Mean-field theory of the transverse-field Ising spin glass in the classical limit

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An infinite-range transverse-field Ising spin glass is studied in the classical limit. At the $T=0$ phase transition the nonlinear susceptibility diverges as a function of the field with an exponent of 1/2 which is equal to the quantum estimates but without a multiplicative logarithmic correction. The replica-symmetric solution fails at $T=0$ below the critical value of the transverse field. Equations that break the replica symmetry are constructed and are shown to be harder to solve than in the absence of the transverse field. In the absence of the transverse field their solution agrees with that of Sommers and Dupont [J. Phys. C **17**, 5785 (1984)]. [S0163-1829(97)06541-7]

I. INTRODUCTION

The transverse-field Ising model, uniform or exchange disordered, undergoes two different phase transitions: thermal and at zero temperature $(T=0)$.[1] The latter takes place at $\Gamma=\Gamma_c$, where Γ is the transverse field and Γ_c is its critical value. The thermal transition is driven by thermal fluctuations and is best understood for $\Gamma=0$. A mean-field account of these fluctuations becomes exact above an upper critical dimensionality D_c. The $T=0$ transition is driven by the ground-state reconstruction and it is affected by quantum zero-point fluctuations. The renewed interest in quantum spin glasses was stimulated by recent experiments[2] on the dipolar Ising magnet $Li_xHo_{1-x}YF_4$, where T_c was varied down to zero by the application of a transverse magnetic field Γ. Therefore it became possible to study the quantum transition by simply tuning the external field.

The $T=0$ transition in quantum spin glasses has been studied in the mean-field approximation.[3] In particular, Miller and Huse[4] have found that the nonlinear susceptibility, $\chi_{\rm nl}$, diverges with an exponent $\tilde{\gamma}=1/2$ at Γ_c which is equal to twice the characteristic value of the exchange couplings (their dispersion). In a related study of the quantum XY spin glass in transverse field, Pazmandi and Domanski[5] have also found $\tilde{\gamma}=1/2$ at the $T=0$ transition.

In this paper, we consider the classical limit of the transverse Ising spin-glass model and study it within the mean-field approximation which should be appropriate for a sufficiently large dimensionality D. By comparing our results to those obtained for the quantum models we demonstrate that the $T=0$ transition is classical in the mean-field limit. Not only $\tilde{\gamma}$ is equal to 1/2 but also Γ_c takes on the classical value. In finite dimensionalities, the quantum fluctuations become relevant. The three-dimensional (3D) Monte Carlo simulations of the quantum spin glasses by Guo, Bhatt, and Huse[6] yield $\tilde{\gamma}\approx 2.8$ which is close to the $\tilde{\gamma}$ of 2.9 for the thermal transition.[7] On the other hand, for $D=2$, Rieger and Young[8] get $\tilde{\gamma}\approx 4.5$. Both of these values are significantly larger than 1/2. Read, Sachdev and Ye[9] have demonstrated that above $D=8$ both of the quantum transverse Ising model and the quantum rotor model are governed by the Gaussian fixed point of the replica theory.

The quantum mean-field result[4] has logarithmic corrections to the power-law behavior of $\chi_{\rm nl}$. Such corrections can be traced as being due to an extra approximation regarding integration over the Matsubara frequency—a feature which will be shown here to disappear in the purely classical treatment.

In the next section of this paper, we discuss the replica-symmetric solution of the mean-field equations for the classical transverse field Ising model. We extend the work of Pirc, Prelovšek, and Tadić[10] to the calculation of the nonlinear susceptibility and demonstrate that the replica-symmetric treatment is inappropriate for $\Gamma<\Gamma_c$: the nonlinear susceptibility is diverging with twice as big an exponent than above Γ_c. It should be noted that the question about the nature of the spin-glass phase at $T=0$ is very subtle. The authors of Ref. 3 and 11 claimed that the replica symmetry is broken in the whole spin-glass phase. Note that no direct result for the $T=0$ case has been presented in Ref. 3. Recently Read, Sachdev, and Ye[9] have demonstrated that the replica symmetry is broken anywhere except at $T=0$. We show in this paper that the replica symmetry is in fact broken at $T=0$ for $\Gamma<\Gamma_c$. The difference between our results and theirs may be due to the fact that we consider the infinite range interaction whereas the interaction in Ref. 9 is short range.

In Sec. III, we construct a mean-field theory that breaks the symmetry of the replicas and attempt to calculate the susceptibility numerically. Unfortunately, the integration stability requires a lot of discretization steps which are beyond our facilities. We demonstrate, however, that our approach agrees with that of Sommers and Dupont[12] in the limit of zero transverse field.

II. REPLICA-SYMMETRIC APPROACH

The transverse-field Ising model is described by the Hamiltonian given by

$$\mathcal{H}=-\sum_{i\neq j} J_{ij}S_i^zS_j^z-\Gamma\sum_i S_i^x+H\sum_i S_i^z. \tag{1}$$

In the classical limit, S_i^z and S_i^x become components of a classical unit vector such that $(S_i^z)^2+(S_i^x)^2=1$.

We will study model (1) in the mean-field approximation. Standard methods yield[7,13] the replica-symmetric equations for magnetization m, Edwards-Anderson parameter q, and quadrupolar parameter in the following form:

$$m=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dy\ \exp(-y^2/2)\ \frac{C_1(q,p)}{C_0(q,p)}, \tag{2}$$

$$q=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dy\ \exp(-y^2/2)\left[\frac{C_1(q,p)}{C_0(q,p)}\right]^2, \tag{3}$$

and

$$p=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dy\ \exp(-y^2/2)\ \frac{C_2(q,p)}{C_0(q,p)}, \tag{4}$$

where

$$C_n(q,p)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\ \cos^n\varphi\ \exp\left[\beta(H+Jq^{1/2}y)\cos\varphi + \beta\Gamma\ \sin\varphi+\beta^2J^2\ \frac{(p-q)}{2}\cos^2\varphi\right]. \tag{5}$$

Here H denotes an external field, $J^2=\Sigma_j[J_{ij}^2]_{\rm av}$, $\beta=1/k_BT$, and $[\cdots]_{\rm av}$ denotes an average over random couplings J_{ij}. For simplicity we consider units in which $J=1$. According to Pirc, Prelovšek, and Tadić[10] the susceptibility χ is given by $\chi=\beta(p-q)$ and we expect that it is finite at zero temperature. The integral expression for $C_n(q,p)$, Eq. (5), can be reduced to the following form:

$$C_n(q,p)=\frac{(-1)^n}{\pi}\int_{-1}^{1}\frac{dS}{\sqrt{1-S^2}}\ S^n\exp[\beta g(S)], \tag{6}$$

where

$$g(S)=\beta\left[xS+\frac{\chi}{2}\ S^2+\frac{1}{\beta}\ \ln\cosh(\beta\Gamma\sqrt{1-S^2})\right] \tag{7}$$

with $x=H+q^{1/2}y$.

For large $\beta(T\to 0)$ the integral (6) can be calculated by the steepest descent method. Thus one obtains

$$C_n(q,p)\approx\frac{(-1)^n}{\pi}\ \frac{1}{\sqrt{1-S_0^2}}\ S_0^n\exp[\beta g(S_0)], \tag{8}$$

where S_0 is the value of S at which $g(S)$ has a maximum. The necessary condition for the maximum $g(S)$ is

$$g'(S)=-x+S\chi-\frac{\Gamma S}{\sqrt{1-S^2}}=0. \tag{9}$$

For $x=0$ Eq. (9) has two solutions: $S_0=0$ and $\sqrt{1-\Gamma^2/\chi^2}$ when $\Gamma>\chi$ and $\Gamma<\chi$, respectively. We are interested in the case $\Gamma>\chi$, which is true near the transition point.

In order to investigate critical properties of the system at $T=0$ we assume that the transverse field Γ is very close to its critical value Γ_c and that $H\ll 1$. Under these conditions, x is a small parameter and we can solve Eq. (9) perturbatively with respect to x. Up to third order we have

$$S_0(x)=\frac{-x}{\Gamma-\chi}+\frac{x^3\Gamma}{2(\Gamma-\chi)^4}+\cdots\ . \tag{10}$$

On substituting this form of $S_0(x)$ into Eq. (9) and then to Eqs. (2)–(4) one obtains

$$m=\frac{H}{\Gamma-\chi}-\frac{3}{2}\ \frac{\Gamma qH}{(\Gamma-\chi)^4}-\frac{\Gamma H^3}{2(\Gamma-\chi)^4} \tag{11}$$

and

$$q=p=\frac{(H^2+q)}{(\Gamma-\chi)^2}-\frac{\Gamma}{(\Gamma-\chi)^5}\ (H^4+6H^2q+3q^2). \tag{12}$$

For a small H we introduce the following definitions:

$$m=\chi H+\chi_{\rm nl}H^3+\mathcal{O}(H^5), \tag{13}$$

where χ and $\chi_{\rm nl}$ are the linear and nonlinear susceptibilities, respectively, and

$$q=q_0+q_2H^2+\mathcal{O}(H^4). \tag{14}$$

By expanding the right-hand sides of Eqs. (11) and (12) up to the third- and second-order terms in H, respectively, we get the following equations:

$$\chi=\frac{1}{\Gamma-\chi}-\frac{3\Gamma q_0}{2(\Gamma-\chi)^4}, \tag{15a}$$

$$q_0=\frac{q_0}{(\Gamma-\chi)^2}-\frac{3\Gamma q_0^2}{(\Gamma-\chi)^5}, \tag{15b}$$

$$q_2=\frac{1}{(\Gamma-\chi)^2}-\frac{6\Gamma q_0}{(\Gamma-\chi)^5}+\frac{q_2}{(\Gamma-\chi)^2}\left[1-\frac{6\Gamma q_0}{(\Gamma-\chi)^3}\right] \tag{15c}$$

and

$$\chi_{\rm nl}=-\frac{3\Gamma q_2}{2(\Gamma-\chi)^4}. \tag{15d}$$

Now we will solve Eqs. (15a)–(15d) for the paramagnetic phase with $q_0=0$ as $\Gamma\to\Gamma_c$. For the linear susceptibility, in agreement with Ref. 10, we get

$$\chi=\frac{\Gamma-\sqrt{\Gamma^2-4}}{2}\approx 1-(\Gamma-\Gamma_c)^{1/2}+\mathcal{O}(\Gamma-\Gamma_c) \tag{16}$$

with $\Gamma_c=2$ (a self-consistent derivation of Γ_c is given in Appendix A). The critical behavior of parameters q_2 and $\chi_{\rm nl}$ as $\Gamma\to\Gamma_c^+$ is given by

$$q_2=\frac{1}{2}\ (\Gamma-\Gamma_c)^{-1/2} \tag{17}$$

and

$$\chi_{\rm nl}=-\frac{3}{2}(\Gamma-\Gamma_c)^{-1/2}. \tag{18}$$

So $\tilde{\gamma}=1/2$. This result for $\chi_{\rm nl}$ is different and the critical exponent coincides with the quantum result except for the lack of any logarithmic correction, as discussed in the Introduction. Note that the nonlinear susceptibility has the negative divergence as in the quantum case.[6,8]

We now apply the replica-symmetric theory to the spin-glass phase, i.e., when $\Gamma<\Gamma_c$. In the spin-glass phase the nonzero solution of Eq. (15b) is

$$q_0=\frac{1}{3\Gamma}(\Gamma-\chi)^3[1-(\Gamma-\chi)^2]. \tag{19}$$

On substituting Eq. (19) into Eq. (15a) we get

$$\chi=\frac{2\Gamma-\sqrt{\Gamma^2-3}}{3}\approx 1-\frac{1}{6}(\Gamma_c-\Gamma)^2 \tag{20}$$

as $\Gamma\to\Gamma_c^-$. With the help of Eqs. (19) and (20) we get

$$q_0\approx\frac{\Gamma_c-\Gamma}{3} \tag{21}$$

when $\Gamma\to\Gamma_c^-$. This result coincides with the one obtained in Ref. 10. From Eq. (15c) we get, for $\Gamma\to\Gamma_c^-$,

$$q_2=\frac{1}{2(\Gamma_c-\Gamma)}. \tag{22}$$

By taking into account Eqs. (22) and (15d) we obtain

$$\chi_{\rm nl}\approx-\frac{3}{2}\frac{1}{\Gamma_c-\Gamma}. \tag{23}$$

From Eq. (23) it is seen that $\tilde{\gamma}=1$ as $\Gamma\to\Gamma_c^-$ which is different from $\tilde{\gamma}=1/2$ from above Γ_c. The fact that $\tilde{\gamma}$ below Γ_c becomes larger than above Γ_c itself signifies that the replica-symmetry solution is not adequate and one has to apply the replica-symmetry-breaking theory. Before doing this, we study the stability of the replica-symmetry (RS) solution in more detail using the de Almeida–Thouless approach.[14]

Following Ref. 13 we consider the following quantity:

$$\lambda=P-2Q+R. \tag{24}$$

The definitions of P, Q, and R are given in Appendix B. If $\lambda>0$ then the RS solution is stable. Otherwise it is unstable. The case of $\lambda=0$ corresponds to the de Almeida–Thouless line. After tedious calculations (see Appendix B) we have

$$\lambda=1-\beta^2\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2)[C_2/C_0-(C_1/C_0)^2]^2. \tag{25}$$

If we set $\lambda=0$ then we have the de Almeida–Thouless line.

Using the definition of C_n from Eq. (5), λ has the following simple form (see Appendix B):

$$\lambda=1-(\Gamma-\chi)^{-2}. \tag{26}$$

Now we consider λ close to Γ_c. For $\Gamma\to\Gamma_c^+$ we have

$$\Gamma-\chi\approx 1+(\Gamma-\Gamma_c)^{1/2},$$
$$\lambda\approx 2(\Gamma-\Gamma_c)^{1/2}>0 \tag{27}$$

and the replica-symmetry solution is, as expected, stable above Γ_c.

For $\Gamma\to\Gamma_c^-$ simple calculations give

$$\Gamma-\chi\approx 1-(\Gamma_c-\Gamma),$$
$$\lambda\approx-2(\Gamma_c-\Gamma)<0. \tag{28}$$

Therefore the replica-symmetry solution is unstable below Γ_c. In agreement with the result of Ref. 11, this conclusion is also valid for the corresponding quantum model. The proof in the quantum case is, however, much more involved.

III. REPLICA-SYMMETRY-BREAKING SOLUTION

Below the de Almeida–Thouless line the replica-symmetry solution is no longer valid. Then the nonergodic phase should be characterized by the Parisi order-parameter function $q(x)$. Using the Parisi ansatz,[7,13,15] the Duplantier method,[16] and the Sommers-Dupont variational procedure[12] one can obtain the following equations for $q(x)$ and magnetization m:

$$m=\int_{-\infty}^{\infty}P(0,y)m(0,y)dy, \tag{29}$$

$$q(x)=\int_{-\infty}^{\infty}P(x,y)(m(x,y))^2dy. \tag{30}$$

The distribution function $P(x,y)$ and $m(x,y)$ obey the following equations:

$$\dot{P}(x,y)=\frac{\dot{q}(x,y)}{2}P''(x,y)+\dot{\Delta}(x)(m(x,y)P(x,y))' \tag{31}$$

with the boundary

$$P(0,y)=(2\pi q(0))^{-1/2}\exp[-(y-H)^2/2q(0)] \tag{32}$$

and

$$\dot{m}(x,y)=\dot{q}(x)m''(x,y)+\dot{\Delta}(x)m(x,y)m'(x,y) \tag{33}$$

with the initial condition

$$m(1,y)=\frac{D_1(y)}{D_0(y)}. \tag{34}$$

Here a dot and a prime denotes differentiation with respect to x and y, respectively. Function $D_n(y)$ is as follows:

$$D_n(y)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,\cos^n\varphi\,\exp\left[\frac{y}{T}\cos\varphi+\frac{[p-q(1)]}{2T^2}\times\cos^2\varphi+\frac{\Gamma}{T}\sin\varphi\right]. \tag{35}$$

We have also obtained the equations for gauge function $\Delta(x)$ and quadrupolar parameter p

$$\frac{p-q(1)}{T}+\Delta(x)=\int dy\ P(x,y)m'(x,y), \tag{36}$$

$$p=\int dy\ P(1,y)\ \frac{D_2(y)}{D_0(y)}. \tag{37}$$

To solve the diffusionlike equations (31) and (33) numerically we transform them into the following integral equations:[17,18]

$$m(x,y)=\int_{-\infty}^{\infty} G(x,y;1,y')m(1,y')dy' - \int_x^1 dx'\dot{\Delta}(x')\int_{-\infty}^{\infty} dy' G(x,y;x',y') \times m(x',y')m'(x',y'), \tag{38}$$

$$P(x,y)=(2\pi q(x))^{-1/2}\exp[-(y-H)^2/2q(x)] + \int_0^x \dot{\Delta}(x')\int_{-\infty}^{\infty} dy' G(x',y';x,y) \times (m(x',y')P(x',y'))', \tag{39}$$

where the Green function is given by

$$G(x,y;x',y')=[2\pi(q(x')-q(x))]^{-1/2} \times \exp[-(y-y')^2/2(q(x')-q(x))]. \tag{40}$$

We now can solve Eqs. (38), (39), (36), and (30) iteratively. Note that because $\Delta(x=1)=0$ we can use Eq. (36) to find χ (at $x=1$) and $\Delta(x)$ (at $x\neq 0$). At $T=0$ we have to use

$$\chi=\frac{p-q(1)}{T} \tag{41}$$

and the boundary value $m(1,y)$ in the following way. The form of Eq. (34) suggests that $m(1,y)$ may be obtained by the steepest descent method and $m(1,y)=S_0$, where at S_0 the function

$$A(S)=yS+\chi S^2+\Gamma(1-S^2)^{1/2} \tag{42}$$

has a maximum.

When $P(x,y)$ and $m(x,y)$ are found we can use Eq. (29) to obtain the magnetization and therefore the nonlinear susceptibility

$$\chi_{\rm nl}=[m(2\Delta H)-2m(\Delta H)+2m(-\Delta H) -m(-2\Delta H)]/(2\Delta H)^3. \tag{43}$$

The iterative procedure is to take place in the order: Eq. (38) → Eq. (39) → Eq. (29) → Eq. (30) → Eq. (36) → Eq. (38). In order to perform the iterations numerically, we discretized the variables x and y, dividing the interval $[0,1]$ into N_x pieces and the interval $[-5,5]$ into N_y pieces [y is chosen in this interval because $P(x,y)$ should be very small for larger $|y|$], respectively. The convergence of the numerical iteration is monitored by the maximum variance of all of the variables m, q, P, and Δ. In order to check the numerical algorithm we have checked the results obtained by Sommers and Dupont[12] for the standard Ising spin glass for $T=0$ and $H=0$ (here the replica-symmetry-breaking equations are similar to ours but with different boundary conditions). If we use the same model gauge $\Delta(x)=1-x$ as in Ref. 12, then we can reproduce their results as shown in the inset of Fig. 1. We can show that even in the Ising spin-glass case (without the transverse field) it is much harder to solve the replica-symmetry equations at $T=0$ without assuming an analytic form of the gauge. It may be seen in Fig. 1, where the maximal variance of all of the variables m, q, P, and Δ is plotted versus the iteration steps.

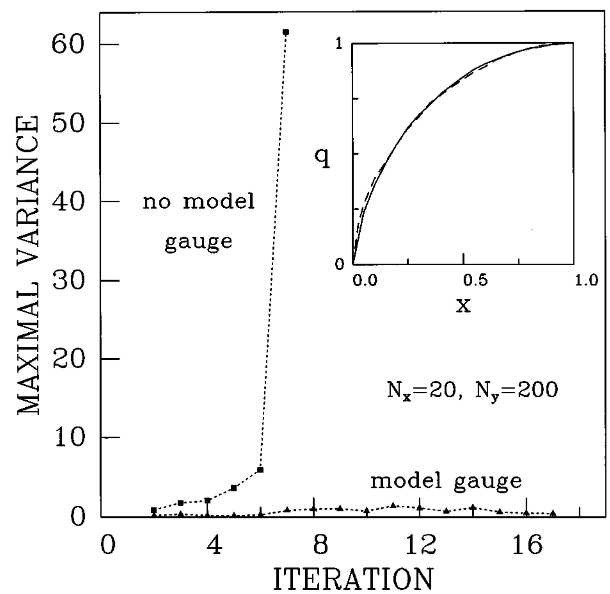

FIG. 1. The dependence of the maximum variance of all of the variables m, q, P, and Δ on iteration steps for the standard Ising spin glass at $T=0$ and $H=0$ ($N_x=20$, $N_y=200$). The choice of the gauge $\Delta(x)=1-x$ guarantees much better stability of the numerical procedure. The inset shows the dependence of the Parisi function $q(x)$ on x. The solid line corresponds to our results obtained for $N_x=20$. The results of Sommers and Dupont (Ref. 12) (dashed line) correspond to $N_x=160$.

The difficulty mentioned above remains in the transverse-field case where one could not fix the gauge $\Delta(x)$ but solve the equations self-consistently at $T=0$. We can show that even the choice of $N_x=80$ and $N_y=800$ does not guarantee the stability of the numerical procedure. Figure 2 shows the dependence of the maximum variance of all of the variables m, q, P, and Δ on iteration steps for several different values of N_x and N_y. The Hamiltonian parameters are chosen to be $\Gamma=1.8$ and $H=0.1$. The minimal value of the maximum variance is about 0.02 for $N_x=80$ and $N_y=800$. Further iteration leads to the instability (the situation does not change if one chooses other sets of parameters). Unfortunately, due to limitation of computational facilities we could not go to larger values of N_x and N_y. We expect, however, that the correct solution of the replica-symmetry-breaking equations would give the nonlinear susceptibility with the exponent of 1/2 below the transition line.

We have also considered the classical limit of the transverse field XY model. Similar to the Ising case, the critical field at $T=0$ is found to be equal to $\Gamma_c=2$ and the nonlinear

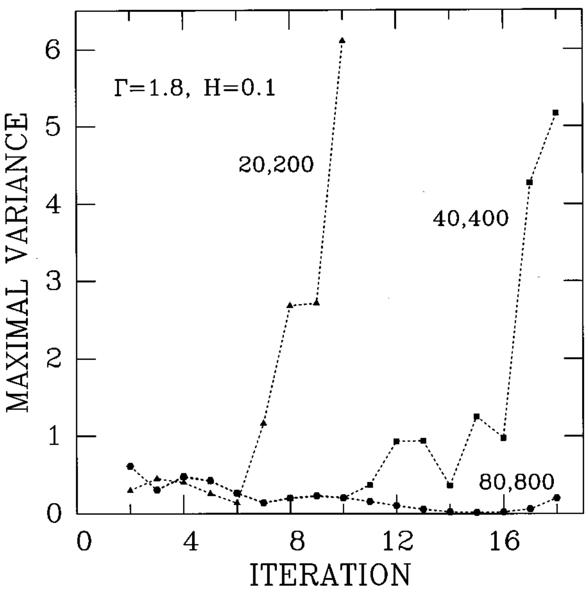

FIG. 2. The dependence of maximum variance on iteration steps. The values of N_x and N_y are shown next to the curves. We chose $\Gamma=1.8$ and $H=0.1$.

susceptibility exponent $\tilde{\gamma}=1/2$. Our results agree with those of Pazmandi and Domanski.[5] It would be interesting to study the classical limit of the quantum models in finite dimensions to explore the role of quantum fluctuations.

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APPENDIX A

The critical value of Γ at $T=0$ can be calculated by taking into account the form of linear susceptibility χ (16). It is seen that χ is real as $\Gamma>\Gamma_c=2$. However there exists an alternative method to calculate Γ_c. We start from the following relation for the transition point valid at any temperature:

$$T_c=\frac{C_2(0,p)}{C_0(0,p)} \tag{A1}$$

and

$$p_c=T_c, \tag{A2}$$

where $C_n(q,p)$ is defined by Eq. (5). By plugging Eq. (A2) into the definition of $C_n(0,p)$ one obtains

$$C_n(0,p)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\ \cos^n\varphi \times\exp\left[\beta_c\left(\Gamma_c\sin\varphi+\frac{\cos^2\varphi}{2}\right)\right]. \tag{A3}$$

After the following change of variable: $\cos\varphi=z\sqrt{T_c}$, where T_c is expressed in J unit, we get

$$C_n(0,p)=\frac{(-1)^n}{\pi}T_c^{(n+1)/2}\times\int_{-\sqrt{\beta_c}}^{\sqrt{\beta^c}}\frac{dz}{\sqrt{1-T_cz^2}}z^n \times\exp\left(\frac{1}{2}z^2\right)\cosh(\beta_c\Gamma_c\sqrt{1-T_cz^2}). \tag{A4}$$

For $T_c\to 0$, i.e., large β_c the asymptotic form of $C_n(0,p)$ is

$$C_n(0,p)\approx\frac{(-1)^n}{\pi}T_c^{(n+1)/2}\cosh(\beta_c\Gamma_c)\times\int_{-\infty}^{\infty}dz\ z^n\exp\left[-\frac{(\Gamma_c-1)}{2}z^2\right]. \tag{A5}$$

With the help of Eq. (A5) the condition (A1) takes the form for $\Gamma_c>1$

$$1=\frac{1}{\Gamma_c-1}. \tag{A6}$$

Hence $\Gamma_c=2$.

APPENDIX B

In order to derive Eq. (25) for λ we use the following expression of the free energy per one spin $(H=0)$:

$$f=\lim_{n\to 0}\frac{1}{n}\left\{\frac{\beta}{4}\sum_{\alpha=1}^{n}p_\alpha^2+\frac{\beta}{2}\sum_{\alpha>\alpha'}^{n}q_{\alpha\alpha'}^2-\beta^{-1}\times\ln\mathrm{Tr}\exp\left[\beta\Gamma\sum_{\alpha=1}^{n}S_\alpha^x+\frac{\beta^2}{2}\sum_{\alpha=1}^{n}p_\alpha(S_\alpha^z)^2+\beta^2\sum_{\alpha>\alpha'}^{n}q_{\alpha\alpha'}S_\alpha^zS_{\alpha'}^z\right]\right\}, \tag{B1}$$

where the replica index $\alpha=1,...,n$; p_α and $q_{\alpha\alpha'}$ are quadrupolar and spin-glass parameters, respectively. The symbol Tr in Eq. (B1) means that

$$\mathrm{Tr}\{A\}=\frac{1}{(2\pi)^n}\int_0^{2\pi}\prod_{\alpha=1}^{n}d\phi_\alpha A. \tag{B2}$$

One can show that P, Q, and R in Eq. (24) may be written as follows:

$$P=n\beta^{-1}\partial^2f/\partial q_{\alpha\alpha_1}^2 =1-\beta^2[\langle(S_\alpha^z)^2S_{\alpha_1}^z)^2\rangle-\langle S_\alpha^zS_{\alpha_1}^z\rangle^2], \tag{B3}$$

$$Q=n\beta^{-1}\partial^2f/\partial q_{\alpha\alpha_1}\partial q_{\alpha\alpha_2} =-\beta^2[\langle(S_\alpha^z)^2S_{\alpha_1}^zS_{\alpha_2}^z\rangle-\langle S_\alpha^zS_{\alpha_1}^z\rangle\langle S_\alpha^zS_{\alpha_2}^z\rangle] \tag{B4}$$

with $\alpha_1\neq\alpha_2$, and

$$R=n\beta^{-1}\partial^2f/\partial q_{\alpha\alpha_1}\partial q_{\alpha_2\alpha_3}$$

$$= -\beta^2[\langle S^z_\alpha S^z_{\alpha_1} S^z_{\alpha_2} S^z_{\alpha_3}\rangle - \langle S^z_\alpha S^z_{\alpha_1}\rangle\langle S^z_{\alpha_2} S^z_{\alpha_3}\rangle]. \quad \text{(B5)}$$

In Eqs. (B3)–(B5) the symbol $\langle\cdots\rangle$ denotes averaging within the framework of the replica-symmetry theory with the effective Hamiltonian H_{eff}

$$H_{\text{eff}} = \frac{\chi}{2}\sum_{\alpha=1}^{n}(S^z_\alpha)^2 + \frac{\beta q}{2}\left(\sum_{\alpha=1}^{n} S^z_\alpha\right)^2 + \Gamma\sum_{\alpha=1}^{n} S^x_\alpha. \quad \text{(B6)}$$

After standard transformations we get

$$P = 1 - \beta^2\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2) \times [C_2(q,p)C_0(q,p)]^2 + \beta^2 q^2, \quad \text{(B7)}$$

$$Q = -\beta^2\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2) \times \frac{C_2(q,p)}{C_0(q,p)}[C_1(q,p)/C_0(q,p)]^2 + \beta^2 q^2, \quad \text{(B8)}$$

and

$$R = -\beta^2\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2) \times [C_1(q,p)/C_0(q,p)]^4 + \beta^2 q^2, \quad \text{(B9)}$$

where $C_n(q,p)$ is given by Eq. (5) with $H=0$. Plugging (B7)–(B9) into Eq. (24) we obtain Eq. (25).

Using Eq. (5) (with $H=0$) and changing the variable $\cos\phi=\beta^{-1/2}z$ one has

$$C_n(q,p) = \frac{(-1)^n}{\pi}\beta^{-(n+1)/2}\int_{-\sqrt{\beta}}^{\sqrt{\beta}} dz(1-\beta^{-1}z^2)^{1/2}z^n \times \cosh[\beta\Gamma(1-\beta^{-1}z^2)^{1/2}]\exp(\chi z^2/2 - \sqrt{\beta q}yz). \quad \text{(B10)}$$

In the $\beta\to\infty$ limit $C_n(q,p)$ has the form

$$C_n(q,p) = \frac{(-1)^n}{\pi}\beta^{-(n+1)/2}\cosh(\beta\Gamma) \times \int_{-\infty}^{\infty} dz\, z^n \exp\left(-\frac{\Gamma-\chi}{2}Z^2 - \sqrt{\beta q}yz\right). \quad \text{(B11)}$$

With the help of the last equation we obtain

$$\{C_2(q,p)/C_0(q,p) - [C_1(q,p)/C_0(q,p)]^2\}^2 = \frac{\beta^{-2}}{(\Gamma-\chi)^2}. \quad \text{(B12)}$$

Substituting Eq. (B12) into Eq. (25) we come to the stability condition (26) in the main text.

[1] See, e.g., H. Rieger and A. P. Young, in *Coherent Approaches to Fluctuations*, edited by M. Suzuki and N. Kawashima (World Scientific, Singapore, 1995).
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