Stability of commensurate phases of a planar model with competing interactions

Jair L. Cadorin* and Carlos S. O. Yokoi
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, SP, Brazil
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We study a one-dimensional planar model with competing interactions at zero temperature in the limit of low and high external magnetic fields. The stability of commensurate phases with wave number $q_0=2\pi(P/Q)$, where P/Q is an irreducible fraction, is discussed in the continuum approximation. We show that for small fields $H\rightarrow 0+$, the helical phase with period $Q>4$ has a width proportional to $H^{Q/2}$. Similarly, for high fields $H\rightarrow H_c-$, where H_c is the critical field for the parafan transition, the fan phase with period $Q\geqslant 4$ has a width proportional to $|H-H_c|^{(Q-2)/4}$ if Q is an even number and $|H-H_c|^{(Q-1)/2}$ if Q is an odd number. [S0163-1829(97)05642-7]

I. INTRODUCTION

Since the beginning of the 1960's many materials, most notably rare-earth elements and their compounds, have been found to display a variety of modulated magnetic ordering.[1,2] In particular, helical or spiral structures, which may transform into a fan structure by the application of an external magnetic field, have been observed in Tb, Dy, Ho, and the manganese compounds MnO_2, $MnAu_2$, and MnP.[3] The zero temperature properties of such systems have often been modeled by a one-dimensional classical planar (or XY) spin with competing first- and second-neighbor interactions with energy given by

$$E(\{\theta_n\})=-J_1\sum_n \cos(\theta_{n+1}-\theta_n)-J_2\sum_n \cos(\theta_{n+2}-\theta_n)-H\sum_n \cos\theta_n, \tag{1.1}$$

where θ_n is the angle nth spin makes with the magnetic field.[4–6] For $J_2>0$ there is no competition and the model can be trivially solved. In this work we will consider only the frustrated case $J_2<0$ and the competition parameter

$$\alpha=\frac{J_1}{J_2} \tag{1.2}$$

in the range $-4\leqslant\alpha\leqslant 4$ where modulated structures are favored.

Nagamiya, Nagata, and Kitano[4] investigated the zero temperature properties of the model using an approximate analytic theory based on low- and high-field expansions. Their phase diagram in the $H-\alpha$ plane consists of commensurate helical and fan phases with wave numbers $q=\pi,2\pi/3,\pi/2$, and large regions of incommensurate helical phases at low fields and fan phases at high fields. They assumed that the transitions between commensurate and incommensurate phases, as well as between helical and fan phases, were first order. Later Robinson and Erdös[5] analyzed the model using numerical methods, and obtained a qualitatively similar phase diagram. Carazza, Rastelli, and Tassi[6] investigated the model at finite temperatures using transfer matrix method and at zero temperature by numerical minimization of the energy. They found that the transition between helical and fan phases is continuous for wave numbers $q>\pi/2$ and discontinuous otherwise. Harris, Rastelli, and Tassi[7] investigated the phase locking of the commensurate configurations for the low-temperature and low-field regime of the three-dimensional extension of the model (1.1). Also we mention that the model (1.1) with the inclusion of anisotropy energy was considered by Kitano and Nagamiya[8] and more recently by Sasaki.[9]

In this work we study the lock-in and stability of commensurate helical and fan phases at low and high fields, respectively. The stability limit of commensurate phases is determined using the continuum approximation along the lines of the work of Theodorou and Rice[10] for the Frenkel-Kontorova model[11] and of Bak and von Böhm[12] for the ANNNI[13,14] model.

II. HELICAL PHASES AT LOW FIELDS

At zero field and $J_2<0$ the energy (1.1) is minimized by a ferromagnetic structure for $\alpha\leqslant -4$, antiferromagnetic structure for $\alpha\geqslant 4$, and a perfect helical structure $\theta_n=qn+\phi$, where ϕ is an arbitrary phase and the wave number q is given by

$$q=\cos^{-1}\left(-\frac{\alpha}{4}\right), \tag{2.1}$$

for $-4\leqslant\alpha\leqslant 4$.[4] Therefore a commensurate helical phase has a zero width of stability at zero field, but a small field will tend to pin the structure relative to the field and give it a finite width of stability.

Let us consider a commensurate helical structure of wave number $q_0=2\pi(P/Q)$, where P/Q is an irreducible fraction. We will assume that a small field will in general distort the perfect helical structure found at zero field.[4] An exception is the helical structure $q_0=\pi/4$ which is unstable against a "double spin-flop state"[4,6] consisting of two identical interpenetrating spin-flop states. The distorted helix can be described by

$$\theta_n=q_0n+\phi_n, \tag{2.2}$$

where ϕ_n is periodic with period Q admitting the Fourier expansion

$$\phi_n=\sum_q \phi_q e^{iqn}=\phi_0+\sum_q{}' \phi_q e^{iqn}, \tag{2.3}$$

where the prime in the sum Σ_q' indicates that the harmonic $q=0$ was excluded. With the notations

$$\phi_0=\phi \quad \text{and } \varphi_n=\sum_q{}' \phi_q e^{iqn}, \tag{2.4}$$

we have

$$\theta_n=q_0 n+\phi+\varphi_n . \tag{2.5}$$

In the above expression ϕ is the phase that pins the helical structure relative to the magnetic field, whereas φ_n describes the distortion of the helical structure due to the field. For small fields it is natural to assume that $|\varphi_n|\ll 1$. Within the usual approximation[10,12] that keeps only the main harmonic $q=\pm q_0$ in the expansion (2.4) of φ_n we have,

$$\varphi_n=\phi_{q_0}e^{iq_0 n}+\text{ c.c.}=2A\cos(q_0 n+\alpha), \tag{2.6}$$

where we have written $\phi_{q_0}=Ae^{i\alpha}$.

We observe that the energy (1.1) is given by the real part of

$$\boldsymbol{E}=-\sum_n [J_1 e^{i(\theta_{n+1}-\theta_n)}+J_2 e^{i(\theta_{n+2}-\theta_n)}+He^{i\theta_n}]. \tag{2.7}$$

Inserting θ_n given by Eq. (2.5) into the above expression we may write the expansion

$$\boldsymbol{E}=\sum_{l=0}^{\infty} \boldsymbol{E}_l , \tag{2.8}$$

where

$$\boldsymbol{E}_l=-\frac{i^l}{l!}\left[J_1 e^{iq_0}\sum_n (\varphi_{n+1}-\varphi_n)^l+J_2 e^{i2q_0}\sum_n (\varphi_{n+2}-\varphi_n)^l+He^{i\phi}\sum_n e^{iq_0 n}\varphi_n^l\right]. \tag{2.9}$$

Substituting the Fourier expansion (2.4) for φ_n we find

$$N^{-1}\boldsymbol{E}_l=-\frac{i^l}{l!}\sum_{q_1\ldots q_l}{}' \phi_{q_1}\cdots\phi_{q_l}\left\{\left[J_1 e^{iq_0}\prod_{j=1}^{l}(e^{iq_j}-1)+J_2 e^{i2q_0}\prod_{j=1}^{l}(e^{i2q_j}-1)\right]\Delta\left(\sum_{j=1}^{l} q_j\right)+He^{i\phi}\Delta\left(q_0+\sum_{j=1}^{l} q_j\right)\right\}, \tag{2.10}$$

where N is the number of spins, and the function $\Delta(q)$ is defined by

$$\Delta(q)=\begin{cases}1 & \text{if } q=0(\text{mod}2\pi)\\ 0 & \text{otherwise.}\end{cases} \tag{2.11}$$

Therefore the only terms that contribute to the sum of Eq. (2.10) are the normal (N) terms for which the argument of the function Δ is zero and the umklapp (U) terms for which the argument is a multiple of 2π.

Let us initially consider only the contribution from normal terms. Within the approximation (2.6) which takes into account only the main harmonic we have $q_j=\pm q_0$ and the equation $\Sigma_{j=1}^l q_j=0$ can be satisfied for even l of the form $l=2k$ in $\binom{2k}{k}$ different ways. With $\phi_{q_0}=Ae^{i\alpha}$ we obtain for the real part the result

$$N^{-1}E_{2k}^{(\mathrm{N})}=-\frac{(-1)^k 2^k}{(k!)^2}[J_1\cos q_0(1-\cos q_0)^k+J_2\cos 2q_0(1-\cos 2q_0)^k]A^{2k}. \tag{2.12}$$

Analogously, the equation $q_0+\Sigma_{j=1}^l q_j=0$ can be satisfied for odd l of the form $l=2k+1$ in $\binom{2k+1}{k}$ different ways, and we obtain

$$N^{-1}E_{2k+1}^{(\mathrm{N})}=\frac{(-1)^k}{(k+1)(k!)^2}\sin(\phi-\alpha)HA^{2k+1}. \tag{2.13}$$

Inserting results (2.12) and (2.13) into the real part of (2.8) the normal contribution to the energy is obtained as

$$N^{-1}E^{(\mathrm{N})}=\sum_{l=0}^{\infty} a_l A^l, \tag{2.14}$$

where the coefficients a_l are given by

$$a_{2k}=-\frac{(-1)^k 2^k}{(k!)^2}[J_1\cos q_0(1-\cos q_0)^k+J_2\cos 2q_0(1-\cos 2q_0)^k], \tag{2.15a}$$

$$a_{2k+1}=\frac{(-1)^k}{(k+1)(k!)^2}\sin(\phi-\alpha)H. \tag{2.15b}$$

The first three coefficients a_l are

$$a_0=-\frac{1}{2}J(q_0), \tag{2.16a}$$

$$a_1=\sin(\phi-\alpha)H, \tag{2.16b}$$

$$a_2=[2J(q_0)-J(2q_0)-J(0)]/2, \tag{2.16c}$$

where

$$J(nq)=2J_1\cos nq+2J_2\cos 2nq. \tag{2.17}$$

Keeping terms up to A^2 in Eq. (2.14) and minimizing with respect to A we obtain

$$A=-\frac{a_1}{2a_2}=-\frac{H}{2J(q_0)-J(0)-J(2q_0)}\sin(\phi-\alpha), \tag{2.18}$$

and the normal contribution to the energy correct to H^2 is found to be

$$N^{-1}E^{(\mathrm{N})}=-\frac{1}{2}J(q_0)-\frac{H^2}{2[2J(q_0)-J(0)-J(2q_0)]}\times\sin^2(\phi-\alpha). \tag{2.19}$$

The energy is minimized for $\phi-\alpha$ an odd multiple of $\pi/2$. Choosing $\alpha=\phi+\pi/2$ for the amplitude A in Eq. (2.18) to be positive, we finally have

$$\theta_n=q_0n+\phi-2A\sin(nq_0+\phi). \tag{2.20}$$

These results are in agreement with Nagamiya, Nagata, and Kitano.[4] We observe that the phase ϕ is arbitrary, meaning that there is no pinning of the commensurate phase to the field when only the normal contributions are taken into account.

In order to determine the phase ϕ we have to consider the umklapp contributions to the energy. Since the amplitude A is proportional to H, Eq. (2.10) shows that there are two lowest order umklapp contributions of order H^Q to the energy. The first one comes from the term $l=Q-1$ with $q_j=q_0(j=1,\ldots,Q-1)$, and the second from $l=Q$ with $q_j=q_0$ or $q_j=-q_0$ $(j=1,\ldots,Q)$. Thus the umklapp contribution to the energy in the lowest order is found to be

$$N^{-1}E^{(\mathrm{U})}=(-1)^QV\cos Q\phi, \tag{2.21}$$

where V is given by

$$V=\frac{1}{(Q-1)!}HA^{Q-1}-\frac{(-1)^{Q/2}2^{Q+1}}{Q!}A^Q\left[(-1)^PJ_1\sin^Q\frac{q_0}{2}\cos q_0+J_2\sin^Qq_0\cos 2q_0\right], \tag{2.22}$$

for even Q and

$$V=\frac{1}{(Q-1)!}HA^{Q-1}-\frac{(-1)^{(Q+1)/2}2^{Q+1}}{Q!}A^Q\left[(-1)^PJ_1\sin^Q\frac{q_0}{2}\sin q_0+J_2\sin^Qq_0\sin 2q_0\right], \tag{2.23}$$

for odd Q. It is clear that the value of ϕ which minimizes the umklapp energy (2.21) is an even or odd multiple of π/Q depending on the sign of V. We conclude, therefore, that the phase ϕ in the expression (2.20) for θ_n is fixed in general by the umklapp energy of order H^Q, in agreement with the results of Ref. 7. We observe that for $Q>3$ higher harmonics ϕ_{mq_0} $(m>1)$, of order H^m,[4] which were neglected in the expansion (2.6) may give extra contributions of the same order to the umklapp energy. Thus the expression (2.23) for the pinning potential vanishes for $q_0=2\pi/5$ and $4\pi/5$, but this is due to the neglect of the second harmonic in the expansion (2.20). Taking into account the second harmonic contribution we find a nonzero pinning potential, in agreement with numerical study of these phases.[6] On the other hand, the vanishing of pinning potential (2.23) for the phase $q_0=\pi/3$ is an exact result because in the expansion (2.20) no higher order harmonics are present in this case. It can be shown that this phase is exceptional, presenting a continuous degeneracy of the ground state for arbitrary fields[6] in a way analogous to the ground state of the triangular antiferromagnetic planar model in a field.[15–18] Finally, the pinning potential (2.22) also vanishes for the helical configuration $q_0=\pi/2$, but this result is spurious because the helical configuration is unstable in the presence of a field, being replaced by a ‘‘double spin-flop phase’’ as soon as the field is turned on.[4,5,7] Therefore our result that assumed helical structure cannot be applied to this phase.

We will now study the stability of the helical structures determined above against creation of defects or solitons.[10,12] We will assume that the defect can be described in terms of a smooth variation of the phase ϕ with constant amplitude A given by Eq. (2.18), that is,

$$\theta_n=q_0n+\phi_n+\varphi_n, \tag{2.24}$$

where

$$\varphi_n=-2A\sin(q_0n+\phi_n). \tag{2.25}$$

Substituting θ_n given by Eq. (2.24) in the expression (2.7) for the energy we can write it as an expansion of the form (2.8) with

$$\boldsymbol{E}_l=-\frac{i^l}{l!}\left[J_1e^{iq_0}\sum_n e^{i(\phi_{n+1}-\phi_n)}(\varphi_{n+1}-\varphi_n)^l + J_2e^{i2q_0}\sum_n e^{i(\phi_{n+2}-\phi_n)}(\varphi_{n+2}-\varphi_n)^l + H\sum_n e^{iq_0n+i\phi_n}\varphi_n^l\right]. \tag{2.26}$$

We can again classify the various terms of the expansion into normal and umklapp terms. Let us first consider the normal contribution. For $l=0$ expanding the differences $\phi_{n+1}-\phi_n$ and $\phi_{n+2}-\phi_n$ up to second order, and using $\phi_{n+1}-\phi_n\approx d\phi/dn$ and $\phi_{n+2}-\phi_n\approx 2d\phi/dn$ we obtain

$$E_0^{(\mathrm{N})}=-N(J_1\cos q_0+J_2\cos 2q_0)$$

$$+\int_{-\infty}^{\infty}\left\{[J_1\sin q_0+2J_2\sin 2q_0]\frac{d\phi}{dn}+\frac{1}{2}[J_1\cos q_0+4J_2\cos 2q_0]\left(\frac{d\phi}{dn}\right)^2\right\}dn. \tag{2.27}$$

Notice that the term with the field H gives a negligible contribution because of the strongly oscillating factor e^{iq_0n}. Analogously for $l=1$ and $l=2$ we get

$$N^{-1}E_1^{(\mathrm{N})}=-HA, \tag{2.28a}$$

$$N^{-1}E_2^{(\mathrm{N})}=A^2[2J_1\cos q_0(1-\cos q_0)+2J_2\cos 2q_0(1-\cos 2q_0)]. \tag{2.28b}$$

We observe that there are also terms involving the derivatives of ϕ in the expression of $E_2^{(\mathrm{N})}$, but since these are of order A^2 they can be neglected in comparison with similar terms in $E_0^{(\mathrm{N})}$. Therefore we obtain in order A^2,

$$E^{(\mathrm{N})}=E_{\mathrm{C}}^{(\mathrm{N})}+\int_{-\infty}^{\infty}\left\{[J_1\sin q_0+2J_2\sin 2q_0]\frac{d\phi}{dn}+\frac{1}{2}[J_1\cos q_0+4J_2\cos 2q_0]\left(\frac{d\phi}{dn}\right)^2\right\}dn, \tag{2.29}$$

where $E_{\mathrm{C}}^{(\mathrm{N})}$ is the normal energy corresponding to the commensurate phase with no defects.

We will next consider the umklapp contribution to the energy. As in the commensurate case, in the lowest order they arise from the terms $l=Q-1$ and $l=Q$ in the expansion. Ignoring all the terms containing strongly oscillating factors we get

$$E^{(\mathrm{U})}=(-1)^QV\int_{-\infty}^{\infty}dn\ \cos Q\phi(n), \tag{2.30}$$

where V is given by Eqs. (2.22) or (2.23). Finally, adding the normal contribution (2.29) and the umklapp contribution (2.30) we have for the difference between defective and commensurate configurations the result

$$\Delta E=E-E_{\mathrm{C}}=\int_{-\infty}^{\infty}\left\{-\delta\left(\frac{d\phi}{dn}\right)+\frac{a}{2}\left(\frac{d\phi}{dn}\right)^2+|V|[1\pm\cos(Q\phi)]\right\}dn, \tag{2.31}$$

where $E_{\mathrm{C}}=E_{\mathrm{C}}^{(\mathrm{N})}+E_{\mathrm{C}}^{(\mathrm{U})}$ is the energy of the commensurate phase, V is given by Eqs. (2.22) or (2.23), the sign in front of cosine is $+$ if $(-1)^QV$ is positive and $-$ otherwise, and δ and a are given by

$$\delta=-J_1\sin q_0-2J_2\sin 2q_0, \tag{2.32a}$$

$$a=J_1\cos q_0+4J_2\cos 2q_0. \tag{2.32b}$$

The value of the parameter $\alpha=J_1/J_2$ for which the phase q_0 is stable at zero field is $\alpha_0=-4\cos q_0$. In terms of α and α_0 we have

$$\delta=\frac{|J_2|}{4}\sqrt{16-\alpha_0^2}(\alpha-\alpha_0), \tag{2.33a}$$

$$a=\frac{|J_2|}{4}(\alpha\alpha_0-2\alpha_0^2+16)\approx(16-\alpha_0^2)/4. \tag{2.33b}$$

Following the analysis of Frank and Van der Merwe,[19] expression (2.31) implies that the commensurate phase is stable against defect creation as long as

$$|\delta|<\frac{2}{\pi^2}\sqrt{a|V|}. \tag{2.34}$$

Since $\delta\propto\alpha-\alpha_0$ and $V\propto H^Q$, we conclude that the boundary of the commensurate phase is given by

$$|\alpha-\alpha_0|\propto H^{Q/2}. \tag{2.35}$$

Therefore our calculations show that all the commensurate helical phases will in general have a width of stability $\Delta\alpha$ for $H>0$. This width will be proportional to $H^{Q/2}$, and therefore will be very narrow for long period structures. An exception is the period three helical phase $q_0=2\pi/3$ for which the pinning potential V vanishes, and we expect a zero width of stability. In the case of spin-flop phase $q_0=\pi$ we have $\delta=0$, which implies that the instability of this phase is not caused by defect creation and the transition is presumably of first order. Similarly we expect that the period four ''double spin-flop phase'' $q_0=\pi/2$ at low fields will undergo a first-order transition to the nearby helical phases.

III. FAN PHASES AT HIGH FIELDS

The modulated structures found at high fields are the fan phases in which the spins oscillate around the direction parallel to the field.[4] We will study the pinning of the commensurate fan phases and the stability of these phases against the defect creation in much the same way as the previous section, but the results will be different due to the difference in the spin structures.

Let us consider a commensurate fan phase with wave number given by $q_0=2\pi(P/Q)$ where P/Q is an irreducible fraction. Close to the transition to the paramagnetic phase the angles that the spins make with the field will be small, that is, $|\theta_n|\ll 1$. Expanding the cosines in the expression (1.1) for the energy we obtain

$$E=\sum_{l=0}^{\infty}E_l, \tag{3.1}$$

where

$$E_l=-\frac{(-1)^l}{(2l)!}\sum_n[J_1(\theta_{n+1}-\theta_n)^{2l}+J_2(\theta_{n+2}-\theta_n)^{2l}+H\theta_n^{2l}]. \tag{3.2}$$

Introducing the Fourier representation $\theta_n=\Sigma_q\theta_qe^{iqn}$ expression (3.2) becomes

$$N^{-1}E_l=-\frac{(-1)^l}{(2l)!}\sum_{q_1\ldots q_{2l}}\theta_{q_1}\cdots\theta_{q_{2l}}[J_1(e^{iq_1}-1)\cdots(e^{iq_{2l}}-1)+J_2(e^{2iq_1}-1)\cdots(e^{2iq_{2l}}-1)+H]\Delta(q_1+\cdots+q_{2l}). \tag{3.3}$$

In our approximate calculations we will keep only the first harmonic in the Fourier expansion for θ_n,

$$\theta_n=\theta_{q_0}e^{iq_0n}+\theta_{-q_0}e^{-iq_0n}=2A\cos(q_0n+\phi), \tag{3.4}$$

where we have written $\theta_{q_0}=Ae^{i\phi}$.

Let us first compute the contributions due to normal terms in the expansion (3.3). Since $q_i=\pm q_0$, there are $\binom{2l}{l}$ different ways of satisfying the condition $q_1+\cdots+q_{2l}=0$. Therefore the normal contribution to E_l is given by

$$N^{-1}E_l^{(\mathrm{N})}=-\frac{(-1)^l}{(2l)!}|\theta_{q_0}|^{2l}\binom{2l}{l}[J_1|e^{iq_0}-1|^{2l}+J_2|e^{2iq_0}-1|^{2l}+H]. \tag{3.5}$$

Using $\theta_{q_0}=Ae^{i\phi}$ we can write the total normal contribution to the energy (3.1) in the form

$$N^{-1}E^{(\mathrm{N})}=\sum_{l=0}^{\infty}a_lA^{2l}, \tag{3.6}$$

where

$$a_l=-\frac{(-1)^l}{(l!)^2}2^l\left[J_1(1-\cos q_0)^l+J_2(1-\cos 2q_0)^l+\frac{H}{2^l}\right]. \tag{3.7}$$

The first three coefficients a_l are given by

$$a_0=-J_1-J_2-H, \tag{3.8a}$$

$$a_1=-[J(q_0)-J(0)-H], \tag{3.8b}$$

$$a_2=-\frac{1}{4}[3J(0)-4J(q_0)+J(2q_0)+H], \tag{3.8c}$$

where we used the definition (2.17) for $J(nq)$. Considering terms up to order A^4 in the expansion (3.6) and minimizing with respect to A we obtain in leading order

$$A=\left\{\frac{2[J(q_0)-J(0)-H]}{4J(q_0)-3J(0)-J(2q_0)-H}\right\}^{1/2}. \tag{3.9}$$

Thus the solution for A is possible only for

$$H\leqslant J(q_0)-J(0)=H_c, \tag{3.10}$$

where H_c is the critical field for the parafan transition. The normal contribution to the energy in leading order is given by

$$N^{-1}E^{(\mathrm{N})}=N^{-1}E_{\mathrm{P}}-\frac{(H_c-H)^2}{4J(q_0)-3J(0)-J(2q_0)-H}, \tag{3.11}$$

where E_{P} is the energy of the paramagnetic phase. These results are in agreement with Ref. 4. We observe that the phase ϕ in the spin structure (3.4) is not fixed if we consider only the normal contributions.

Let us consider the contributions from the umklapp terms. In the approximation (3.4) in which only the main harmonic is taken into account, that is, $q_i=\pm q_0$, the condition that $q_1+\cdots+q_{2l}$ be a multiple of 2π is satisfied in the lowest order for

$$l=L=\begin{cases}Q/2 & \text{if } Q \text{ is even,}\\ Q & \text{if } Q \text{ is odd.}\end{cases} \tag{3.12}$$

Therefore we find for the lowest order umklapp contribution to the energy the result

$$N^{-1}E^{(\mathrm{U})}=N^{-1}E_L^{(\mathrm{U})}=V\cos 2L\phi, \tag{3.13}$$

where

$$V=-\frac{2^{L+1}}{(2L)!}A^{2L}[J_1(1-\cos q_0)^L\cos Lq_0+J_2(1-\cos 2q_0)^L+(-1)^L2^{-L}H]. \tag{3.14}$$

In order for the energy (3.13) to be a minimum the phase ϕ should be an even or odd multiple of $\pi/2L$ depending on the sign of V. Thus the phase ϕ is in general determined by the umklapp terms of order $A^{2L}\sim(H_c-H)^L$, where $L=Q/2$ for even Q and $L=Q$ for odd Q. We observe that the other harmonics that were neglected in the expansion (3.4) may give contributions of the same order to the pinning potential, but this should not alter our conclusions. An exception is the case $Q=3$ which is known to have zero pinning energy for all fields.[6]

We will next examine the stability of commensurate fan phases against the defect creation. These defects will be described by structures of the form

$$\theta_n=Ae^{i(q_0n+\phi_n)}+\text{c. c.}=2A\cos(q_0n+\phi_n), \tag{3.15}$$

where A is given by Eq. (3.9). We will begin by calculating the contribution due to normal terms. Inserting the expression (3.15) into the expression for the energy (3.2) and neglecting all sums involving strongly oscillating terms we find

$$E^{(\mathrm{N})}=E_C^{(\mathrm{N})}+b\sum_n(\phi_{n+1}-\phi_n)+c\sum_n(\phi_{n+1}-\phi_n)^2+d\sum_n(\phi_{n+2}-\phi_n)+e\sum_n(\phi_{n+2}-\phi_n)^2, \tag{3.16}$$

where $E_C^{(\mathrm{N})}$ is the normal energy in the commensurate phase and the coefficients b, c, d, and e are given in order A^2 by

$$b=2J_1\sin q_0A^2,\quad c=J_1\cos q_0A^2, \tag{3.17a}$$

$$d=2J_2\sin 2q_0A^2,\quad e=J_2\cos 2q_0A^2. \tag{3.17b}$$

The umklapp contribution to the energy can be computed in the similar way. Substituting θ_n given by Eq. (3.15) into expression (3.2) with $l=L$ given by Eq. (3.12) we find that the lowest order contribution to the umklapp energy can be written in the form

$$E^{(\mathrm{U})}=E_C^{(\mathrm{U})}+|V|\sum_n (1\pm\cos 2L\phi_n), \tag{3.18}$$

where $E_C^{(\mathrm{U})}$ is the umklapp energy of the commensurate phase and the sign in front of cosine is + if $V>0$ and − otherwise.

Adding the normal (3.16) and umklapp (3.18) contributions, and making the continuum approximations $\phi_{n+1}-\phi_n\approx d\phi/dn$ and $\phi_{n+2}-\phi_n\approx 2d\phi/dn$, the energy difference ΔE between the defective and commensurate structures becomes

$$\Delta E=\int_{-\infty}^{\infty}dn\left[-\delta\frac{d\phi}{dn}+\frac{a}{2}\left(\frac{d\phi}{dn}\right)^2+|V|(1\pm\cos 2L\phi)\right], \tag{3.19}$$

where

$$a=2c+8e=2A^2[J_1\cos q_0+4J_2\cos 2q_0], \tag{3.20a}$$

$$\delta=-b-2d=-2A^2[J_1\sin q_0+2J_2\sin 2q_0]. \tag{3.20b}$$

In terms of the parameter $\alpha=J_1/J_2$ we have

$$a=A^2|J_2|(\alpha\alpha_0-2\alpha_0^2+16)/2, \tag{3.21a}$$

$$\delta=A^2|J_2|\sqrt{16-\alpha_0^2}(\alpha-\alpha_0)/2, \tag{3.21b}$$

where $\alpha_0=-4\cos q_0$. Using the result (2.34) for the stability limit of the commensurate phase, we conclude that the fan phase is stable inside the region $|\alpha-\alpha_0|\propto|H-H_c|^{(L-1)/2}$. Remembering that L is given by Eq. (3.12) we finally find

$$|\alpha-\alpha_0|\propto\begin{cases}|H_c-H|^{(Q-2)/4} & \text{for even } Q,\\ |H_c-H|^{(Q-1)/2} & \text{for odd } Q.\end{cases} \tag{3.22}$$

Thus we conclude that every commensurate fan phase with period $Q\geqslant 4$ has a finite width of stability for $H<H_c$. Again, our results show that these widths are very narrow for long period phases. We note, however, that the exponents characterizing the width for the fan phases in expression (3.22) are different from those of helical phases given in Eq. (2.35). For the spin-flop phase $Q=2$ we have $\delta=0$, which indicates that the instability of this phase is not caused by creation of defects and should undergo a first-order transition. Also the phase $q_0=2\pi/3$ has a zero pinning energy[6] and we expect a zero width of stability.

IV. SUMMARY AND DISCUSSIONS

We have determined the exponents governing the phase boundaries of commensurate helical phases at low fields and commensurate fan phases at high fields using the continuum approximation for the stability analysis. We found that for $H\to 0+$ a helical phase with period $Q>4$ has a width proportional to $H^{Q/2}$. Similarly, for $H\to H_c-$, where H_c is the critical field for the parafan transition, a fan phase with period $Q\geqslant 4$ has a width proportional to $|H-H_c|^{(Q-2)/4}$ if Q is an even number and $|H-H_c|^{(Q-1)/2}$ if Q is an odd number. Exceptions to the above results are the spin-flop phase $q_0=\pi$ which our calculations indicate undergoes a first-order transition both at low and high fields; the period $Q=3$ phase $q_0=2\pi/3$ which has a zero pinning energy[6] and should presumably have a zero width of stability for arbitrary fields; and the period four ''double spin-flop phase'' $q_0=\pi/2$ at low fields which we expect to undergo a first-order transition.

Our calculations were based on the same kind of approximations that is known to give good results for the Frenkel-Kontorova[10] and ANNNI models.[12] The approximation of taking only the first harmonic of the modulation and ignoring the second and higher order harmonics in the expansions such as Eq. (3.4) was made to keep the calculations for general commensurate phase P/Q not too involved, but it should not change the exponents governing the phase boundaries. On the other hand, the continuum approximation for the stability analysis seems to be necessary in order to get analytic results, and may in some cases lead to results that are qualitatively incorrect concerning the nature of the transition. Thus the continuum approximation predicts a continuous transition for the (2,2)-antiphase of the ANNNI model,[12] but numerical calculations seem to indicate that the transition is first order[20]. Even though this might be the case for some phases, the exponents governing the phase boundaries might still be correct because if the transition is weakly first order we do not expect significant change in its location. Our analytic results seem to agree qualitatively with numerical calculations,[21] but a detailed comparison seems to be difficult because the widths of commensurate phases are very narrow, implying that very long period structures have to be determined, challenging both numerical precision and computational resources.

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*Present address: Escola Técnica Federal de Santa Catarina, Unidade de Ensino Descentrarizada de São José, Rua José Lino Kretzer 608, 88103-902, São José, SC.

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