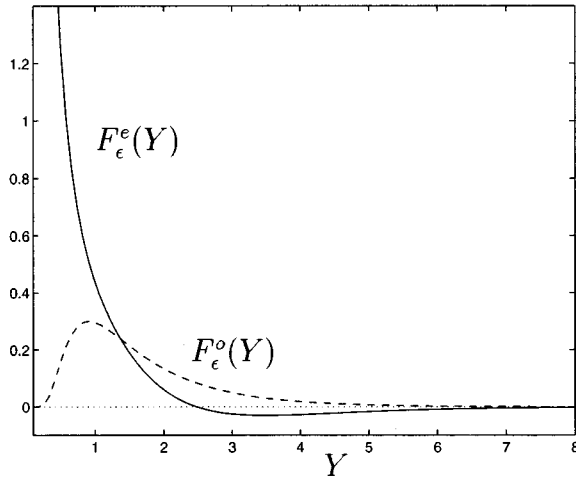
FIG. 2. Scaling functions for the fermion Green's function. The solid and dashed lines are the scaling functions, $F_\epsilon^e(Y)$ and $F_\epsilon^o(Y)$, for correlators $C(x)$ between two sites on the same and different sublattices, respectively.

tion,'' with multifractal scaling exponent (see the next section) $y(q=1)=3/2$. For distances $x \gg \xi_\epsilon$ (and $M=0$), even the rare wave functions are localized, and $C(x)$ decays exponentially. The full scaling function, which describes the crossover between these two limits, is plotted in Fig. 2. Note the *change of sign* for $Y \approx 2.5$—this may be interpreted physically as arising from the first node in the dominant rare wave function at energy ϵ.

The different sublattice Green's function behaves quite differently at short distances. Using the Poisson summation formula, it is straightforward to show that

$$F_\epsilon^o(Y,1) \sim \frac{\sqrt{\pi}}{2Y^{3/2}} \left[1 - \frac{\pi^2}{4Y} \right] e^{-\pi^2/4Y} + O(e^{-9\pi^2/4Y}),$$
(5.42)

for $Y \ll 1$. This implies that $C(x=2n+1)$ is *much smaller* than the same sublattice correlator. The suppression is due to the fact that sublattice mixing of wave functions occurs only at finite ϵ. The exact zero-energy ''wave function,'' discussed in detail in the next section, lies *entirely* on a single sublattice.

VI. MULTIFRACTAL WAVE FUNCTIONS

The above results reveal that the ensemble-averaged single-particle Green's function decays as a power law with an exponent 3/2 for distances smaller than the correlation lenth. When $m_0=0$, the relevant scale is $\xi_\epsilon = |\ln\epsilon|^2/\pi^2$, which diverges at the band center. Such power-law spatial scaling is in striking contrast to the behavior of the *typical* Green's function, which decays as a stretched exponential, $G_{\text{typ}} \sim \exp(-ax^{1/2})$ at criticality. This stretched exponential form follows directly from the exact zero-energy wave function, Eq. (2.13), whose logarithm undergoes a one-dimensional random walk.

Power-law scaling of the *average* Green's function at criticality can also be understood (albeit more subtly) in terms of Eq. (2.13). To see this it is useful to consider

ensemble-averaged correlation functions of the zero-energy wave function. As with higher-dimensional localization transitions, such as the plateau transition in the IQHE, this critical wave function is expected to exhibit multifractal scaling characteristics. As shown in a recent paper, such correlators in this 1D case can be computed exactly, via a mapping to Liouville quantum mechanics. This mapping exploits the equivalence between imaginary time quantum mechanics and the one-dimensional random walk. Slightly generalizing this work, we compute below the full multifractal spectrum for the 1D critical case. This calculation is instructive since it reveals a link between the supersymmetry calculations and Liouville quantum mechanics.

To extract average wave-function correlators, it is necessary to consider normalized states. We thus consider a finite system of length L, and normalize the wave function from Eq. (2.13) over the finite interval $|x| < L/2$. Focussing on one component of the spinor wave function, say $\phi_+(x)$, an appropriately normalized wave function, denoted as $\psi(x)$, can be written

$$\psi(x) = \mathcal{N}^{-1/2} e^{z(x)},$$
(6.1)

with normalization

$$\mathcal{N} = \int_{-L/2}^{L/2} dx\, e^{2z(x)},$$
(6.2)

where we have defined $\partial_x z(x) = m(x)$, with the random potential $m(x)$ assumed as before to be Gaussian with $[m(x)m(x')]_{\text{ens}} = 2g\delta(x-x')$. It remains to specify the boundary conditions on $\psi(x)$ at $x = \pm L/2$. For technical reasons it is convenient to ''pin'' the 1d random walker at the ends, taking $z(x=\pm L/2)=0$.

We focus on the ensemble averaged correlation function,

$$\mathcal{W}_q(x,L) = [|\psi(x)\psi(0)|^q]_{\text{ens}},$$
(6.3)

between two points separated by a distance x, assumed much smaller than L. The one-point function $\mathcal{W}_q(L) = \mathcal{W}_q(x=0,L)$, referred to as a participation ratio, is expected to vary as a power law with system size:

$$\mathcal{W}_q(L) \sim \frac{1}{L^{d+\tau(q)}},$$
(6.4)

with $d=1$ the spatial dimensionality. The exponent $\tau(q)$ is often written as $\tau(q)=(q-1)D(q)$. For a plane-wave or nonfractal wave function (as in a 3D metal, say) $D(q)=d$. For an exponentially localized wave function, $D(q) = \tau(q)=0$. A simple fractal would be characterized by a q-independent D, different that the spatial dimension, whereas in a multifractal D depends on q and equivalently $\tau(q)$ is a nonlinear function of q.

The two-point function $\mathcal{W}_q(x,L)$ for x much larger than microscopic lengths (i.e., lattice spacing) yet much smaller than L is also expected to exhibit power-law scaling:

$$\mathcal{W}_q(x,L) \sim \frac{1}{L^{d+\tau(q)}} \frac{1}{x^{y(q)}},$$
(6.5)

with a spectrum of exponents $y(q)$. For some multifractals a relation can be obtained between $\tau(q)$ and $y(q)$, but in general they can be independent exponents.

To extract $\tau(q)$ and $y(q)$ for the 1D critical wave function, we follow closely Shelton and Tsvelik.[23] The correlation function can be expressed as a functional integral over the random walk configurations $z(x)$ as

$$\mathcal{W}_q(x,L) = \frac{1}{Z_0}\int' \mathcal{D}z\, |\psi(x)\psi(0)|^q e^{-S_0}, \qquad (6.6)$$

with

$$Z_0 = \int' \mathcal{D}z\, e^{-S_0}, \qquad (6.7)$$

and an action,

$$S_0 = \frac{1}{4g}\int_{-L/2}^{L/2} dx\, (\partial_x z)^2. \qquad (6.8)$$

Here the prime on the integration indicates the boundary conditions, $z(x=\pm L/2)=0$. In the following we put $g=1$. The functional integral over disorder configurations is nontrivial due to the normalization of the wave functions. Fortunately, the normalization can be exponentiated via the identity,

$$\mathcal{N}^{-q} = \frac{1}{\Gamma(q)}\int_0^\infty d\omega\, \omega^{q-1} e^{-\omega\mathcal{N}}, \qquad (6.9)$$

where it can be absorbed into the action. In this way one obtains

$$\mathcal{W}_q(x,L) = \frac{1}{\Gamma(q)}\int_0^\infty d\omega\, \omega^{q-1}\langle e^{qz(x)} e^{qz(0)}\rangle, \qquad (6.10)$$

where the average is given by

$$\langle e^{qz(x)} e^{qz(0)}\rangle = \frac{1}{Z_0}\int' \mathcal{D}z\, e^{qz(x)} e^{qz(0)} e^{-S}, \qquad (6.11)$$

with the total action

$$S = \int_{-L/2}^{L/2} dx\, [\tfrac{1}{4}(\partial_x z)^2 + \omega e^{2z}]. \qquad (6.12)$$

If x is viewed as an imaginary time coordinate, this average is seen to be equivalent to a path integral representation of the quantum mechanics of a particle with coordinate z moving in an exponential potential. Passing to the operator representation of this quantum mechanics by defining a quantum Hamiltonian,

$$H = -\partial_z^2 + \omega e^{2z}, \qquad (6.13)$$

the above average can be written as a quantum expectation value,

$$\langle e^{qz(x)} e^{qz(0)}\rangle = \frac{\langle 0|e^{-LH/2} e^{qz(x)} e^{qz} e^{-LH/2}|0\rangle}{\langle 0|e^{-LH_0}|0\rangle}, \qquad (6.14)$$

with $e^{qz(x)} = e^{xH} e^{qz} e^{-xH}$, and $H_0 = -\partial_z^2$. Here $|0\rangle$ is a position ket with $z=0$.

Evaluating the wave-function correlator has thus been reduced to solving for the quantum mechanics of a particle moving in an exponential potential—Liouville quantum mechanics. This form is identical to that which arose in the bosonic sector of the supersymmetric calculation of the fermion Green's function. In that case, the coordinate z was related to the boson number via $n = e^z$, and ω was a small imaginary part of the energy. The supersymmetric calculation can thus be viewed as a supersymmetric version of Liouville quantum mechanics. In earlier work, Kogan, Mudry, and Tsvelik showed that the wave function correlators for a two-dimensional particle described by a Dirac equation with random vector potential (for, which the exact zero-energy wave function can also be written down explicitly) could be formulated in terms of Liouville *field theory*. Perhaps such a 2D localization critical point can be formulated in terms of a supersymmetric Liouville field theory.

For simplicity we evaluate the above quantum expectation value within the hard-wall approximation, which should give the correct scaling behavior for the wave-function correlator. As before, we replace the exponential potential by a hard-wall at z_w, with $\omega e^{2z_w} = c$, for a constant c of order 1. The value of c affects the overall prefactor in the correlator. We choose $c = 1/2$, which gives the correct normalization, $\mathcal{W}_{q=1}(L) = 1/L$. Since the quantum particle is contrained to have $z < z_w$, when $\omega > 1$ (and z_w is negative) the particle's wave function vanishes at $z=0$, so that the quantum expectation value in Eq. (6.14) vanishes identically. We can thus restrict the integration over ω to $\omega < 1$.

The denominator in Eq. (6.14) is the propagator for a free random walker (i.e., without the hard wall) and can be readily evaluated giving $(4\pi L)^{-1/2}$. To evaluate the numerator it is convenient to let $z \to z_w - z$, so that the quantum particle is then constrained to have $z > 0$. Within the hard-wall approximation, the correlator can then be expressed in terms of the free Hamiltonian as

$$\mathcal{W}_q(x,L) = \frac{\sqrt{4\pi L}}{\Gamma(q)2^q}$$

$$\times \int_0^1 \frac{d\omega}{\omega}\langle z_w| e^{-LH_0/2} e^{-qz(x)} e^{-qz} e^{-LH_0/2}|z_w\rangle, \qquad (6.15)$$

with z restricted positive and $z_w = |\ln\omega|/2$ [the factor of 2 difference between this definition and the one used in the SUSY calculations is a consequence of a different choice of normalization of the field $z(x)$]. To evaluate this quantum average we introduce a complete set of standing waves,

$$\langle z|k\rangle = \sqrt{\frac{2}{\pi}}\sin(kz), \quad (k>0), \qquad (6.16)$$

which are eigenstates $H_0|k\rangle = k^2|k\rangle$, and appropriately normalized on the interval $z>0$: $\langle k|k'\rangle = \delta(k-k')$. Inserting the resolution of the identity,

$$1 = \int_0^\infty dk\, |k\rangle\langle k|, \qquad (6.17)$$

into Eq. (6.15), evaluating the matrix elements in closed form and performing the integration over ω gives for large L the final result:

$$\mathcal{W}_q(x,L) = \frac{\widetilde{\mathcal{W}}(q^2 x)}{2^{q-1} q^3 \Gamma(q)} \frac{1}{L}, \qquad (6.18)$$

with

$$\widetilde{\mathcal{W}}(x) = \frac{16}{\pi} \int_0^\infty \frac{dk k^2}{(1+k^2)^4} e^{-xk^2}. \qquad (6.19)$$

The crossover function $\widetilde{\mathcal{W}}(x)$ interpolates between one at $x=0$ and $\widetilde{\mathcal{W}}(x) \sim x^{-3/2}$ for $x \gg 1$. The crossover length, $x_c = 1/q^2$, is the characteristic distance over which the logarithm of the wave function $\psi(x)$ changes by order $1/q$ (or equivalently the "time" it takes the 1D random walk to move a distance $\delta z \sim 1/q$). This crossover scale clearly depends on the strength of the disorder. For the original lattice tight-binding model, when the random hopping strengths are comparable to the mean hopping strength, $\delta t_n / t$ of order 1, this crossover length is of the order of the tight-binding lattice spacing. Thus it is clear that the *form* of the crossover function $\widetilde{\mathcal{W}}(x)$ for x of order 1, cannot be universal. In fact, the precise form above is particular to the hard-wall approximation, and an evaluation using the exact Liouville eigenfunctions would give another form, although they agree in their universal large x behavior.

For large x, $\mathcal{W}_q(x,L) \sim x^{-3/2} L^{-1}$. By comparing with Eq. (6.5), the exponents $\tau(q)$ and $y(q)$ are seen to be *independent* of q, with $\tau(q)=0$ and $y(q)=3/2$. A vanishing $\tau(q)$ is characteristic of an exponentially localized wave function. However, the two-point correlator for an exponentially localized wave function also decays exponentially ($y(q)=\infty$), in contrast to the present 1D wave function, which exhibits power-law correlations. The 1D critical wave function is *typically* quasilocalized (centered around a maximum) with stretched exponential decay. However, the *average* two-point correlator at separation x is dominated by the rare wave function, which has a secondary maximum close in magnitude but separated spatially (by distance x) from its primary maximum. The likelihood of this involves the extremal statistics of a 1D random walk near a global maximum (absorbing wall), which is being described mathematically above by quantum mechanics near an exponential (or hard-wall) potential. As expected, the two-point wave function correlator at $q=1$ decays with the same exponent, $y(q=1)=3/2$, as the average Green's function obtained with supersymmetry.

VII. SUMMARY AND CONCLUSIONS

In this paper we have presented a detailed SUSY analysis of the critical properties of the zero-energy delocalization transition in the 1D random hopping model. This 1D random critical point has been of interest for many years, originating with the pioneering paper by Dyson in 1953 on a related model of a 1D harmonic chain with random spring constants.[24] Most of the prior work, using a variety of different approaches, has focussed on properties derivable from the mean local Green's function, specifically the density of states and the typical localization length, $\widetilde{\xi}$.[18] By employing

a real-space RG approach to analyze phase transitions in a class of closely related random spin chains, D.S. Fisher has recently extended the analysis to extract the spatial dependence of mean correlation functions.[19,20] An important element is the emergence of a *second* correlation length ξ that determines the spatial decay of *mean* (rather than typical) correlation functions. To the growing body of knowledge concerning this 1D random critical point, we add several new results in this paper. (i) Using SUSY we have computed the *exact* two-parameter scaling functions for the mean fermion Green's function. (ii) By employing Liouville quantum mechanics, we have extracted the set of multifractal scaling exponents $\tau(q)$ and $y(q)$ that characterize the critical wavefunction pair correlators.

Together, these two results encapsulate the important universal scaling characteristics of this 1D random critical point. The spatial dependence of the mean fermion Green's function is controlled by two lengths, a mean localization length that diverges upon approaching the band center as $\xi_\epsilon \sim |\ln\epsilon|^2$, and a mean "staggering length," varying as $\xi_M \sim M^{-2}$ when the strength M of a staggering in the hopping strengths is taken to zero. These two lengths are to be contrasted with their counterparts, denoted $\widetilde{\xi}_\epsilon$ and $\widetilde{\xi}_M$, which charaterize the spatial decay of the *typical* (rather than ensemble-averaged) Green's function. From the singular behavior of the density of states, one can infer that the typical localization length diverges more slowly, as $\widetilde{\xi}_\epsilon \sim |\ln\epsilon|$. Likewise, the typical staggering length that follows rather directly from the nature of the exact (decaying) zero-energy wave function diverges more slowly, $\widetilde{\xi}_M \sim M^{-1}$, than its mean counterpart.

For spatial separations $x \ll \xi_\epsilon, \xi_M$, between two points on the same sublattice, the mean fermion Green's function varies as an inverse power law of x with universal exponent 3/2. This result also follows from an analysis of the zero-energy wave function whose logarithm undergoes a 1D random walk, with the exponent 3/2 being related to *extremal* properties of the random walker. In contrast, the *typical* Green's function for $x \ll \widetilde{\xi}_\epsilon, \widetilde{\xi}_M$ is expected to decay as a stretched exponential, $\mathcal{G}_{\text{typ}} \sim e^{-c\sqrt{x}}$, reflecting the *typical* behavior of the random walker.

A key motivation for the present paper was to investigate in detail features that emerge in a SUSY formulation of a random critical point. The calculation proceeded by expressing mean correlators in terms of quantum-mechanical expectation values for a SUSY Hamiltonian, which involved a *single* supaspin. This Hamiltonian has a number of notable features: (i) It is non-Hermitian, with distinct left and right eigenfunctions. (ii) It has a unique zero-energy ground state, as dictated by supersymmetry, and the excited states are organized into supersymmetric doublets and quadruplets. (iii) The right (or left) eigenstates alone do *not* span the Hilbert space—the Hamiltonian is thus "defective." (iv) The Hilbert space is infinite, due to the noncompact SU(1,1) bosonic subalgebra of the supaspin group. (v) The eigenstates explore the outer reaches of the noncompact manifold, in a manner that can be described by Liouville quantum mechanics.

It is our hope that a thorough undertanding of these unusual features will be helpful in extending the SUSY approach to attack two-dimensional random critical points,

such as the IQHE plateau transition. It is tantalizing to speculate that some appropriate supersymmetric version of Liouville field theory might give a correct description of delocalization transitions in 2D.

APPENDIX A

The mapping from the random transverse field Ising chain to a free-fermion model was introduced by Shankar and Murthy. We briefly recapitulate this mapping. In terms of Majorana fermions,

$$\eta_{1,n} = \frac{1}{\sqrt{2}} \prod_{m<n} \sigma_m^x \sigma_n^y, \tag{A1}$$

$$\eta_{2,n} = \frac{1}{\sqrt{2}} \prod_{m<n} \sigma_m^x \sigma_n^z, \tag{A2}$$

which satisfy $\{\eta_{i,m}, \eta_{j,n}\} = \delta_{ij}\delta_{mn}$, the random Ising Hamiltonian, Eq. (2.27), can be rewritten as

$$\mathcal{H}_I = \sum_n \left[-2iK_{1,n}\eta_{1,n}\eta_{2,n} + 2iK_{2,n}\eta_{1,n}\eta_{2,n+1} \right]. \tag{A3}$$

A continuum limit can be taken by putting $x = ndx$, $K_1 = dx/2$, and $K_2 = (1/2 + m(x))dx$, and converting the sums to integrals. This gives

$$\mathcal{H}_c = \int dx\, \eta [\sigma^x i \partial_x + m(x)\sigma^y]\eta, \tag{A4}$$

where we have defined a two-component Majorana field, $\eta = (\eta_1, \eta_2)$. For spatially uniform m this model describes criticality in the pure 2D Ising model, with the phase transition occuring at $m = 0$.

To complete the mapping it is convenient to consider a path integral representation of the partition function, $Z = \mathrm{Tr}\, \exp(-\beta H)$, which can be written as a functional integral over Grassmann fields, $\eta(x, \tau)$, with associated Euclidian action:

$$S = \int dx \frac{d\omega}{2\pi} \eta(x, \omega)[i\omega + \sigma^x i \partial_x + m\sigma^y]\eta(x, -\omega). \tag{A5}$$

These can be decomposed into new Grassmann fields by defining

$$\eta_\alpha(\omega) = \bar{\psi}_\alpha(\omega), \qquad \eta_\alpha(-\omega) = \psi_\alpha(\omega), \tag{A6}$$

for *positive* ω and $\alpha = 1,2$. In terms of these new fields the action becomes

$$S = \int_0^\infty \frac{d\omega}{2\pi} S_\omega, \tag{A7}$$

with

$$S_\omega = \int dx\, \bar{\psi}[\sigma^x i \partial_x + m(x)\sigma^y + i\omega]\psi. \tag{A8}$$

Notice that the functional integral factorizes into a product over independent frequencies. In the following we focus on only a single frequency. The action at a single frequency can be cast into the form of Eq. (2.6) by a rotation in "spin space" around the y axis by $\pi/2$, which takes $\sigma^x \to \sigma^z$, giving

$$S_\omega = \int dx\, \bar{\psi}(h + i\omega)\psi, \tag{A9}$$

with h the 1D random Hamiltonian in Eq. (2.7). Note that, in this case, a nonzero mass m_0 corresponds simply to the deviation from the Ising critical point.

APPENDIX B

For the special case of the ground-state wave function, we are in fact able to obtain an exact solution *without taking the continuum limit*. This is possible because the difference equation, Eq. (4.6), is linear in n. Here, we specialize to the case $n = 0$, in which this solution is especially simple.

For $M = 0$, the Schrödinger equation decouples on even and odd sublattices and can be solved independently on each. To bring this out, we define

$$\phi_{2n} = \gamma_n^e, \qquad \phi_{2n+1} = \gamma_n^o, \tag{B1}$$

for $n = 0,1,2,\ldots$. The even and odd sublattice fields then obey

$$\omega \gamma_n^o = (n+1)\gamma_{n+1}^o - (2n+1)\gamma_n^o + n\gamma_{n-1}^o, \quad n \geq 0$$

$$\omega \gamma_n^e = (n+1/2)\gamma_{n+1}^e - 2n\gamma_n^e + (n-1/2)\gamma_{n-1}^e, \quad n > 0.$$

To solve them, we define the generating function

$$\hat{\gamma}^P(w) = \sum_{n=0}^\infty \gamma_n^P w^n, \tag{B2}$$

where $P = o, e$. Consider first the odd sector. Multiplying the equation for γ_n^0 by w^n and summing gives

$$(1-w)^2 \frac{d\hat{\gamma}^o}{dw} = (1 - w + \omega)\hat{\gamma}^o. \tag{B3}$$

This is easily solved by separation of variables, to give

$$\hat{\gamma}^o = \frac{C}{1-w} \exp\left[\frac{\omega}{1-w}\right], \tag{B4}$$

where C is an arbitrary constant. Not the strong divergence at $w = 1$. This implies unacceptable behavior for γ_n^o at large n.

The even sector is (fortunately) rather more complicated. The crucial difference is the fact that Eq. (4.6) is valid only for $n > 0$, leaving an extra free parameter. Carrying out the transform in this case gives

$$(1-w)^2 \frac{d\hat{\gamma}^e}{dw} + \left[\frac{1}{2}\left(w - \frac{1}{w} \right) - \omega \right] \hat{\gamma}^e = -\left[\frac{1}{2}\left(\frac{1}{w} - \gamma_1^e \right) + \omega \right],$$
(B5)

where we have imposed the normalization $\gamma_0^e = 1$. Note the appearance of γ_1^e as a parameter in the equation. It must be adjusted to achieve a well-behaved solution.

This inhomogenous equation can be solved by introducing the integrating factor $(1-w)/\sqrt{w}\exp[-\omega/(1-w)]$. The solution is

$$\frac{1-w}{\sqrt{w}} e^{-\omega/(1-w)} \hat{\gamma}^e(w) = -\int^w \frac{dy}{\sqrt{y(1-y)}} \left[\frac{1-y}{2y} + \frac{\delta}{2} + \omega \right]$$
$$\times e^{-\omega/(1-y)},$$
(B6)

where $\delta = 1 - \gamma_1^e$. Performing an integration by parts leads to the form

$$\hat{\gamma}^e(w) = \frac{1}{1-w} [1 + \sqrt{w} e^{\omega/(1-w)} J(w)],$$
(B7)

where

$$J(w) = \int_0^w \frac{dy}{\sqrt{y(1-y)}} \left[\frac{\omega}{1-y} - \frac{\delta}{2} - \omega \right] e^{-\omega/(1-y)}.$$
(B8)

To avoid the strong divergence of γ_n^e, we clearly need

$$J(1) = 0.$$
(B9)

This fixes δ. This implies that

$$J(w) = -\int_w^1 \frac{dy}{\sqrt{y(1-y)}} \left[\frac{\omega}{1-y} - \frac{\delta}{2} - \omega \right] e^{-\omega/(1-y)}.$$
(B10)

We now change variables via $y = 1 - \omega/t$, and also define $s = (1-w)/\omega$. Then

$$J = -\int_{1/st}^{\infty} \frac{dt}{\sqrt{1-\omega/t}} \left[t - \frac{\delta}{2} - \omega \right] e^{-t}.$$
(B11)

We may now take the limit $\omega \rightarrow 0$, with s fixed, and $\delta \gg \omega$. Then

$$J(s) \rightarrow \frac{\delta}{2} E_1(1/s) - e^{-1/s},$$
(B12)

where

$$E_1(x) = \int_x^{\infty} dt e^{-t}/t$$
(B13)

is the exponential-integral function. Plugging back in gives, finally,

$$\hat{\gamma}^e(s) = \frac{\delta}{2\omega s} e^{1/s} E_1(1/s) = \frac{\delta}{2\omega} \int_0^{\infty} dt \frac{e^{-t}}{1+ts}.$$
(B14)

In fact, $\hat{\gamma}^e(s)$ is nothing but the Laplace transform in the limit $\omega \ll 1$,

$$\hat{\gamma}^e(s) = \sum_n \gamma_n^e w^n = \sum_n \gamma_n^e e^{-n\omega s} \rightarrow \int dn \gamma^e(n) e^{-n\omega s}$$
$$= [L\gamma^e](\omega s).$$
(B15)

We can therefore invert it using the inversion formula,

$$\gamma_n^e = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{ns} L\gamma^e(s) = \omega \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{n\omega s} [L\gamma^e](\omega s)$$
$$= \frac{\delta}{2} \int_0^{\infty} dt e^{-t} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{e^{n\omega s}}{1+ts} = \frac{\delta}{2} \int_0^{\infty} dt \frac{1}{t} e^{-n\omega/t-t}$$
$$= \frac{\delta}{2} \int_0^{\infty} \frac{dt}{t} e^{-nt-\omega/t}.$$
(B16)

This is precisely the modified Bessel function solution obtained from the continuum limit (see Appendix E).

APPENDIX C

In this appendix we obtain the density of states without resorting to the hard-wall approximation, by solving exactly the full continuum equation, Eq. (4.14). This can be accomplished by employing an inverse Laplace transform, defining

$$\phi(n,M) = \int_0^{\infty} dt e^{-nt} \tilde{\phi}(t).$$
(C1)

Provided $t^2 \tilde{\phi}(t) \rightarrow 0$ as $t \rightarrow 0$ (as required for a well-behaved solution as $n \rightarrow \infty$), insertion into Eq. (4.14) leads to the simple transformed form,

$$t^2 \frac{d\tilde{\phi}}{dt} + (1-M)t\tilde{\phi} = \frac{\omega}{2} \tilde{\phi}.$$
(C2)

This has the general solution,

$$\tilde{\phi}(t) = \frac{a}{t^{1-M}} e^{-\omega/2t},$$
(C3)

where a is an arbitrary constant. The ''unnormalized'' wave function is thus given by

$$\phi(n,M) = a \int_0^{\infty} \frac{dt}{t^{1-M}} e^{-nt-\omega/2t},$$
(C4)

which is the integral representation of a Bessel function, $\phi(n,M) = 2a(\omega/2n)^{M/2} K_M(\sqrt{2\omega n})$. To evaluate Eqs. (4.15,4.17), we need $\phi(1)$ and $\phi'(1)$. These are determined by making the change of variables $x = t^{-M}$, which yields

$$\phi(1,M) = \frac{a}{M} \int_0^{\infty} dx \exp[-x^{1/M} - (xW^{-1})^{-1/M}],$$
(C5)

with the scaling variable $W = (\omega/2)^M$. In the scaling limit $\omega, M \rightarrow 0$ with W fixed and finite, Eq. (C5) can be simply evaluated. Since each of the arguments in the exponential goes to zero or infinity, the limits of integration are restricted, giving

$$\phi(n=1,M) = \frac{a}{M} (1 - \omega^M),$$
(C6)

(using $2^M \approx 1$). The same change of variables can be used to extract the n derivative, giving in the scaling limit

$$\left. \frac{d\phi}{dn} \right|_{n=1} = -a. \tag{C7}$$

Comparison with the hard-wall forms shows that the constants c_1 and c_2 are identical provided we take $a = 1/z_w = |\ln \omega|^{-1}$.

With the exact solutions of the continuum equations in hand, one can readily evaluate the density of states by inserting the integral representations Eq. (C4), into the expression for $G(i\omega)$ in Eq. (4.25). The n integration can be readily performed. In the scaling limit $M \ll 1$ the remaining two t integrations are simple and yield an identical result to Eqs. (4.26)–(4.27).

APPENDIX D

In this appendix, we obtain the exact excited-state wave functions in the continuum limt in the $N_F = 0,2$, $N_B = -1$ sectors, and show that they lead to the same scaling form for the Green's function as does the hard-wall approximation. Beginning with Eq. (5.7), we make the change of variables,

$$\chi(n,M) = a(\beta) n^{-(1+M)/2} \widetilde{\chi}(n,\beta), \tag{D1}$$

where $a(\beta)$ is a normalization constant to be chosen later in order to maintain the closest possible agreement with the hard-wall solutions in Sec. V. The transformed wave function then satisfies the simpler equation,

$$\left[n^2 \frac{d^2}{dn^2} + n \frac{d}{dn} + \frac{1}{4}(\beta^2 - 2\omega n) \right] \widetilde{\chi}(n), \tag{D2}$$

where $\beta = \sqrt{E - M^2}$. Equation (D2) is a standard equation of classical mathematical physics. Its solutions are modified Bessel functions of imaginary index,

$$\widetilde{\chi}(n) = K_{i\beta}(\sqrt{2n\omega}), \tag{D3}$$

where we have chosen the solution K that decays at infinity. Note that we have assumed $E \geq M^2$, for which β is real. It is straightforward to show that there are no satisfactory solutions with $E < M^2$. Very few results are readily available for these functions at imaginary index. Instead, we will make heavy use of the integral representation,

$$\widetilde{\chi}(n,\beta) = \int_0^\infty e^{-\sqrt{2n\omega}\cosh t} \cos\beta t, \tag{D4}$$

which can be verified by direct substitution into Eq. (D2). A second useful form is obtained by integrating Eq. (D4) by parts:

$$\widetilde{\chi}(n,\beta) = \frac{\sqrt{2n\omega}}{\beta} \int_0^\infty e^{-\sqrt{2n\omega}\cosh t} \sinh t \sin\beta t. \tag{D5}$$

The first task at hand is to determine the spectrum, or allowed values of β. To do this required asymptotic matching for the continuum solution (valid for $n \geq 1$) to the "outer" solution of the difference equation with $\omega = 0$ (valid for $n \ll 1/\omega$). This matching is imposed in the overlap region $1 \ll n \ll 1/\omega$. To study this limit, we let $s = e^t$ in Eq. (D5), which is then dominated by $s \gg 1$. Thus

$$\widetilde{\chi}(n,\beta) \sim \sqrt{n\omega/2} \frac{1}{2i\beta} \int_1^\infty ds (s^{i\beta} - s^{-i\beta}) e^{-\sqrt{n\omega/2}s}. \tag{D6}$$

This gives

$$\chi(n,M) \sim \frac{1}{2i\beta} n^{-(1+M)/2} [(n\omega)^{-i\beta/2} - (n\omega)^{i\beta/2}]. \tag{D7}$$

Comparison to the outer solution, Eq. (5.16),

$$\chi(n,M) \sim n^{-(1+M)/2} [c_+ n^{-i\beta/2} + c_- n^{i\beta/2}], \tag{D8}$$

then gives, as in Sec. V,

$$\beta_k = \pi k/z_w, \tag{D9}$$

$$c_3 = (-1)^{k+1} c_4 \equiv c. \tag{D10}$$

We must next determine the constants c and $a(\beta)$. Normalization requires

$$2|C_k|^2 |a(\beta)|^2 I_\beta = 1, \tag{D11}$$

where

$$I_\beta = \int_1^\infty \frac{dn}{n} |\widetilde{\chi}(n,\beta)|^2. \tag{D12}$$

Performing the integral over n gives

$$I_\beta = \frac{2}{\beta^2} \int_0^\infty dt\, dt' \sin\beta t \sin\beta t' \frac{\sinh t \sinh t'}{(\cosh t + \cosh t')^2} [1 + \sqrt{2\omega}$$
$$\times (\cosh t + \cosh t')] e^{-\sqrt{2\omega}(\cosh t + \cosh t')}. \tag{D13}$$

The next step is to rescale the parameter $t \to t/\beta$, $t' \to r'/\beta$ to give

$$I_\beta = \frac{1}{2\beta^4} \int_0^\infty dt\, dt' \sin t \sin t' \operatorname{sech}^2\left(\frac{t-t'}{2\beta}\right) [1 + \sqrt{2\omega}(\cosh t/\beta$$
$$+ \cosh t'/\beta)] e^{-\sqrt{2\omega}(\cosh t/\beta + \cosh t'/\beta)}. \tag{D14}$$

We are interested in small ω, with $\beta = \pi k/|\ln(\omega)|$. In this limit,

$$\cosh t/\beta = \frac{1}{2}(e^{t/\beta} + e^{-t/\beta}) \approx \frac{1}{2}\omega^{-t/\pi k}, \tag{D15}$$

and an identical result with $t \to t'$. The factor in the exponential in Eq. (D14) therefore becomes

$$\sqrt{2\omega}(\cosh t + \cosh t') \underset{\omega \to 0}{\longrightarrow} \begin{cases} 0, & 0 < t, t' < \pi k/2 \\ \infty, & \text{otherwise.} \end{cases} \tag{D16}$$

Taking this limit therefore acts simply to restrict the limits of integration, and we have

$$I_\beta = \frac{1}{2\beta^4} \int_0^{\pi k/2} dt\, dt' \sin t \sin t' \operatorname{sech}^2\left(\frac{t-t'}{2\beta}\right). \tag{D17}$$

Since the sech is sharply peaked around zero in the $\beta \to 0$ limit, we may effectively set $t' \approx t$ in the second sine to obtain

$$I_\beta = \frac{1}{2\beta^4} \int_0^{\pi k/2} dt \sin^2 t \int_0^{\pi k/2} dt' \, \text{sech}^2 \left(\frac{t - t'}{2\beta} \right). \quad (D18)$$

The t' integral is clearly proportional to β, and performing these integrations exactly gives, in the $\beta \to 0$ limit,

$$I_\beta = \frac{\pi k}{2\beta^3}. \quad (D19)$$

Thus, by choosing

$$a(\beta) = \beta = \frac{\pi k}{|\ln \omega|}, \quad (D20)$$

we obtain the same constant $c = z_w^{-1/2}$ as found for the hard-wall solutions in Sec. V.

We are now in a position to calculate the Fermion Green's function. As in Sec. V, to use the decomposition in Eq. (5.2), we must calculate matrix elements of single-fermion operators between the ground and excited states. In general, using Eqs. (4.13, 5.6),

$$_L\langle k | F_\beta^\dagger | 0 \rangle_R \sim \frac{c_1}{\sqrt{2z_w}} \int_1^\infty \frac{dn}{\sqrt{n}} [(-1)^\beta \phi(n, M) \chi(n, -M)$$
$$+ (-1)^{k+1} \phi(n, -M) \chi(n, M)] \quad (D21)$$

$$= \frac{c_1}{\sqrt{2z_w}} [(-1)^\beta I(M) + (-1)^{k+1} I(-M)], \quad (D22)$$

where the integral

$$I(M) = \frac{\pi k}{z_w} \int_1^\infty \frac{dn}{\sqrt{n}} \phi(n, M) \widetilde{\chi}(n, \beta) n^{M/2}. \quad (D23)$$

To evaluate this integral, we let $t \to e^t$ in the integral representation, Eq. (C4), giving

$$\phi(n, M) = c_1 \frac{\omega}{n}^{M/2} \int_{-\infty}^\infty dt \, e^{-\sqrt{2n\omega}\cosh t} e^{Mt}. \quad (D24)$$

Inserting this and Eq. (D4) above, the n integration can be readily performed, giving

$$I(M) = \frac{\pi k \omega^{M/2}}{\sqrt{2\omega} z_w^2} \int_{-\infty}^\infty dt_1 dt_2 \frac{e^{Mt_1} \cos \beta t_2}{\cosh t_1 + \cosh t_2}. \quad (D25)$$

The limits $M, \beta \to 0$ can be safely taken in the numerator of the integral. The final result is

$$I(M) = \frac{\pi^3 k \omega^{-(1-M)/2}}{\sqrt{2} z_w^2}. \quad (D26)$$

Putting this into Eq. (D22) above and thence into Eq. (5.2), one recovers the final result, Eqs. (5.31)–(5.35) quoted in Sec. V, with a different value, $A = \pi^6/4$, for the nonuniversal constant.

APPENDIX E

In this appendix, we solve the difference equation for $\mathcal{J}^x$ eigenstates, Eq. (5.14), in the appropriate sector for the fermion Green's function. Consider the generating function,

$$\hat{\psi}(w, \alpha) = \sum_{n=0}^\infty \psi_n(\alpha) w^n. \quad (E1)$$

Multiplying Eq. (5.14) by w^n and summing gives

$$(1 - w^2) \frac{d}{dw} \hat{\psi}(w, \alpha) - w \hat{\psi}(w, \alpha) = -2i\alpha \hat{\psi}(w, \alpha). \quad (E2)$$

This is easily solved by separation of variables. One finds

$$\hat{\psi}(w, \alpha) = (1 + w)^{-1/2 - i\alpha} (1 - w)^{-1/2 + i\alpha}, \quad (E3)$$

choosing $\psi_0(\alpha) = 1$ to fix the overall constant. This can be inverted using the contour integral

$$\psi_n(\alpha) = \oint \frac{dw}{2\pi i} \frac{\hat{\psi}(w, \alpha)}{w^{1+n}}. \quad (E4)$$

Deforming the contour to obtain a real integral gives

$$\psi_n(\alpha) = \frac{\cosh \pi \alpha}{\pi} \int_1^\infty \frac{dw}{w^{1+n}} [(w + 1)^{-1/2 - i\alpha} (w - 1)^{-1/2 + i\alpha}$$
$$+ (-1)^n (w + 1)^{-1/2 + i\alpha} (w - 1)^{-1/2 - i\alpha}]. \quad (E5)$$

For large $n \gg 1$, this integral is dominated by $w \approx 1$, and can be easily evaluated to give the result quoted in Eq. (5.15).

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