Exact renormalization-group study of aperiodic Ising quantum chains and directed walks

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We consider the Ising model and the directed walk on two-dimensional layered lattices and show that the two problems are inherently related: The zero-field thermodynamical properties of the Ising model are contained in the spectrum of the transfer matrix of the directed walk. The critical properties of the two models are connected to the scaling behavior of the eigenvalue spectrum of the transfer matrix which is studied exactly through renormalization for different self-similar distributions of the couplings. The models show very rich bulk and surface critical behaviors with nonuniversal critical exponents, coupling-dependent anisotropic scaling, first-order surface transition, and stretched exponential critical correlations. It is shown that all the non-universal critical exponents obtained for the aperiodic Ising models satisfy scaling relations and can be expressed as functions of varying surface magnetic exponents. [S0163-1829(97)00441-4]

I. INTRODUCTION

The study of layered Ising models (IM's) has been an active field of research during the last decades. One may mention the pioneering works on two-dimensional (2D) periodically[1] and randomly[2–4] layered lattices. Similarly the critical behavior of directed walks (DW's) in inhomogeneous[5] or random media[6] has attracted widespread interest.

Recently, following the discovery of quasicrystals,[7] on the one hand, and the progress in molecular beam epitaxy which allows the production of good quality multilayers, on the other hand,[8] there has been a growing interest in the theoretical study of phase transitions in quasiperiodic systems and, more generally, aperiodic systems.[9] These are deterministic but nonperiodic structures which are called quasiperiodic when the spatial fluctuations are so weak that the Fourier spectrum is still discrete, but point symmetry is incompatible with a periodic structure. Such systems may be considered as intermediates between homogeneous and random ones and, consequently, are expected to display a rich variety of critical behaviors.

A. Previously known results

Most of the early works about phase transitions on aperiodic systems were done on quasiperiodic lattices and did not show any sign of modified critical behavior. Among these, one may mention an approximate renormalization group treatment of the classical IM on the Penrose lattice[10] and Monte Carlo renormalization group studies of the same

problem.[11,12] Universal behavior was also obtained in Monte Carlo simulations of the percolation problem on the Penrose lattice and its dual[13] as well as for the statistics of self-avoiding walks.[14] One may notice, as an exception, the analytical renormalization group study of interfacial roughness, still on the Penrose lattice, where the fluctuating interface "feels" a Fibonacci quasiperiodic potential. In this case, a marginal behavior was obtained for the decay of the transverse correlations.[15]

Probably the most studied system is the aperiodically layered 2D classical IM and its quantum counterpart in the extreme anisotropic limit,[16] the aperiodic Ising quantum chain in a transverse field.

In the classical formulation, the energy of a configuration is given by

$$-\beta H = \sum_{k,l} K_1(k)\sigma_{k,l}\sigma_{k,l+1} + \sum_{k,l} K_2(k)\sigma_{k,l}\sigma_{k+1,l}, \tag{1.1}$$

where the σ's are the spin-1/2 Ising variables, and K_1 and K_2 are the exchange interactions in the vertical and horizontal directions, respectively. Their values are the same in a vertical layer k and are modulated according to some aperiodic sequence in the horizontal direction.

In the extreme anisotropic limit $(K_1 \to \infty, K_2 \to 0)$, the transfer matrix between successive rows in the vertical direction can be written as $\exp(-\tau\mathcal{H})$, where $\tau = 2K_1^*$ is the infinitesimal lattice spacing in the Euclidian time direction. $\mathcal{H}$ is the Hamiltonian of a spin-1/2 quantum Ising chain in a transverse field:

$$\mathcal{H} = -\frac{1}{2}\sum_{k=1}^{L} h_k \sigma_k^z - \frac{1}{2}\sum_{k=1}^{L-1} J_k \sigma_k^x \sigma_{k+1}^x, \qquad (1.2)$$

where the $\sigma_i^{x,z}$'s are Pauli spin matrices. The transverse field h_k (such that $h_k K_1^*$ is the dual coupling $K_1^*(k)$ given by $\exp[-2K_1^*(k)]=\tanh K_1(k)$) plays the role of the temperature. The coupling J_k is the ratio $K_2(k)/K_1^*(k)$. In general, due to universality, the classical and quantum systems have the same critical properties. Only in the presence of a marginal perturbation which leads to nonuniversal exponents are the corresponding quantum exponents obtained by taking an appropriate limit of the classical expressions.

Usually the aperiodic modulation is assumed to involve only the horizontal couplings, i.e., the two-spin interaction J_k and $h_k=1$. The fluctuation around the average $\bar{J}$ at a length scale L is measured by

$$\Delta(L) = \sum_{k=1}^{L} (J_k - \bar{J}). \qquad (1.3)$$

When the aperiodic couplings are generated via substitutions using an inflation rule, this quantity behaves as L^ω, where ω is the wandering exponent of the aperiodic sequence which is linked to the two leading eigenvalues of a substitution matrix.[17,18]

For sequences with bounded fluctuations ($\omega<0$) the aperiodic perturbation does not change the Ising critical behavior. This was shown analytically by Tracy in the case of the Fibonacci sequence, with $\omega=-1$, for the 2D layered IM. The Onsager logarithmic singularity of the specific heat then keeps a nonvanishing finite amplitude.[19] The same conclusion was reached for the Ising quantum chain with generalized Fibonacci modulations of the couplings.[20] The low-energy spectrum of the quantum chain which, through the gap-exponent relation on finite critical chains, gives the values of the critical exponents was shown to be unaffected by a quasiperiodic modulation.[21] Universal behavior was also obtained with the Thue-Morse sequence and its generalizations.[22] In this case, the quantum chain is not quasiperiodic but the fluctuations remain bounded.

For an aperiodic sequence with unbounded fluctuations ($\omega>0$), Tracy[23] noticed that the Onsager singularity is suppressed like in the randomly layered McCoy-Wu model.[2]

The situation was later clarified by Luck[24] who proposed a generalization of the Harris criterion for quenched randomness[25] adapted to the case of aperiodic fluctuations of the couplings (see also Ref. 26). By comparing the mean shift of the local temperature in the 2D layered system (governed by the wandering exponent ω), at the scale of the correlation length of the unperturbed system, to the deviation from the critical temperature, one obtains a crossover exponent $\phi=1+\nu(\omega-1)$. It controls the evolution of the amplitude of the aperiodic modulation when one approaches the critical point. For the 2D IM with $\nu=1$, the crossover exponent is equal to the wandering exponent so that, quite generally, the aperiodic modulation becomes a relevant perturbation and changes the Ising critical behavior when the fluctuations are unbounded, as conjectured by Benza et al.[20] One must notice that the correspondence between relevant perturbations and unbounded fluctuations holds only when

$\nu=1$. The marginal behavior obtained for the Fibonacci sequence with bounded fluctuations in the case of the interface roughness problem[15] follows from Luck's criterion where ν is now the correlation length exponent in the transverse direction $\nu_\perp=1/2$. Finally let us mention that for a randomly layered system the relevance-irrelevance criterion applies with $\omega=1/2$.

In the same work,[24] Luck checked the validity of his criterion for random and aperiodic quantum Ising chains. To treat the aperiodic problem he considers periodic approximants, i.e., a periodic quantum chain with a large unit cell of length L in which the couplings J_k are distributed according to the aperiodic sequence. He deduces the low-energy behavior of the fermionic excitations[27] Λ from a perturbation expansion in Λ. For the unperturbed problem at criticality, the massless excitations have a linear low-energy dispersion relation $\Lambda=vq$ where v is the velocity and q the wave vector. On the periodic approximant a L-dependent velocity v_L is obtained and the properties of the aperiodic system are governed by the limiting behavior of v_L when L goes to infinity. The behavior of the singular part of the ground-state energy, corresponding to the free energy in the 2D classical system, is linked to the low-energy excitation spectrum and its temperature dependence can be obtained through a scaling argument.

For sequences with bounded fluctuations ($\omega<0$), v_L is bounded and nonvanishing in the limit $L\to\infty$ so that the Onsager logarithmic critical singularity is preserved. For unbounded fluctuations ($\omega>0$), the typical velocity vanishes exponentially, leading to an essential singularity for the singular part of the ground-state energy as in the case of random chains. Finally, when the fluctuations grow on a logarithmic scale ($\omega=0$), the typical velocity vanishes as a nonuniversal power of L. The perturbation is marginal, and the specific heat exponent is negative (the logarithmic singularity is suppressed) and varies continuously with the amplitude of the aperiodic modulation. This marginal behavior was checked numerically.

B. Renormalization-group method and main results

The results obtained so far for different aperiodic modulations in different models are all in accordance with Luck's criterion.[28–35] Most of the activity in our groups was concerned with the study of the surface and bulk critical properties of 2D aperiodically layered IM's, either using the 1D quantum formulation or working on a triangular lattice, making use of the star-triangle relation.

Although some relevant perturbations were treated in Refs. 28 and 29, we mainly considered marginal aperiodic perturbations. The continuously varying surface magnetic exponent $x_{m_s}=\beta_s/\nu$ was obtained analytically for different aperiodic sequences whereas the scaling dimension of the surface energy was conjectured on the basis of finite-size scaling studies.[32–35]

The marginal aperiodic models were found to display anisotropic scaling.[30,31,33] The correlation length diverges with different exponents along and through the layers with a ratio $z=\nu_\parallel/\nu$, giving a continuously varying anisotropy exponent. Such a behavior was in fact implicitly contained in Luck's work[24] where a power-law dependence on L was found in

the marginal case. Accurate numerical calculations of the anisotropy exponent z led us to propose a simple scaling relation between z and the surface magnetic exponents on both sides of the system, x_{m_s} and $\bar{x}_{m_s}$. The anisotropic scaling of bulk and surface properties was extensively studied in Ref. 33.

In this paper we present the results of an exact renormalization-group (RG) study of aperiodic and hierarchical Ising and DW models. The introduction of RG techniques into the field of phase transitions and critical phenomena has largely contributed to our understanding of the properties of the critical state. For instance, the RG method has given a natural explanation for the scaling hypothesis and universality. At the same time, it has provided powerful procedures to calculate critical exponents,[36] albeit generally using some approximation, e.g., approximate RG transformations, expansions in a small parameter ($\epsilon, 1/N, \ldots$), or numerical methods. There are few nontrivial problems in statistical mechanics for which the RG transformation can be worked out exactly. One may mention the IM on the triangular lattice[37] or different physical processes on self-similar fractal objects.[38]

Here we develop exact RG solutions for a class of 2D layered Ising and DW models. The novel feature of our approach is that we study both problems within the framework of the same RG transformation. It is based on a hitherto unnoticed connection between the eigenvalue problem for fermionic excitations which enters the solution of the IM (Ref. 27) and the transfer matrix of a DW in two dimensions. Both problems are considered on layered lattices, such that the walk is directed along the translationally invariant direction. The solution of the DW, which means the diagonalization of its transfer matrix (TM), provides in principle all the necessary information to obtain the zero-field thermodynamical properties and correlation functions of the IM.

The critical properties of the two models are connected to the scaling behavior of the eigenstates of the TM at different edges of the spectrum. An exact RG study of the eigenvalue problem of the TM is performed for different self-similar distributions of the couplings and *the critical properties of the IM and the DW are governed by two different fixed points of the same RG transformation.*

Our method is well adapted to the case of self-similar perturbations. It is quite different from the approximate renormalization-group technique recently introduced by Fisher to treat randomly layered systems.[3] In this approach, which leads to exact results in the critical domain, instead of using the transformation to fermions, Fischer works on the Hamiltonian itself, reducing the energy scale of the problem by a systematic elimination of the stronger couplings

The bulk and surface critical properties are examined for several aperiodic and hierarchical sequences. In all of the models we studied the aperiodicity is marginal at the Ising fixed point and induces continuously varying critical exponents. The bulk anisotropy exponent z and the correlation length exponents ν, on the one hand, and the surface energy exponent x_{e_s} and the surface magnetization exponents x_{m_s} and $\bar{x}_{m_s}$, on the other hand, are obtained analytically. We also prove the previously conjectured relation between the anisotropy and surface magnetic exponents, $z = x_{m_s} + \bar{x}_{m_s}$

>1, which holds for sequences which modifiy the critical coupling. A simple scaling picture emerges in which the surface magnetic exponents play a fundamental role: All the nonuniversal exponents (except the bulk magnetic one, not considered in this work) can be expressed as functions of these two surface magnetic exponents. The surface energy exponent is given by $x_{e_s} = z + 2x_{m_s}$ on one side of the chain and $\bar{x}_{e_s} = z + 2\bar{x}_{m_s}$ on the other, whereas the specific heat exponent is given by $\alpha = 1 - z < 0$.

With aperiodic sequences with a vanishing density of modified couplings, which changes the critical behavior only locally near the surface, there is no anisotropy in the bulk of the system ($z = 1$). For sufficiently strong modified couplings, the surface remains ordered at the bulk critical point and then the first excitation alone scales in an anomalous way, with a continuously varying power of the size of the system.

Finally one may mention that these marginal aperiodic Ising sytems are closer to periodic than to randomly layered ones. The varying exponents evolve continuously from their unperturbed values when the aperiodic modulation grows. In particular we checked numerically that the gap is nonvanishing in the disordered phase, i.e., that there is no trace of a Griffiths phase,[39,3] as expected for systems with bounded fluctuations.[24] Even with relevant aperiodic perturbations, which by some aspects are closer to random ones, displaying essential singularities in the singular part of the ground-state energy[24] and in the surface magnetization,[29] the Griffiths phase is absent according to a recent study.[40]

A short account of our results, concerning the bulk critical behavior, has been given in a recent Letter.[41]

The structure of the paper is the following. The relation between the IM and the DW is presented in Sec. II; afterwards the basic properties of aperiodic sequences are recapitulated in Sec. III. In the following sections, Secs. IV–VIII, the RG transformation is worked out for different aperiodic and hierarchical models which were chosen in order to illustrate the different renormalization procedures one may use, in particular in the treatment of the surface properties. Both the bulk and surface critical behaviors are studied. Some relations between the critical exponents of the IM are derived in Sec. IX and the results are discussed in Sec. X. Details about the derivation of the RG equations for the specific models are collected in the Appendixes.

II. RELATION BETWEEN THE ISING QUANTUM CHAIN AND THE DIRECTED WALK MODEL

Using a Jordan-Wigner transformation,[42] the Ising Hamiltonian (1.2) can be rewritten as a quadratic form in fermion operators. It is then diagonalized through a canonical transformation[27] which gives

$$\mathcal{H} = \sum_{q=1}^{L} \Lambda_q \left(\eta_q^{\dagger} \eta_q - \frac{1}{2} \right), \quad (2.1)$$

where $\eta_q^{\dagger}$ and η_q are fermion creation and anihilation operators, respectively. The fermion excitations Λ_q are nonnegative and satisfy the set of equations

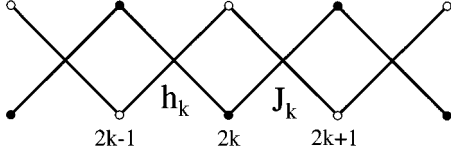$$\Lambda_q \Psi_q(k) = -h_k \Phi_q(k) - J_k \Phi_q(k+1),$$

FIG. 1. Portion of two interpenetrating diagonally layered square lattices involved in the transfer matrix of the directed walk. The fugacities are alternatively h_k and J_k.

$$\Lambda_q \Phi_q(k) = -J_{k-1}\Psi_q(k-1) - h_k\Psi_q(k), \qquad (2.2)$$

with the boundary conditions $J_0 = J_L = 0$. The vectors Φ_q's and Ψ_q's, which are related to the coefficients of the canonical transformation, are normalized. They enter into the expressions of correlation functions and thermodynamical quantities.[27]

Usually one proceeds by eliminating either Ψ_q or Φ_q in Eqs. (2.2) and the excitations are deduced from the solution of one of the following eigenvalue problems:

$$J_{k-1}h_{k-1}\Phi_q(k-1) + (J_{k-1}^2 + h_k^2)\Phi_q(k) + J_k h_k \Phi_q(k+1)$$
$$= \Lambda_q^2 \Phi_q(k),$$

$$J_{k-1}h_k\Psi_q(k-1) + (J_k^2 + h_k^2)\Psi_q(k) + J_k h_{k+1}\Psi_q(k+1)$$
$$= \Lambda_q^2 \Psi_q(k), \qquad (2.3)$$

with the same boundary conditions as above. This last step can be avoided by introducing a $2L$-dimensional vector $\mathbf{V}_q$ with components

$$V_q(2k-1) = -\Phi_q(k), \quad V_q(2k) = \Psi_q(k), \qquad (2.4)$$

and noticing that the relations in Eqs. (2.2) then correspond to the eigenvalue problem for the matrix:

$$T = \begin{pmatrix} 0 & h_1 & 0 & 0 & 0 & 0 & \cdots \\ h_1 & 0 & J_1 & 0 & 0 & 0 & \cdots \\ 0 & J_1 & 0 & h_2 & 0 & 0 & \cdots \\ 0 & 0 & h_2 & 0 & J_2 & 0 & \cdots \\ \vdots & \vdots & & \ddots & & \ddots & \end{pmatrix}. \qquad (2.5)$$

Taking the square of T, odd and even components of $\mathbf{V}_q$ are decoupled, and one recovers the two eigenvalue equations in Eqs. (2.3). The matrix T can be interpreted as the TM of a DW problem on two interpenetrating, diagonally layered square lattices as shown in Fig. 1. The walker makes steps with weights h_k and J_k between first-neighbor sites on one of the two square lattices and the walk is directed in the diagonal direction.

According to Eqs. (2.2), changing Φ_q into $-\Phi_q$ in $\mathbf{V}_q$, the eigenvector corresponding to $-\Lambda_q$ is obtained. Thus all the information about the DW and the IM is contained in that part of the spectrum with $\Lambda_q \geq 0$. Later on we shall restrict ourselves to this sector.

Let us now consider the correlation lengths in the direction parallel to the layers in both problems. For the DW it can be expressed as a function of the two leading eigenvalues of the TM with

$$\xi_\parallel^{DW} = \left[\ln\left(\frac{\Lambda_L}{\Lambda_{L-1}}\right)\right]^{-1} \simeq \frac{\Lambda_L}{\Lambda_L - \Lambda_{L-1}}. \qquad (2.6)$$

Thus $\xi_\parallel^{DW}$ is proportional to the inverse gap at the top of the spectrum of the TM. For the IM in the disordered phase, the correlation length is the inverse of the lowest excitation energy of the Hamiltonian $\mathcal{H}$ in Eq. (2.1):

$$\xi_\parallel^{IM} \sim \Lambda_1^{-1}. \qquad (2.7)$$

The fermionic excitation Λ_1 is also the lowest eigenvalue in the positive sector of the TM.[43]

When any one of the two critical points is approached, the correlation length of the problem diverges and the corresponding part in the TM spectrum displays a scaling behavior. Let us consider a finite system with transverse size $L \gg 1$ and denote by $\Delta\Lambda_i$ either $\Lambda_L - \Lambda_{L-i}$ for the DW or Λ_i for the IM with $i \ll L$. Under a change of the length scale by a factor $b > 1$ such that $L' = L/b$, the gaps are expected to transform as

$$(\Delta\Lambda_i)' = b^{y_\Lambda}\Delta\Lambda_i, \qquad (2.8)$$

where the scaling dimension is generally different for different parts of the spectrum. This leads to the finite-size scaling behavior $\Delta\Lambda_i(L) \sim L^{-y_\Lambda}$ and, according to Eqs. (2.6) and (2.7), the longitudinal correlation lengths behave as $\xi_\parallel \sim L^{y_\Lambda}$. Since $\xi_\perp \sim L$, the anisotropy exponent z, defined through $\xi_\parallel \sim \xi_\perp^z$,[44] is given by

$$z = y_\Lambda. \qquad (2.9)$$

In the case of the DW one is interested in the transverse fluctuations of the walk which are characterized by a wandering exponent w through $\xi_\perp \sim \xi_\parallel^w$ (Refs. 5 and 34) so that

$$w = y_\Lambda^{-1}. \qquad (2.10)$$

III. SUBSTITUTION MATRIX AND RELEVANCE-IRRELEVANCE CRITERION

In the following, except for the hierarchical sequence, we consider sequences generated via substitutions on a finite alphabet such that, in the case of two letters A and B, $A \to S(A)$, $B \to S(B)$. The properties of the sequence are governed by its substitution matrix[17,18]

$$M = \begin{pmatrix} n_A^{S(A)} & n_A^{S(B)} \\ n_B^{S(A)} & n_B^{S(B)} \end{pmatrix}, \qquad (3.1)$$

where the matrix element $n_i^{S(j)}$ gives the number of i in $S(j)$. The matrix elements in M^n give the same numbers in the sequence obtained after n iterations. When the substitution starts with j, the corresponding numbers are contained in column j.

If $\mathbf{U}_\nu$ denotes the right eigenvector of M with eigenvalue Ω_ν, the asymptotic density of i is given by

$$\rho_\infty^{(i)} = \frac{U_1(i)}{\Sigma_j U_1(j)}, \qquad (3.2)$$

where $\mathbf{U}_1$ is the eigenvector corresponding to the leading eigenvalue Ω_1. The length of the sequence after n iterations is related to the leading eigenvalue through $L_n \sim \Omega_1^n$ so that $\Omega_1 > 1$.

In the following, each letter in the sequence is replaced by one digit or more (for examples, $A = 0$, $B = 1$). Thus one obtains a sequence of digits f_k ($k = 1, 2, \ldots, L$). The aperiodic Hamiltonian is defined as in Eq. (1.2) with a constant transverse field $h_k = h$ and a modulation of the couplings following the aperiodic sequence,

$$J_k = J R^{f_k}, \tag{3.3}$$

where J is the unperturbed interaction and R the modulation ratio.

When $f_k = 0, 1$, the cumulated deviation $\Delta(L)$ from the averaged coupling $\overline{J}$ defined in Eq. (1.3) scales with L as

$$\Delta(L) = J(R - 1)(n_L - L\rho_\infty) \sim \delta |\Omega_2|^n \sim \delta L^\omega. \tag{3.4}$$

In this expression,

$$n_L = \sum_{k=1}^{L} f_k, \quad \rho_\infty = \lim_{L \to \infty} \frac{n_L}{L} \tag{3.5}$$

give the number of digits equal to 1 in a sequence with length L and their asymptotic density, which can be deduced from Eq. (3.2), respectively; Ω_2 is the next-to-leading eigenvalue of the substitution matrix, δ measures the amplitude of the aperiodic modulation, and ω is the wandering exponent of the sequence, given by

$$\omega = \frac{\ln |\Omega_2|}{\ln \Omega_1}. \tag{3.6}$$

Thus the mean shift of the coupling strength $\overline{\delta J}(L)$ at a length scale L, proportional to $L^{\omega - 1}$, is governed by the wandering exponent.

The relevance of the perturbation follows when one compares the deviation t from the critical point to the averaged temperature shift $\overline{\delta t} \sim \overline{\delta J}(\xi)$ induced by the aperiodicity at a length scale given by the correlation length $\xi \sim t^{-\nu}$:[24]

$$\frac{\overline{\delta t}}{t} \sim t^{-\phi}, \quad \phi = 1 + \nu(\omega - 1). \tag{3.7}$$

When $\phi > 0$, the ratio is divergent, which indicates a relevant perturbation. When $\phi < 0$, the ratio vanishes at the critical point and the perturbation is irrelevant. Finally, when $\phi = 0$, the perturbation is marginal and may lead to a nonuniversal behavior. The same conclusions can be reached by calculating the scaling dimension of the modulation amplitude δ, which is equal to ϕ/ν.[28]

For a strongly anisotropic unperturbed system ν in Eq. (3.7) has to be replaced by the exponent $\nu_\perp$ of the correlation length in the direction perpendicular to the layers in the 2D system.[34]

The critical transverse field of the inhomogeneous IM is generally given by[45]

$$h_c = \lim_{L \to \infty} \prod_{k=1}^{L} (J_k)^{1/L}. \tag{3.8}$$

FIG. 2. Matrix elements $T_{k,k+1}$ as a function of k for the period-doubling sequence. Components of the eigenvector to be decimated out in the RG transformation are denoted by crosses. The heights of solid vertical bars indicate the strength of the couplings; the grey bars stand for the field.

Introducing the reduced coupling $\lambda = J/h$, its critical value on the aperiodic quantum chain follows from Eqs. (3.3), (3.5), and (3.8) as

$$\lambda_c = R^{-\rho_\infty}. \tag{3.9}$$

IV. PERIOD-DOUBLING SEQUENCE

A. Definition and general properties

The period-doubling sequence[46] follows from the substitutions $A \to S(A) = AB$, $B \to S(B) = AA$. Here we make the identification $A = 0$ and $B = 1$, i.e., $J_A = J$ and $J_B = JR$ according to Eq. (3.3). Thus starting on A, after n iterations, one obtains the following sequences of digits f_k:

$$n = 0, \quad 0,$$

$$n = 1, \quad 01,$$

$$n = 2 \quad 0100,$$

$$n = 3, \quad 01000101,$$

$$n = 4, \quad 01\underline{0}00\underline{1}0\underline{1}0\underline{1}000\underline{1}0\underline{0}. \tag{4.1}$$

The eigenvalues of the substitution matrix are $\Omega_1 = 2$ and $\Omega_2 = -1$ so that the wandering exponent ω, given by Eq. (3.6), vanishes. The asymptotic density $\rho_\infty = \rho_\infty^{(B)} = 1/3$ follows from Eq. (3.2) and leads to the Ising critical coupling $\lambda_c = R^{-1/3}$, according to Eq. (3.9).

One easily verifies on the last line of Eqs. (4.1) that the f_k's satisfy the relations

$$f_{2k} = 1 - f_k, \quad f_{2k+1} = 0. \tag{4.2}$$

B. Bulk critical behavior

We now proceed to the exact renormalization of the eigenvalue equations, associated with the matrix (2.5), which follow from Eqs. (2.2) and (2.4). We first treat the bulk problem on a semi-infinite system. To recover the period-doubling sequence of interactions after one renormalization step, we eliminate triplets of interactions (J, RJ, J) indicated by crosses in Fig. 2. Using reduced couplings $\lambda_k = J_k/h$ and a reduced eigenvalue $\hat{\Lambda} = \Lambda/h$, the RG equations, as derived in Appendix A, are given by

$$\hat{\Lambda}' = \hat{\Lambda} \frac{c - d}{R\lambda^3}, \quad \lambda' = \frac{c}{R\lambda^2}, \tag{4.3}$$

where c and d are defined in Eqs. (A2).

According to Eqs. (A3) and (A4), the components of the eigenvectors transform as

$$V'(2k)=V(8k), \quad V'(2k+1)=V(8k+1). \quad (4.4)$$

The bulk IM fixed point corresponds to

$$\hat{\Lambda}^*=0, \quad \lambda^*=-R^{-1/3}, \quad (4.5)$$

which, using Eqs. (A2), leads to

$$c^*=-1, \quad d^*=1+R^{2/3}+R^{-2/3}. \quad (4.6)$$

Thus the eigenvalues of the linearized transformation are given by

$$b^z=\left.\frac{\partial\hat{\Lambda}'}{\partial\hat{\Lambda}}\right|^*=(R^{1/3}+R^{-1/3})^2, \quad b^{y_t}=\left.\frac{\partial\lambda'}{\partial\lambda}\right|^*=4, \quad (4.7)$$

and, with $b=4$, one obtains

$$z=\frac{\ln(R^{1/3}+R^{-1/3})}{\ln2}, \quad y_t=\nu^{-1}=1, \quad (4.8)$$

thus confirming the conjecture of Ref. 30.

The top of the spectrum, which governs the behavior of the DW, scales to a fixed point with $\hat{\Lambda}\to\infty$ and $\lambda\to\infty$. Thus it is convenient to write the RG equations in terms of the new variables $\kappa=1/\lambda$ and $a=\hat{\Lambda}/\lambda$, leading to

$$\kappa'=\kappa^4\frac{R}{A}, \quad a'=a\left(1-\kappa^2\frac{B}{A}\right),$$

$$A=(a^2-R^2)(a^2-1)^2+2\kappa^2a^2(1-a^2)+\kappa^4a^2,$$

$$B=(a^2-R^2)(a^2-1)+\kappa^2(1-2a^2)+\kappa^4. \quad (4.9)$$

From Eqs. (4.9) the DW fixed point is given by

$$\kappa^*=0, \quad \Delta^*=a^{*2}-R^2=0. \quad (4.10)$$

The scaling behavior at the DW fixed point is different for the homogeneous model, with $R=1$, and for the aperiodic one, with $R\neq1$. First we start with the homogeneous model, where the separatrix in the (Δ,κ) plane is linear at the fixed point: $\Delta(\kappa)=a^*\kappa$. According to Eqs. (4.9), a point with coordinates $\Delta=\alpha\kappa, \kappa\ll1$, which lies close to the separatrix, is repelled by the fixed point as

$$\kappa'=\frac{\kappa}{\alpha^3-2\alpha}, \quad \Delta'=\Delta\left[1-2\frac{\alpha^2-1}{\alpha^2(\alpha^2-2)}\right], \quad (4.11)$$

and thus $\alpha'=\Delta'/\kappa'$ is given by

$$\alpha'=\alpha^4-4\alpha^2+2. \quad (4.12)$$

The fixed-point value $\alpha^*=2$ determines the equation of the separatrix while the leading eigenvalue of the transformation, $\epsilon_1=\partial\alpha'/\partial\alpha|^*=16$, is connected to the gap exponent through $y_\Lambda=\ln\epsilon_1/\ln b=2$. Consequently, the wandering exponent is given by

$$w=\frac{1}{y_\Lambda}=\frac{1}{2} \quad (R=1), \quad (4.13)$$

in agreement with known results.[5]

For the aperiodic model the separatrix has a quadratic dependence $\Delta(\kappa)=\beta^*\kappa^2$ when $\kappa\ll1$, with $\beta^*=(\sqrt{2}R-2R^2)/(1-R^2)$, in contrast to the linear behavior for the homogeneous model. The scaling behavior of a point with coordinates $\Delta=\beta\kappa^2, \kappa\ll1$, close to the separatrix, can be deduced from Eqs. (4.9) as $\kappa'\sim\kappa^2$ and $\Delta'\sim\Delta$, thus $\Delta'/\kappa'\sim\Delta^{1/2}/\kappa$. Consequently, at a fixed $\kappa=\kappa'\ll1$ we obtain $\Delta'\sim\Delta^{1/2}$, which represents a strong repulsion. This type of scaling behavior is compatible with an essential singularity in the gaps at the top of the spectrum:

$$\Delta\Lambda_i\sim\exp(-CL^\sigma), \quad (4.14)$$

with $\sigma=1/2$ since the rescaling factor is $b=4$. From Eq. (4.14) the parallel correlation length of the DW is given by $\xi_\parallel^{DW}\sim(\Delta\Lambda_1)^{-1}\sim\exp(CL^{1/2})$, and thus the transverse fluctuations of the walk grow anomalously, on a logarithmic scale:

$$\langle[X(t)-X(0)]^2\rangle^{1/2}\sim\ln^2(t). \quad (4.15)$$

Here $X(t)$ denotes the position of the walker at time t. We note that the same asymptotic behavior is found in the Sinai model of a one-dimensional random walk in a random environment.[47]

C. Surface critical behavior

We now turn to the renormalization of the surface block, looking for the scaling behavior of the surface temperature t_s. In order to do so, we apply a modified transverse field $h_1=ht_s$ on the first site. As shown in Appendix A the RG transformation now generates an auxiliary variable θ, in terms of which the recursion relations are given by

$$t_s'^2=t_s^2\frac{c-d}{c-dt_s^2}, \quad \theta'^2=\theta^2\frac{c-d}{c-dt_s^2}, \quad (4.16)$$

where c and d are the parameters defined in Eqs. (A2). The auxiliary variable θ, which does not enter into the renormalization of t_s, may be discarded.

For the bulk Ising-fixed-point values of c and d given in Eqs. (4.6), the transformation of the surface temperature gives two surface fixed points with

$$\left.\frac{\partial t_s'^2}{\partial t_s^2}\right|^*=(R^{1/3}+R^{-1/3})^{-2}<1, \quad t_s^{*2}=1, \quad (4.17a)$$

$$\left.\frac{\partial t_s'^2}{\partial t_s^2}\right|^*=(R^{1/3}+R^{-1/3})^2>1, \quad t_s^{*2}=0. \quad (4.17b)$$

Thus the attractive fixed point in the critical surface, corresponding to Eq. (4.17a), governs the surface critical behavior and leads to the scaling dimension of the surface temperature,

$$y_{t_s}=-\frac{\ln(R^{1/3}+R^{-1/3})}{\ln2}, \quad (4.18)$$

in agreement with the conjecture of Ref. 33. The same quantity at the repulsive fixed point, corresponding to Eq. (4.17b), is given by

$$\widetilde{y}_{t_s} = \frac{\ln(R^{1/3} + R^{-1/3})}{2\ln2}. \qquad (4.19)$$

We now consider the critical behavior of the surface magnetization of the aperiodic IM. For a semi-infinite layered system, the surface magnetization is simply given by the first component of the normalized eigenvector $\mathbf{\Phi}_1$ corresponding to the lowest fermionic excitation, which vanishes in the ordered phase:[48]

$$m_s = \Phi_1(1), \quad \sum_{k=1}^{\infty} \Phi_1^2(k) = 1. \qquad (4.20)$$

According to Eq. (2.4), its scaling dimension x_{m_s} can be deduced from the renormalization of the odd components of $\mathbf{V}$ at the Ising fixed point which corresponds to $\Lambda^* = 0$.

Like $V(8k+7)$ in Eqs. (A2), the odd components inside each block can be expressed as functions of $V(8k+1)$ and $V(8k+8)$ using Eqs. (A1b)–(A1g). At the fixed point, taking Eqs. (4.4) into account, one obtains

$$\Phi^*(4k+1) = \Phi'^*(k+1), \quad \Phi^*(4k+2) = R^{1/3}\Phi'^*(k+1),$$

$$\Phi^*(4k+3) = -R^{-1/3}\Phi'^*(k+1),$$

$$\Phi^*(4k+4) = \Phi'^*(k+1). \qquad (4.21)$$

Thus the normalization of $\mathbf{\Phi}^*$ leads to

$$\sum_{k=0}^{\infty} \sum_{l=1}^{4} \Phi^{*2}(4k+l) = (R^{1/3} + R^{-1/3})^2 \sum_{k=0}^{\infty} \Phi'^{*2}(k+1) = 1. \qquad (4.22)$$

Near the critical point, the surface magnetization transforms as

$$m_s' = \frac{\Phi'^*(1)}{\sqrt{\Sigma_{k=0}^{\infty}[\Phi'^*(k+1)]^2}} = b^{x_{m_s}}m_s, \qquad (4.23)$$

so that, using Eq. (4.22), $m_s' = (R^{1/3} + R^{-1/3})m_s$ and, with $b = 4$,

$$x_{m_s} = \frac{\ln(R^{1/3} + R^{-1/3})}{2\ \ln2}, \qquad (4.24)$$

in agreement with an analytical result for the surface magnetization.[28] The thermal and magnetic surface scaling dimensions are related through

$$x_{m_s} = -\frac{1}{2}y_{t_s}, \qquad (4.25)$$

a relation conjectured in Ref. 33. Furthermore, comparing Eqs. (4.24) and (4.19), one may verify that

$$x_{m_s} = \widetilde{y}_{t_s}. \qquad (4.26)$$

These relations, which are generally valid for the IM, will be discussed in Sec. IX.

V. HIERARCHICAL SEQUENCE

A. Definition and general properties

In the generalized hierarchical sequence associated with an integer $m > 1$, the positions k of the digits f_k satisfy the relation[49,31]

$$k = m^{f_k}(ml+p), \quad l = 0,1,\dots, \quad p = 1,2,\dots,m-1. \qquad (5.1)$$

With $m = 2$, the Huberman-Kerszberg sequence is recovered.[50]

We recently noticed that these hierarchical sequences can be also generated via substitution, using an alphabet with an infinite number of letters. Let us put the letters in correspondance with the natural numbers; the f_k's then follow from $n \to S(n)$ with[51]

$$S(n) = \overbrace{0\,0\,\dots\,0}^{m-1}(n+1). \qquad (5.2)$$

Starting with $n = 0$, repeated applications of Eq. (5.2) for $m = 2$ leads to the following sequence at the fourth step:

$$0\underline{1}0\underline{2}0\underline{1}0\underline{3}0\underline{1}0\underline{2}0\underline{1}0\underline{4}. \qquad (5.3)$$

According to Eq. (3.3), it corresponds to the interactions $J_1 = J$, $J_2 = JR$, $J_3 = J$, $J_4 = JR^2,\dots$.

One may notice that the underlined terms f_{2k} give back the original sequence with f_k replaced by $f_k + 1$. The same property remains true for f_{mk} with any value of m so that

$$f_{mk} = f_k + 1, \quad f_{mk+1} = f_{mk+2} = \dots = f_{mk+m-1} = 0. \qquad (5.4)$$

The Ising critical coupling is still given by Eqs. (3.5) and (3.9) where n_L can be evaluated recursively. Using Eqs. (5.4) for a sequence with $L = m^p$, we obtain

$$n_{m^p} = \sum_{k=1}^{m^{p-1}} f_{mk} = n_{m^{p-1}} + m^{p-1}$$

$$= m^{p-1} + m^{p-2} + \dots + 1 = \frac{m^p - 1}{m-1}, \qquad (5.5)$$

which leads to

$$\rho_\infty = \frac{1}{m-1}, \quad \lambda_c = R^{-1/(m-1)}. \qquad (5.6)$$

B. Bulk critical behavior

In the exact renormalization group transformation, we decimate out those sites of the lattice, which have connection by a J coupling. In such a way, blocks of $2(m-1)$ sites are eliminated as indicated by crosses in Fig. 3. Using the reduced variables $\lambda = J/h$ and $\hat{\Lambda} = \Lambda/h$, a lengthy calculation detailed in Appendix B leads to the transformation

$$\hat{\Lambda}' = \hat{\Lambda}\frac{D_{2m-2}}{\lambda^{m-1}} + \frac{D_{2m-3}}{\lambda^{m-1}}, \quad \lambda' = \lambda R\frac{D_{2m-2}}{\lambda^{m-1}}, \qquad (5.7)$$
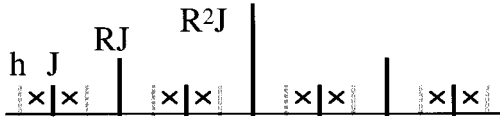
FIG. 3. As in Fig. 2 for the hierarchical sequence with $m=2$.

where the D's are determinants defined in Eq. (B3).

With $m=2$ and $m=3$ one obtains

$$\hat{\Lambda}'=\frac{\hat{\Lambda}}{\lambda^2}(\hat{\Lambda}^2-\lambda^2-1), \quad \lambda'=R(\hat{\Lambda}^2-\lambda^2) \quad (m=2),$$
(5.8a)

$$\hat{\Lambda}'=\frac{\hat{\Lambda}}{\lambda}[(\hat{\Lambda}^2-\lambda^2)^2-2\hat{\Lambda}^2+\lambda^2+1],$$

$$\lambda'=\frac{R}{\lambda}[(\hat{\Lambda}^2-\lambda^2)^2-\hat{\Lambda}^2] \quad (m=3).$$
(5.8b)

The RG transformation in Eqs. (5.7) has an Ising fixed point with $\hat{\Lambda}^*=0$. To study the behavior of the system close to this fixed point, we expand the determinants D_{2m-2} and D_{2m-3} to linear order in $\hat{\Lambda}$:

$$D_{2m-2}=(-\lambda^2)^{m-1}+O(\hat{\Lambda}^2),$$

$$D_{2m-3}=(-1)^{m-1}\hat{\Lambda}\,\frac{1-\lambda^{2m-2}}{1-\lambda^2}+O(\hat{\Lambda}^3).$$
(5.9)

Putting these expressions into Eqs. (5.7), we obtain the location of the Ising fixed point:

FIG. 4. Schematic RG phase diagram for the hierarchical model. There are two nontrivial fixed points on the separatrix, at $\Lambda^*=0$ for the IM and at a nonvanishing value of Λ^* for the DW. The two fixed points generate fixed lines (not shown here), parametrized by the modulation ratio R, in an extended parameter space. These fixed lines govern the marginal behavior of the IM and the DW, respectively. For the period-doubling, three-folding, and paper-folding sequences the scaling is anomalous at the DW fixed point which is shifted to infinity.

$$\hat{\Lambda}^*=0, \quad \lambda^*=(-1)^{m-1}R^{-1/(m-1)}.$$
(5.10)

The eigenvalues of the linearized transformation are given by

$$b^z=\frac{\partial\hat{\Lambda}'}{\partial\hat{\Lambda}}\bigg|^*=\frac{|\lambda^*|^m-|\lambda^*|^{-m}}{|\lambda^*|-|\lambda^*|^{-1}}, \quad b^{y_t}=\frac{\partial\lambda'}{\partial\lambda}\bigg|^*=m,$$
(5.11)

and with $b=m$ one obtains

$$z=\frac{\ln(|R^{m/(m-1)}-R^{-m/(m-1)}|)-\ln(|R^{1/(m-1)}-R^{-1/(m-1)}|)}{\ln m}, \quad y_t=\nu^{-1}=1,$$
(5.12)

in agreement with the conjecture of Ref. 31.

Another fixed point of the transformation with $\hat{\Lambda}^*>0$ governs the critical behavior of the DW as shown in Fig. 4. The position of this fixed point and the values of the corresponding critical exponents can be calculated in a closed form only for $m=2$ and $m=3$.

The DW fixed point for $m=2$ is deduced from Eq. (5.8a) as

$$\hat{\Lambda}^*=\frac{\sqrt{1-R+R^2}}{1-R}, \quad \lambda^*=\frac{R}{1-R}.$$
(5.13)

The leading eigenvalue of the linearized transformation is given by

$$\epsilon_1=R^{-1}+R+\frac{1}{2}+\left[\left(R^{-1}+R+\frac{1}{2}\right)^2-2\right]^{1/2},$$
(5.14)

and thus the wandering exponent of the walk is

$$w=\frac{1}{y_\Lambda}=\frac{\ln 2}{\ln\epsilon_1},$$
(5.15)

since the rescaling factor is $b=2$.

The DW fixed point for $m=3$, which follows from Eq. (5.8b), is located at

$$\hat{\Lambda}^*=\sqrt{\frac{1+R^2}{1-R}}, \quad \lambda^*=\frac{\sqrt{2}R}{1-R}.$$
(5.16)

The leading eigenvalue reads

$$\epsilon_1=2(R^{-1}+R+1)+[4(R^{-1}+R+1)^2-3]^{1/2},$$
(5.17)

and the wandering exponent of the walk is given by

$$w=\frac{1}{y_\Lambda}=\frac{\ln 3}{\ln\epsilon_1},$$
(5.18)

since the rescaling factor is now $b=3$.

C. Surface critical behavior

At the surface of the system we define a modified surface field $h_1 = h t_s$ with scaling dimension y_{t_s} and introduce an auxiliary variable θ to take into account the asymmetry of the renormalized couplings. In terms of the reduced variables, Eqs. (B1) have to be supplemented for $n=0$ with the first two equations

$$-\hat{\Lambda} V(1) + \theta t_s V(2) = 0,$$

$$\frac{t_s}{\theta} V(1) - \hat{\Lambda} V(2) + \lambda V(3) = 0. \tag{5.19}$$

Then, besides the recursion relations of the bulk variables in Eqs. (5.7), we have two more relations for the surface fields:

$$t_s' = t_s \left[\frac{D_{2m-2} + D_{2m-4}}{D_{2m-2} + t_s^2 D_{2m-4}} \right]^{1/2},$$

$$\theta' = \theta \left[\frac{D_{2m-2} + D_{2m-4}}{D_{2m-2} + t_s^2 D_{2m-4}} \right]^{1/2}, \tag{5.20}$$

where D_{2m-4} is defined through $D_{2m-3} = \hat{\Lambda} D_{2m-4}$. Here, as before, the auxiliary variable θ does not enter into the renormalization of t_s and may be discarded.

As one can see from Eqs. (5.20), there are two surface fixed points at $t_s^* = 0$ and at $t_s^* = 1$, from which the latter is stable, both on the IM and the DW critical surfaces. Evaluating the linearized RG transformation around the stable fixed point, one obtains, for the scaling dimension of the surface temperature,

$$y_{t_s} = -\frac{\ln(1 + R^{2/(m-1)} + R^{4/(m-1)} + \cdots + R^2)}{\ln m} \quad \text{(IM)} \tag{5.21}$$

for the IM as expected from numerical results[31] and

$$y_{t_s} = -\frac{\ln R}{\ln m} \quad \text{(DW)}, \tag{5.22}$$

for the DW.

The scaling dimension of the surface temperature for the DW is related to the anomalous diffusion exponent d_w on the hierarchical lattice. According to exact results,[49,38] the mean-square displacement of a diffusive particle in a hierarchical environment is asymptotically given by $\langle X^2(t) \rangle \sim t^{2/d_w}$, where

$$d_w = \begin{cases} 1 - \ln R/\ln m, & R < 1/m, \\ 2, & R > 1/m. \end{cases} \tag{5.23}$$

Thus one has $d_w = 1 + y_{t_s}$ for anomalous diffusion, i.e., with $R < 1/m$.

One can also deduce the scaling dimension of the surface magnetization of the IM from the rescaling of the surface component of the eigenvector $\Phi_1(1)$. Following the same way as for the period-doubling sequence in Sec. IV C, with the fixed-point parameters in Eqs. (5.10), one obtains
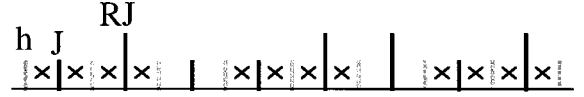
FIG. 5. As in Fig. 2 for the three-folding sequence.

$$x_{m_s} = \frac{\ln(1 + R^{2/(m-1)} + R^{4/(m-1)} + \cdots + R^2)}{2 \ln m} \quad \text{(IM)}, \tag{5.24}$$

in agreement with Ref. 52 for $m=2$ and Ref. 31 for any value of m. One can easily check that the scaling relations in Eqs. (4.25) and (4.26) are satisfied for the hierarchical IM too.

VI. THREE-FOLDING SEQUENCE

A. Definition and general properties

The three-folding sequence[53] is generated through the substitutions $A \to S(A) = ABA$ and $B \to S(B) = ABB$. Starting on A with $A=0$ and $B=1$, at the third iteration, one obtains

$$0100110100100110110100110100. \tag{6.1}$$

The substitution matrix has eigenvalues $\Omega_1 = 3$ and $\Omega_2 = 1$, leading to the wandering exponent $\omega = 0$. The asymptotic density is $\rho_\infty = 1/2$ and gives $\lambda_c = R^{-1/2}$ for the Ising critical coupling. The same sequence is recovered when one keeps every third term, underlined in Eq. (6.1). The digits in between are always 0 and 1 so that the following relations are obtained:

$$f_{3k} = f_k, \quad f_{3k+1} = 0, \quad f_{3k+2} = 1. \tag{6.2}$$

B. Bulk critical behavior

To proceed to the bulk renormalization one considers blocks of six eigenvalue equations from which four, indicated by crosses in Fig. 5, are eliminated so that the rescaling factor is now $b=3$. It is convenient to use the reduced eigenvalue $\widetilde{\Lambda} = \Lambda/J$ and the temperaturelike parameter $\mu = h/J$ as well as an auxiliary variable κ which is needed to take into account the form of the couplings after renormalization. The decimation described in Appendix C leads to the renormalized variables

$$\widetilde{\Lambda}' = \widetilde{\Lambda} \left[\left(1 - \frac{c}{e} \right) \left(1 - \frac{d}{e} \right) \right]^{1/2}, \quad \mu' = \frac{R\mu^3}{e}, \quad \kappa'^2 = \kappa^2 \frac{c-e}{d-e}, \tag{6.3}$$

where c, d, and e are the parameters defined in Eqs. (C2). The components of the eigenvector $\mathbf{V}$ transform as

$$V'(2k) = V(6k), \quad V'(2k+1) = V(6k+1). \tag{6.4}$$

At the IM fixed point

$$\widetilde{\Lambda}^* = 0, \quad \mu^* = R^{1/2}, \tag{6.5}$$

and, according to Eqs. (C2),

$$c^* = -R^2(R+1), \quad d^* = -R(R+1), \quad e^* = R^2.$$
$$(6.6)$$

The auxiliary parameter κ, which does not enter into the renormalization of the physical variables, need not be further considered. The eigenvalues of the linearized RG transformation follow from Eqs. (6.3) with

$$\left.\frac{\partial\widetilde{\Lambda}'}{\partial\widetilde{\Lambda}}\right|^* = [(2+R)(2+R^{-1})]^{1/2}, \quad \left.\frac{\partial\mu'}{\partial\mu}\right|^* = 3. \quad (6.7)$$

Thus, with $b=3$, one obtains the bulk scaling dimensions

$$z = \frac{\ln[(2+R)(2+R^{-1})]}{2\ln 3}, \quad y_t = \nu^{-1} = 1, \quad (6.8)$$

as expected, according to the numerical study of Ref. 33.

The top of the spectrum again scales to infinity ($\Lambda \to \infty$, $J \to \infty$) such that the DW fixed point is located at $\widetilde{\Lambda}^* = 1$, $\mu^* = 0$. The equation of the separatrix is given by $\Delta = \widetilde{\Lambda}^2 - 1 = \gamma^* \mu$ when $\mu \to 0$, i.e., close to the fixed point. The scaling behavior of a point with coordinates $\Delta = \gamma\mu$, $\mu \ll 1$ is of the form $\Delta' \sim \Delta$ and $\mu' \sim \mu^2$, like for the period-doubling sequence. Thus the highest gap in the TM spectrum displays an essential singularity of the form given above in Eq. (4.14) but $\sigma = \ln 2/\ln 3$ since the rescaling factor is now $b=3$. The transverse fluctuations scale similarly to Eq. (4.15).

C. Surface critical behavior

At the surface we define a temperaturelike parameter $\mu_s = h_1/J$ with scaling dimension y_{t_s} and introduce as above an auxiliary variable θ to take into account the asymmetry of the renormalized couplings.

A comparison of Eqs. (C6) with Eqs. (C5) leads to the renormalized parameters

$$\mu_s'^2 = \mu_s^2\left(\frac{\mu^2 R}{e}\right)^2 \frac{c-e}{c(\mu_s/\mu)^2 - e}, \quad \theta'^2 = \theta^2 \frac{c-e}{c(\mu_s/\mu)^2 - e}. \quad (6.9)$$

With the bulk Ising-fixed-point values given in Eqs. (6.5) and (6.6), two surface fixed points are obtained with

$$\left.\frac{\partial\mu_s'^2}{\partial\mu_s^2}\right|^* = (2+R)^{-1} < 1, \quad \mu_s^{*2} = \mu^{*2} = R,$$

$$\left.\frac{\partial\mu_s'^2}{\partial\mu_s^2}\right|^* = (2+R) > 1, \quad \mu_s^{*2} = 0. \quad (6.10)$$

At the stable fixed point $\mu_s^{*2} = R$, with $b=3$, the scaling dimension of the surface temperaturelike parameter reads

$$y_{t_s} = -\frac{\ln(2+R)}{\ln 3}, \quad (6.11)$$

a result previously conjectured on the basis of a finite-size scaling study.[33] The same quantity at the unstable fixed point is given by

$$\widetilde{y}_{t_s} = \frac{\ln(2+R)}{2\ln 3}. \quad (6.12)$$

FIG. 6. As in Fig. 2 for the paper-folding sequence.

The surface magnetization exponent x_{m_s} follows as above from the behavior under renormalization of the odd components $V(6k+1)$, $V(6k+3)$, and $V(6k+5)$ which follows from Eqs. (C1b)–(C1e) and (6.4). At the Ising fixed point Eq. (2.4) leads to

$$\Phi^*(3k+1) = \Phi'^*(k+1),$$

$$\Phi^*(3k+2) = -R^{1/2}\Phi'^*(k+1),$$

$$\Phi^*(3k+3) = \Phi'^*(k+1). \quad (6.13)$$

Making use of Eq. (4.22) with $b=3$, the scaling dimension of the surface magnetization is given by

$$x_{m_s} = \frac{\ln(2+R)}{2\ln 3}, \quad (6.14)$$

in agreement with a direct calculation of the surface magnetization.[33] Again x_{m_s} satisfies the scaling relations (4.25) and (4.26).

VII. PAPER-FOLDING SEQUENCE

A. Definition and general properties

The paper-folding sequence[54] results from the recurrent folding of a sheet of paper onto itself, right over left. After unfolding, one obtains a succession of up and down folds to which one associates a digit, 0 and 1, respectively. After four steps, this process leads to the following sequence:

$$00\underline{1}00\underline{1}1\underline{0}00\underline{1}1\underline{0}1\underline{1}. \quad (7.1)$$

The sequence on the right of the central fold is the mirror image of the left part, with each digit f_k replaced by its complement $1-f_k$. As a consequence, the asymptotic density is $\rho_\infty = 1/2$ and the Ising critical coupling is $\lambda_c = R^{-1/2}$.

The same sequence can be generated using the four letter substitutions $A \to S(A) = AC$, $B \to S(B) = DB$, $C \to S(C) = DC$, and $D \to S(D) = AB$ with the identification $A = 00$, $B = 11$, $C = 10$, and $D = 01$. The leading eigenvalues of the substitution matrix, $\Omega_1 = 2$ and $\Omega_2 = 1$, lead to a vanishing wandering exponent, $\omega = 0$.

The even terms f_{2k}, underlined in Eq. (7.1), reproduce the sequence itself whereas odd terms are alternatively 0 and 1. Thus one has

$$f_{2k} = f_k, \quad f_{2k+1} = \frac{1}{2}[1+(-1)^k]. \quad (7.2)$$

B. Bulk critical behavior

The renormalization of the paper-folding problem is slightly more involved than the preceding ones. In the decimation process, as shown in Fig. 6 one eliminates blocks of two sites which interact alternatively via J or RJ. As a con-

sequence, alternating transverse fields h_α' and h_β' are generated at odd and even lattice sites, respectively. Furthermore, some auxiliary asymmetry parameters are needed to retrieve eigenvalue equations with their original form after renormalization. Altogether the exact RG transformation involves six variables.

The renormalized parameters, following from the decimation process detailed in Appendix D, are given by

$$\widetilde{\Lambda}' = \widetilde{\Lambda}(c_\alpha d_\beta)^{1/2}, \quad \mu_\alpha' = \frac{\mu_\alpha \mu_\beta}{\widetilde{\Lambda}^2 - 1}, \quad \mu_\beta' = \frac{\mu_\alpha \mu_\beta R}{\widetilde{\Lambda}^2 - R^2},$$

$$\kappa'^2 = \kappa^2 \frac{c_\alpha}{d_\beta}, \quad \kappa_\alpha' = \frac{d_\alpha}{c_\alpha}, \quad \kappa_\beta' = \frac{c_\beta}{d_\beta},$$

$$c_i = \left(\kappa_i - \frac{\mu_i^2}{\widetilde{\Lambda}^2 - R^2} \right), \quad d_i = \left(\kappa_i - \frac{\mu_i^2}{\widetilde{\Lambda}^2 - 1} \right), \quad (i = \alpha, \beta).$$

$$(7.3)$$

The fixed point values of interest for the IM are

$$\widetilde{\Lambda}^* = 0, \quad \mu_\alpha^* = -R, \quad \mu_\beta^* = -1, \quad \kappa_\alpha^{*2} = R^2, \quad \kappa_\beta^{*2} = R^{-2}. \tag{7.4}$$

A linearization of the RG transformation, Eqs. (7.3), near this fixed point gives

$$\left. \frac{\partial \widetilde{\Lambda}'}{\partial \widetilde{\Lambda}} \right|^* = [(1+R)(1+R^{-1})]^{1/2},$$

$$\left. \frac{\partial \mu_\alpha'}{\partial \mu_\alpha} \right|^* = \left. \frac{\partial \mu_\beta'}{\partial \mu_\beta} \right|^* = 1, \quad \left. \frac{\partial \mu_\alpha'}{\partial \mu_\beta} \right|^* = R, \quad \left. \frac{\partial \mu_\beta'}{\partial \mu_\alpha} \right|^* = R^{-1}. \tag{7.5}$$

The first line leads to the anisotropy exponent

$$z = \frac{\ln(R^{1/2} + R^{-1/2})}{\ln 2}, \tag{7.6}$$

previously conjectured in Ref. 33, whereas the leading eigenvalue in the linearized transformation of the temperaturelike variables, which is equal to 2, gives the correlation length exponent $\nu = 1$.

C. Surface critical behavior

Using Eqs. (D3) and (D5) together with Eq. (2.4), it may be verified that the normalization of Φ^* here involves two components, $\Phi'^*(2k+1)$ and $\Phi'^*(2k+2)$. Thus the renormalization of the surface magnetization, based on the renormalization of the eigenvectors, becomes equivalent to a direct calculation of m_s. In this case it is more convenient to introduce, besides the surface temperature, a surface field h_s conjugated to m_s in the original Hamiltonian and to study its scaling behavior.

This can be achieved, while keeping the free-fermion character of the Hamiltonian, through the addition of a surface term $-\frac{1}{2} h_s \sigma_0^x \sigma_1^x$ in Eq. (1.2). Since there is no transverse field acting on the first site, σ_0^x, which commutes with $\mathcal{H}$, is conserved. The eigenstates of the Hamiltonian then belong to one of the two sectors corresponding to the eigen-

values ± 1 of σ_0^x. Thus the supplementary term takes the form $\mp \frac{1}{2} h_s \sigma_1^x$ and corresponds to a surface field $\pm h_s$ acting on σ_1^x, the sign depending on the sector.

The decimation of the surface block described in Appendix D gives the renormalized parameters

$$\mu_s'^2 = \mu_s^2 \frac{d_\alpha \mu_\beta^2}{d_s(\widetilde{\Lambda}^2 - 1)^2}, \quad \zeta_s'^2 = \zeta_s^2 \frac{d_\alpha c_\beta}{d_s \kappa_\beta}, \quad \theta'^2 = \theta^2 \frac{d_\alpha}{d_s}, \tag{7.7}$$

where $\zeta_s = h_s / J$ and $\mu_s = h_1 / J$ are reduced surface variables whereas θ takes into account the asymmetry introduced by μ_s. c_i and d_i are the bulk parameters defined previously in Eqs. (7.3) and $d_s = \kappa_\alpha - \mu_s^2 / (\widetilde{\Lambda}^2 - 1)$.

Let us first consider the scaling behavior of μ_s, i.e., of the surface thermal perturbation. With the bulk values given in Eqs. (7.4) one obtains two Ising surface fixed points with

$$\left. \frac{\partial \mu_s'^2}{\partial \mu_s^2} \right|^* = (1+R)^{-1} < 1, \quad \mu_s^{*2} = R^2,$$

$$\left. \frac{\partial \mu_s'^2}{\partial \mu_s^2} \right|^* = (1+R) > 1, \quad \mu_s^{*2} = 0. \tag{7.8}$$

The stable fixed point corresponds to $\mu_s^{*2} = R^2$ and, with $b = 2$, the scaling dimension of the surface temperature is given by

$$y_{t_s} = -\frac{\ln(1+R)}{\ln 2}, \tag{7.9}$$

as expected from numerical results,[33] whereas

$$\widetilde{y}_{t_s} = \frac{\ln(1+R)}{2 \ln 2}, \tag{7.10}$$

at the unstable fixed point.

The stable fixed point values of the parameters in the equation for the surface field variable ζ_s' lead to the transformation

$$\zeta_s'^2 = \zeta_s^2 (1 + R^{-1}). \tag{7.11}$$

Thus, in the extended parameter space, there is a flow from $\zeta_s^{*2} = 0$ to $\zeta_s^{*2} = +\infty$ and the critical behavior is governed by the fixed point with a vanishing surface field which is unstable in the direction of ζ_s^2. Then Eq. (7.11) gives the scaling dimension of the surface field for the Ising problem,

$$y_{h_s} = \frac{\ln(1+R^{-1})}{2 \ln 2} \tag{7.12}$$

or, using Eq. (7.6),

$$x_{m_s} = z - y_{h_s} = \frac{\ln(1+R)}{2 \ln 2}, \tag{7.13}$$

in agreement with the scaling relations (4.25) and (4.26) and the analytical result of Ref. 33.

FIG. 7. As in Fig. 2 for the Fredholm sequence.

VIII. FREDHOLM SEQUENCE

A. Definition and general properties

The Fredholm sequence[54] is obtained via substitution on three letters, $A \rightarrow S(A) = AB$, $B \rightarrow S(B) = BC$, and $C \rightarrow S(C) = CC$. We start the substitution process with A and here, for convenience, we number the sequence starting on $k = 0$. With $A = 0$, $B = 1$, and $C = 0$, after four iterations one obtains the sequence

$$0\underline{1}1\underline{0}10\underline{0}010\underline{0}0000\underline{0}, \tag{8.1}$$

which is the characteristic sequence of the powers of 2, f_k being equal to 1 for $k = 2^p$. Even underlined terms reproduce the sequence and odd terms, except f_1, vanish. This gives the relations

$$f_{2k} = f_k, \quad f_{2k+1} = 0 \quad (k > 0), \quad f_1 = 1. \tag{8.2}$$

The leading eigenvalues of the substitution matrix are $\Omega_1 = 2$ and $\Omega_2 = 1$, and hence the wandering exponent once more vanishes.

The number of digits equal to 1, n_L, grows logarithmically with the length L, and thus the asymptotic density is $\rho_\infty = 0$. The Ising critical coupling in Eq. (3.9) keeps its unperturbed value $\lambda_c = 1$. This aperiodic perturbation modifies the surface critical behavior but does not change the bulk properties, except near line defects which introduce local marginal perturbations in the 2D IM.[55]

B. Bulk critical behavior

The quantum chain is assumed to start on $k = 1$; i.e., we ignore the first digit in the sequence (8.1). As indicated in Fig. 7 in the renormalization process, odd interaction terms J_{2k+1} which, according to Eqs. (8.2), are equal to J in the bulk, are eliminated so that $b = 2$. With the same notation as before for the reduced parameters, the renormalized variables follow from Eqs. (E1) and (E3) with

$$\hat{\Lambda}' = \frac{\hat{\Lambda}}{\lambda}(\hat{\Lambda}^2 - \lambda^2 - 1), \quad \lambda' = (\hat{\Lambda}^2 - \lambda^2), \tag{8.3a}$$

$$V'(2k) = V(4k), \quad V'(2k+1) = V(4k+1). \tag{8.3b}$$

Near the Ising fixed point, corresponding to

$$\hat{\Lambda}^* = 0, \quad \lambda^* = -1, \tag{8.4}$$

the eigenvalues of the linearized transformation

$$\left.\frac{\partial \hat{\Lambda}'}{\partial \hat{\Lambda}}\right|^* = 2, \quad \left.\frac{\partial \lambda'}{\partial \lambda}\right|^* = 2 \tag{8.5}$$

lead to the unperturbed Ising values for the anisotropy and correlation length exponents, $z = \nu = 1$.

C. Surface critical behavior

With the Fredholm sequence, the decimation of the surface block introduces a multiplicative renormalization of the first component of the eigenvector such that (see Appendix E)

$$V'(1) = \theta V(1), \quad \theta^2 = \frac{(\hat{\Lambda}^2 - \lambda^2)(\kappa \hat{\Lambda}^2 - \lambda^2 R^2 - t_s^2)}{(\hat{\Lambda}^2 - \lambda^2 - 1)(\kappa \hat{\Lambda}^2 - \lambda^2 R^2)}. \tag{8.6}$$

The transformation of the other parameters follow from Eqs. (E4) and (E6) as

$$t_s'^2 = t_s^2 \frac{(\hat{\Lambda}^2 - \lambda^2 - 1)(\hat{\Lambda}^2 - \lambda^2)R^2}{(\kappa \hat{\Lambda}^2 - \lambda^2 R^2)(\kappa \hat{\Lambda}^2 - \lambda^2 R^2 - t_s^2)},$$

$$\kappa' = \frac{(\hat{\Lambda}^2 - \lambda^2)(\kappa \hat{\Lambda}^2 - \lambda^2 R^2 - \kappa)}{(\hat{\Lambda}^2 - \lambda^2 - 1)(\kappa \hat{\Lambda}^2 - \lambda^2 R^2)}. \tag{8.7}$$

As before $t_s = h_1/h$ is the surface temperature and κ an auxiliary variable generated by the transformation.

For the Ising fixed-point values of the bulk parameters given in Eqs. (8.4) and $(\kappa \hat{\Lambda}^2)^* = 0$, one obtains two surface fixed points with

$$\left.\frac{\partial t_s'^2}{\partial t_s^2}\right|^* = \frac{2}{R^2}, \quad t_s^{*2} = 0, \quad \theta^* = \frac{1}{2},$$

$$\left.\frac{\partial t_s'^2}{\partial t_s^2}\right|^* = \frac{R^2}{2}, \quad t_s^{*2} = 2 - R^2, \quad \theta^{*2} = \frac{1}{R^2}. \tag{8.8}$$

The first fixed point is stable when $R > R_c = \sqrt{2}$ and, since $t_s^* = 0$ corresponds to a vanishing transverse field on the first spin, the surface is ordered at the critical point. The second fixed point only exists in the regime $R < R_c$ where it is stable. With $b = 2$, the scaling dimension of the surface temperature is given by

$$y_{t_s} = \frac{1}{2} - \frac{\ln R}{\ln 2}, \quad R > \sqrt{2},$$

$$y_{t_s} = -1 + 2\frac{\ln R}{\ln 2}, \quad R < \sqrt{2}. \tag{8.9}$$

These expressions were conjectured in Ref. 32 on the basis of a finite-size scaling study.

The scaling dimension of the surface magnetization follows from the transformation of the odd components of the eigenvector at the appropriate Ising surface fixed point. In the bulk Eqs. (2.4), (E2), and (8.3b) lead to

$$\Phi^*(2k+1) = \Phi^*(2k+2) = \Phi'^*(k+1), \quad k > 0. \tag{8.10}$$

In the surface block, using Eqs. (E5) and (8.6), one obtains

$$\Phi^*(1) = \frac{\Phi'^*(1)}{\theta^*}, \quad \Phi^*(2) = \frac{t_s^* \Phi'^*(1)}{\theta^* R}. \tag{8.11}$$

Thus the normalization of Φ^* gives

$$\sum_{k=0}^{\infty} \sum_{l=1}^{2} \Phi^{*2}(2k+l) = \left(1 + \frac{t_s^{*2}}{R^2}\right) \frac{\Phi'^{*2}(1)}{\theta^{*2}}$$

$$+ 2\sum_{k=1}^{\infty} \Phi'^{*2}(k+1) = 1.$$

(8.12)

According to Eqs. (8.8), the coefficient of $\Phi'^{*2}(1)$ is equal to 2 at both fixed points so that $\Sigma_{k=0}^{\infty}\Phi'^{*2}(k+1) = \frac{1}{2}$. The surface magnetization transforms according to Eq. (4.23), i.e., like

$$m_s' = \sqrt{2}\,\theta^* m_s.$$

(8.13)

With $b=2$, this leads to the following scaling dimensions in the two regimes:

$$x_{m_s} = 0, \quad R > \sqrt{2},$$

$$x_{m_s} = \frac{1}{2} - \frac{\ln R}{\ln 2}, \quad R < \sqrt{2},$$

(8.14)

as given by a direct calculation of the surface magnetization.[32] The value $x_{m_s} = 0$ when $R > R_c$ is consistent with the vanishing surface transverse field at the fixed point. There is surface order when the critical point is approached from the low-temperature phase and, since the surface is one dimensional, the local magnetization vanishes discontinuously when the bulk disorders. When $R < R_c$ the strength of the perturbed couplings is not sufficient to maintain the surface order and the transition is continuous. In this latter case the scaling relations (4.25) and (4.26) are still verified.

D. Aperiodic perturbation in the bulk

Let us consider the aperiodic perturbation which follows from the junction of the Fredholm perturbation in one half-space to its symmetric counterpart in the other, i.e., using the symmetrized sequence

$$\cdots 1000101111010001 \cdots$$

(8.15)

The second half of the sequence is assumed to start on $k = 1$, leaving out the term $k = 0$ in the sequence (8.1) as for the surface perturbation. In this way one obtains a symmetric defect in the bulk with a vanishing asymptotic density so that λ_c remains equal to 1.

The simple relation between the local magnetization and the components of the eigenvector corresponding to the lowest excitation no longer holds in the bulk and one cannot introduce a local field term conjugated to σ^x in the Hamiltonian (1.2) without breaking its free-fermionic character. Thus we shall only consider the renormalization of the local temperaturelike variable $\tau_d = h_1/h$. For the other parameters we keep the same notation as in Sec. VIII B for the surface.

The decimation of the central block of eigenvalue equations is illustrated in Fig. 8. The renormalized local variables follow from Eqs. (E7) and (E8) as

$$\tau_d' = \tau_d \frac{\lambda R^2(\hat{\Lambda}^2 - \lambda^2)}{(\kappa\hat{\Lambda}^2 - \lambda^2 R^2)^2 - \tau_d^2\hat{\Lambda}^2},$$
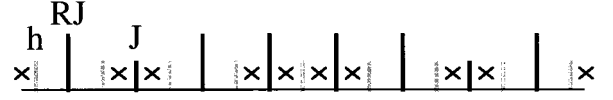

FIG. 8. As in Fig. 2 for the Fredholm defect in the bulk.

$$\kappa' = \frac{\hat{\Lambda}^2 - \lambda^2}{\hat{\Lambda}^2 - \lambda^2 - 1} \left[1 - \frac{\kappa(\kappa\hat{\Lambda}^2 - \lambda^2 R^2) - \tau_d^2}{(\kappa\hat{\Lambda}^2 - \lambda^2 R^2)^2 - \tau_d^2\hat{\Lambda}^2}\right].$$

(8.16)

In the critical surface, with $\hat{\Lambda}^* = 0$ and $\lambda^* = -1$ at the Ising fixed point, the RG transformation of the local variables takes the form

$$\tau_d' = \frac{\tau_d}{R^2}, \quad \kappa' = \frac{1}{2}\left(1 + \frac{\kappa R^2 + \tau_d^2}{R^4}\right).$$

(8.17)

When $R = 1$, $\tau_d' = \tau_d$, which leads to a line of fixed points parametrized by τ_d, $\kappa^* = 1 + \tau_d$. The scaling dimension of the local temperature vanishes as expected for a thermal line defect in the 2D IM.[55]

When $R \neq 1$, two fixed points are obtained with

$$\left.\frac{\partial \tau_d'}{\partial \tau_d}\right|^* = \frac{1}{R^2}, \quad \tau_d^* = 0, \quad \kappa^* = \frac{R^2}{2R^2 - 1},$$

$$\left.\frac{\partial(\tau_d^{-1})'}{\partial(\tau_d^{-1})}\right|^* = R^2, \quad \tau_d^{*-1} = 0, \quad \kappa^{*-1} = 0.$$

(8.18)

In the critical surface, the fixed point at $\tau_d^* = 0$ is stable when $R > 1$. The transverse field at the center of the defect vanishes at this fixed point. Thus the defect is ordered at the critical point, like for the surface, but the critical value of R is now $R_c = 1$ instead of $\sqrt{2}$. There is no need to compensate for missing bonds as it is the case at a surface. When $R < 1$, the second fixed point at $\tau_d^* = +\infty$ becomes stable and leads to a second-order transition at the defect. The appropriate scaling field t_d, associated with the defect temperature, which vanishes at the fixed point is now τ_d^{-1}. Hence, with $b = 2$, Eqs. (8.18) give the following scaling dimensions in the regimes of first- and second-order local transition, respectively:

$$y_{t_d} = -2\frac{\ln R}{\ln 2}, \quad R > 1,$$

$$y_{t_d} = 2\frac{\ln R}{\ln 2}, \quad R < 1.$$

(8.19)

IX. RELATIONS BETWEEN ISING MODEL CRITICAL EXPONENTS

Apart from the correlation length exponent $\nu = 1$, all the critical exponents obtained for the different aperiodic models are varying with the amplitude of the modulation, and thus the critical behavior of these models is nonuniversal. However, some kind of ''weak universality'' still holds and there are relations between critical exponents which follow from the fact that the systems at the critical point obey anisotropic

scaling.[44] A detailed analysis of the scaling behavior can be found in Ref. 33.

One can notice other exponent relations which are specific for the marginally aperiodic IM's. One such relation connects the scaling dimensions of the energy and magnetization densities at the surface as

$$x_{e_s} = z - y_{t_s} = z + 2x_{m_s}. \tag{9.1}$$

It follows from Eq. (4.25) and anisotropic scaling. It was conjectured in Ref. 33 on the basis of an assumption for the scaling behavior of $\Phi(1)$ for low-lying excitations. One can find another relation which surprisingly connects bulk and surface quantities in the form

$$z = x_{m_s} + \bar{x}_{m_s}, \tag{9.2}$$

where $\bar{x}_{m_s}$ is the scaling dimension of the surface magnetization on the right-hand side (RHS) of the system. Here we argue that the relation in Eq. (9.2) is generally true for marginally aperiodic layered IM's.

In the following derivation, we consider the quantum Ising chain Hamiltonian $\mathcal{H}$ given in Eq. (1.2) with homogeneous transverse field $h_k = 1$. The dynamical exponent of the model z is related to the scaling behavior of the lowest gap of the spectrum of the critical Hamiltonian in the form

$$E_1 - E_0 = \Lambda_1 \sim L^{-z}, \tag{9.3}$$

in a finite system of size L.

The asymptotic size dependence of $\Lambda_1(L)$ is calculated in the following approximation. First we determine the leading k dependence of the eigenvectors $\Phi_1(k)$ and $\Psi_1(k)$ from Eqs. (2.3) in such a way that the RHS's of the equations are omitted. This approximation is justified, at the critical point or in the ordered phase, by the fact that the second difference operators on the LHS of the equations are $O(L^{-2})$ whereas Λ_1^2 on the RHS is $O(L^{-2z})$ with $z > 1$ for marginal aperiodic systems at criticality or exponentially small in the ordered phase. In this approximation we obtain

$$\Phi_1(L+1-k) \simeq \Phi_1(L) \prod_{i=1}^{k-1} (-\lambda_{L-i}) \left[1 + \sum_{i=1}^{k-1} \prod_{j=1}^{i} \lambda_{L-j}^{-2} \right],$$

$$\Psi_1(k) \simeq \Psi_1(1) \prod_{i=1}^{k-1} (-\lambda_i) \left[1 + \sum_{i=1}^{k-1} \prod_{j=1}^{i} \lambda_j^{-2} \right]. \tag{9.4}$$

Then the size dependence of Λ_1 is estimated from the linear equations in Eqs. (2.2) as

$$\Lambda_1(L) = -\frac{\Psi_1(1)}{\Phi_1(1)} \simeq -\frac{\Psi_1(1)}{\Phi_1(L)} \prod_{i=1}^{L-1} (-\lambda_i)^{-1}$$

$$\times \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{L-j}^{-2} \right]^{-1}, \tag{9.5a}$$

$$\Lambda_1(L) = -\frac{\Phi_1(L)}{\Psi_1(L)} \simeq -\frac{\Phi_1(L)}{\Psi_1(1)} \prod_{i=1}^{L-1} (-\lambda_i)^{-1}$$

$$\times \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^{-2} \right]^{-1}. \tag{9.5b}$$

Multiplying both sides of Eqs. (9.5a) and (9.5b), one arrives at the result

$$\Lambda_1(L) \sim m_s(L) \bar{m}_s(L) \prod_{i=1}^{L-1} \lambda_i^{-1}, \tag{9.6}$$

where the finite-size surface magnetizations on both sides of the system are given by[48]

$$m_s(L) = \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_j^{-2} \right]^{-1/2},$$

$$\bar{m}_s(L) = \left[1 + \sum_{i=1}^{L-1} \prod_{j=1}^{i} \lambda_{L-j}^{-2} \right]^{-1/2}. \tag{9.7}$$

The relation in Eq. (9.6), which connects the asymptotic behavior of the lowest excitation energy and the finite-size behavior of the surface magnetizations, is valid for general distributions of the couplings, provided the lowest gap in the system goes to zero faster than $1/L$.

In the following, we apply Eq. (9.6) to marginally aperiodic systems at the critical point where, according to rigorous results,[18] $\Pi_{i=1}^{L-1}(\lambda_i)_c = O(1)$ for aperiodic perturbations leading to a shift of the critical coupling. The finite-size surface magnetizations behave as $m_{sc}(L) \sim L^{-x_{m_s}}$ and $\bar{m}_{sc}(L) \sim L^{-\bar{x}_{m_s}}$ and thus, from Eqs. (9.6) and (9.3), one obtains the scaling relation given in Eq. (9.2).

The aperiodic sequences studied in this paper which change the bulk critical behavior are of two kinds: either symmetric with $\lambda_k = \lambda_{L-k}$ (period doubling) or such that a perturbed coupling at k corresponds to an unperturbed coupling at $L-k$, which leads to $\rho_\infty = 1/2$ and, according to Eq. (3.9), $[\lambda_k(R)]_c = [\lambda_{L-k}(R^{-1})]_c$ (paper folding, three folding). For symmetric sequences, $\bar{m}_{sc}(L,R) = m_{sc}(L,R)$, and therefore $\bar{x}_{m_s} = x_{m_s}$. Otherwise, $\bar{m}_{sc}(L,R) = m_{sc}(L,R^{-1})$ and, consequently, $\bar{x}_{m_s}(R) = x_{m_s}(R^{-1})$. *Thus the knowledge of a single exponent $x_{m_s}(R)$ is sufficient to obtain all the varying exponents studied in this paper.* Furthermore, for the period-doubling sequence $x_{m_s}(R)$ is symmetric under the exchange of R into R^{-1} according to Eq. (4.24). It follows that for all the aperiodic sequences one may rewrite the nonuniversal anisotropy exponent in Eq. (9.2) as $z(R) = x_{m_s}(R) + x_{m_s}(R^{-1})$.

For marginal aperiodic sequences which do not change the bulk critical behavior, i.e., leave $z = 1$ and $\lambda_c = 1$, the scaling relation (9.1) does not hold. In this case $\Pi_{i=1}^{L-1}(\lambda_i)_c = R^{-n_L}$ with the number of perturbed couplings growing logarithmically with L. For the Fredholm sequence $n_L = \ln L/\ln 2$ so that the product of the couplings in Eq. (9.6) scales as $L^{-\ln R/\ln 2}$. When the left surface is ordered at the critical point, i.e., for $R > R_c = \sqrt{2}$, and the right surface is free, we have $x_{m_s} = 0$, $\bar{x}_{m_s} = 1/2$ and the lowest excitation does not scale as L^{-1} like the rest of the spectrum, but with a continuously varying exponent:

$$\Lambda_1 \sim L^{-1/2 - \ln R/\ln 2}. \tag{9.8}$$

When the surface magnetization vanishes at the critical point ($R < R_c$), the R dependence of x_{m_s} in Eqs. (8.14) just com-

pensates that appearing in the product of the couplings and one recovers the normal L^{-1} behavior for Λ_1.

Finally, according to Eq. (4.26), the scaling dimension of the surface magnetization x_{m_s} is equal to the scaling dimension of the surface temperature $\widetilde{y}_{t_s}$ at the unstable fixed point. This last relation is a consequence of the self-duality of the Ising quantum chain.[16] Using the dual Pauli spin matrices defined through

$$\tau_k^z = \sigma_{k-1}^x \sigma_k^x, \quad \sigma_k^z = \tau_k^x \tau_{k+1}^x, \tag{9.9}$$

the original Hamiltonian in Eq. (1.2) is transformed into its dual:

$$\widetilde{\mathcal{H}} = -\frac{1}{2} \sum_{k=1}^{L} h_k \tau_k^x \tau_{k+1}^x - \frac{1}{2} \sum_{k=1}^{L-1} J_k \tau_{k+1}^z, \tag{9.10}$$

with a vanishing transverse field on the first spin. As already shown at the beginning of Sec. VII C, τ_1^x commutes with $\widetilde{\mathcal{H}}$ and may be replaced by its eigenvalues ± 1. Thus, in the surface term $-\frac{1}{2} h_1 \tau_1^x \tau_2^x = \mp \frac{1}{2} h_1 \tau_2^x$, $h_1 = h t_s$ now plays the role of a surface field acting on τ_2^x.

The unstable fixed point at $t_s^* = 0$, with its associated scaling dimension $\widetilde{y}_{t_s}$, governs the critical behavior of the dual surface magnetization. In the duality transformation the couplings $\lambda_k = h/J_k$ are changed into λ_k^{-1} so that the surface magnetizations on both sides in Eqs. (9.7) are exchanged. It follows that the scaling dimension of $\langle \tau_2^x \rangle$ is $\bar{x}_{m_s}$ and the dimension of the surface field is given by $\widetilde{y}_{t_s} = z - \bar{x}_{m_s}$, which, according to Eq. (9.2), leads to Eq. (4.26).

X. CONCLUSION

In this paper we have presented a unified statistical-mechanical description of the IM and the DW on layered two-dimensional lattices, taking the extreme anisotropic limit for the IM. The critical properties of the two problems were deduced from the scaling behavior of the spectrum of the transfer matrix of the DW, which is studied through exact RG transformations. For a given value of the aperiodicity parameter R the RG transformations have two nontrivial fixed points, as shown in Fig. 4. The bottom of the spectrum scales to the IM fixed point, which controls the critical behavior of the IM, whereas the top of the spectrum scales to another fixed point, which describes the critical properties of the DW.

The aperiodic sequences we considered have different effects on the critical properties of the two models according to Luck's relevance-irrelevance criterion described in Sec. III.[56] For the IM, the crossover exponent in Eq. (3.7) with $\nu = 1$ is $\phi_{IM} = 0$ whereas for the DW, with $\nu_\perp = 1/2$, it is $\phi_{DW} = 1/2$. Consequently, the nonperiodic perturbation is marginal at the homogeneous ($R = 1$) IM fixed point whereas it is relevant for the $R = 1$ DW fixed point. These statements are in accordance with the exact results.

For the IM, the marginal perturbation creates a line of fixed points, which is parametrized by R, and the critical properties are continuously varying, even at $R = 1$. The non-periodicity also induces a continuously varying anisotropic scaling behavior. However, the different varying exponents

are not independent: Knowledge of the scaling dimensions of the surface magnetization is sufficient to completely describe the nonuniversal critical behavior studied in this work.

Considering the DW problem, here the line of fixed points is discontinuous at $R = 1$, in accordance with the relevant nature of the perturbation. For the hierarchical models the line of fixed points is characterized by finite coordinates and the corresponding critical behavior is of power-law form with R-dependent exponents. On the other hand, for the aperiodic models (period doubling, three folding, and paper folding) the line of DW fixed points is shifted to infinity and the scaling behavior is anomalous: The transverse fluctuations of the walk grow on a logarithmic scale.

Finally we discuss the local critical behavior at extended defects, located either at the surface or in the bulk, which are generated by the Fredholm sequence. In both cases two fixed points exchange there stability at a critical value R_c of the modulation amplitude R. This critical value separates two regimes for the local transition: For $R > R_c$ the local magnetization vanishes discontinuously at the bulk critical point while for $R < R_c$ the transition is continuous. In both cases one obtains critical exponents which vary continuously with the marginal parameter R.

It has been already noticed[32] that the surface Fredholm perturbation is closely connected to the Hilhorst–van Leeuwen model.[57] In the same way, the bulk Fredholm defect is connected to the Bariev model.[58] In these models, the perturbation of the couplings decays as a power of the distance l from the center of the defect with $\delta\lambda(l) = \alpha l^{-1}$ in the marginal case, for the 2D IM.

The varying exponents obtained analytically and numerically in Ref. 32 for the surface Fredholm perturbation as well as those otained via exact RG transformations in Sec. VIII for the surface and bulk Fredholm defects can be put in correspondance with the exponents of the Hilhorst–van Leeuwen and Bariev models with α replaced by $\ln R/\ln 2$. Up to now, the values of the Bariev model's exponents had been conjectured on the basis of conformal methods using gap-exponent relations after a conformal transformation of the inhomogeneous infinite system onto an inhomogeneous infinite strip with periodic boundary conditions.[59] Our RG results for the bulk Fredholm defect and the correspondence between both models strongly support the validity of this procedure.

APPENDIX A: RENORMALIZATION OF THE PERIOD-DOUBLING SEQUENCE

The renormalization of the period-doubling sequence in the bulk involves the following set of equations:

$$\lambda R^{f_{4k}} V(8k) - \hat{\Lambda} V(8k+1) + V(8k+2) = 0, \tag{A1a}$$

$$V(8k+1) - \hat{\Lambda}V(8k+2) + \lambda V(8k+3) = 0, \quad \text{(A1b)}$$

$$\lambda V(8k+2) - \hat{\Lambda}V(8k+3) + V(8k+4) = 0, \quad \text{(A1c)}$$

$$V(8k+3) - \hat{\Lambda}V(8k+4) + \lambda R V(8k+5) = 0, \quad \text{(A1d)}$$

$$\lambda R V(8k+4) - \hat{\Lambda}V(8k+5) + V(8k+6) = 0, \quad \text{(A1e)}$$

$$V(8k+5) - \hat{\Lambda}V(8k+6) + \lambda V(8k+7) = 0, \quad \text{(A1f)}$$

$$\lambda V(8k+6) - \hat{\Lambda}V(8k+7) + \lambda V(8k+8) = 0, \quad \text{(A1g)}$$

$$V(8k+7) - \hat{\Lambda}V(8k+8) + \lambda R^{f_{4k+4}} V(8k+9) = 0. \quad \text{(A1h)}$$

Among these eight equations we eliminate the six central ones, which amounts to rescale the system by a factor $b = 4$. This is accomplished by evaluating $V(8k+2)$ and $V(8k+7)$ as functions of $V(8k+1)$ and $V(8k+8)$, in the linear system given by Eqs. (A1b)–(A1g), with the result

$$V(8k+2) = \frac{d\hat{\Lambda}}{c} V(8k+1) + \frac{R\lambda^3}{c} V(8k+8),$$

$$V(8k+7) = \frac{R\lambda^3}{c} V(8k+1) + \frac{d\hat{\Lambda}}{c} V(8k+8),$$

$$c = \hat{\Lambda}^2(\hat{\Lambda}^2 - \lambda^2 - 1)^2 - \lambda^2 R^2 (\hat{\Lambda}^2 - \lambda^2)^2,$$

$$d = (\hat{\Lambda}^2 - 1)(\hat{\Lambda}^2 - \lambda^2 - 1) - \lambda^2 R^2 (\hat{\Lambda}^2 - \lambda^2). \quad \text{(A2)}$$

Inserting these values into Eqs. (A1a) and (A1h), after multiplication by $c/(R\lambda^3)$ we obtain

$$\frac{c}{R\lambda^2} R^{f_{4k}} V(8k) - \hat{\Lambda} \frac{c-d}{R\lambda^3} V(8k+1) + V(8k+8) = 0,$$

$$V(8k+1) - \hat{\Lambda} \frac{c-d}{R\lambda^3} V(8k+8) + \frac{c}{R\lambda^2} R^{f_{4k+4}} V(8k+9) = 0, \quad \text{(A3)}$$

which are the renormalized equations

$$\lambda' R^{f_k} V'(2k) - \hat{\Lambda}' V'(2k+1) + V'(2k+2) = 0,$$

$$V'(2k+1) - \hat{\Lambda}' V'(2k+2) + \lambda' R^{f_{k+1}} V'(2k+3) = 0, \quad \text{(A4)}$$

after rescaling by $b = 4$. Noticing that, according to Eqs. (4.2), $f_{4l} = 1 - f_{2l} = f_l$, R remains unchanged and one obtains the RG transformation as given in Eq. (4.3).

At the surface, in terms of the reduced variables, the same set of equations as in Eqs. (A1) with $k = 0$ is obtained, except for the two first equations which now read

$$-\hat{\Lambda} V(1) + \theta t_s V(2) = 0, \quad \text{(A5a)}$$

$$\frac{t_s}{\theta} V(1) - \hat{\Lambda} V(2) + \lambda V(3) = 0. \quad \text{(A5b)}$$

The auxiliary variable θ is needed to take into account the asymmetry resulting from the renormalization after one step. In this way the variables $\hat{\Lambda}$, λ, t_s, and θ build a closed set under renormalization.

As above, the components $V(2)$ and $V(7)$ can be deduced from the six central equations and read

$$V(2) = \frac{d\hat{\Lambda} t_s}{c\theta} V(1) + \frac{R\lambda^3}{c} V(8),$$

$$V(7) = \frac{R\lambda^3 t_s}{c\theta} V(1) + \frac{d\hat{\Lambda}}{c} V(8), \quad \text{(A6)}$$

where c and d are defined in Eqs. (A2). Equations (A5a) and (A1h), after multiplication by appropriate factors, then give

$$-\hat{\Lambda} \frac{c-d}{R\lambda^3} V(1) + \theta t_s \frac{c-d}{c - dt_s^2} V(8) = 0,$$

$$\frac{t_s}{\theta} V(1) - \hat{\Lambda} \frac{c-d}{R\lambda^3} V(8) + \frac{c}{R\lambda^2} V(9) = 0. \quad \text{(A7)}$$

These equations give the renormalized forms of Eqs. (A5a) and (A5b) and provide the RG recursions given in Eq. (4.16).

APPENDIX B: RENORMALIZATION OF THE HIERARCHICAL SEQUENCE

In the bulk, the set of eigenvalue equations we consider is the following:

$$\lambda R^n V(2m^n) - \hat{\Lambda} V(2m^n + 1) + V(2m^n + 2) = 0, \quad \text{(B1a)}$$

$$V(2m^n + 1) - \hat{\Lambda} V(2m^n + 2) + \lambda V(2m^n + 3) = 0, \quad \text{(B1b)}$$

$$\lambda V(2m^n + 2) - \hat{\Lambda} V(2m^n + 3) + V(2m^n + 4) = 0, \quad \text{(B1c)}$$

$$\vdots$$

$$\lambda V(2m^n + 2m - 2) - \hat{\Lambda} V(2m^n + 2m - 1) + V(2m^n + 2m)$$
$$= 0, \quad \text{(B1d)}$$

$$V(2m^n + 2m - 1) - \hat{\Lambda} V(2m^n + 2m) + \lambda R V(2m^n + 2m + 1)$$
$$= 0. \quad \text{(B1e)}$$

Among the $2m$ equations one eliminates the $2m - 2$ central ones, which amounts to rescaling the system by a factor of $b = m$. Then we are left with two equations between the components $V(2m^n)$, $V(2m^n + 1)$, $V(2m^n + 2m)$, and $V(2m^n + 2m + 1)$ of the form

$$\frac{\lambda R^n}{r} V(2m^n) - \frac{\hat{\Lambda} - s}{r} V(2m^n + 1) + V(2m^n + 2m) = 0,$$

$$V(2m^n + 1) - \frac{\hat{\Lambda} - s}{r} V(2m^n + 2m) + \frac{\lambda R}{r} V(2m^n + 2m + 1)$$
$$= 0. \quad \text{(B2)}$$

Here $r=\lambda^{m-1}/D_{2m-2}$, $s=-D_{2m-3}/D_{2m-2}$, and D_{2m-2} denotes the $(2m-2)\times(2m-2)$ determinant

$$D_{2m-2}=\begin{vmatrix} -\hat{\Lambda} & \lambda & & & & \\ \lambda & -\hat{\Lambda} & 1 & & & \\ & 1 & -\hat{\Lambda} & \lambda & & \\ & & & \ddots & & \\ & & & 1 & -\hat{\Lambda} & \lambda \\ & & & & \lambda & -\hat{\Lambda} \end{vmatrix}, \quad (B3)$$

while D_{2m-3} is the lower central minor of D_{2m-2}. Then from Eqs. (B2) we deduce the RG transformation given in Eqs. (5.7).

APPENDIX C: RENORMALIZATION OF THE THREE-FOLDING SEQUENCE

The eigenvalue equations take the following form in the bulk:

$$R^{f_{3k}}V(6k)-\kappa\widetilde{\Lambda}V(6k+1)+\mu V(6k+2)=0, \quad (C1a)$$

$$\mu V(6k+1)-\frac{\widetilde{\Lambda}}{\kappa}V(6k+2)+V(6k+3)=0, \quad (C1b)$$

$$V(6k+2)-\kappa\widetilde{\Lambda}V(6k+3)+\mu V(6k+4)=0, \quad (C1c)$$

$$\mu V(6k+3)-\frac{\widetilde{\Lambda}}{\kappa}V(6k+4)+RV(6k+5)=0, \quad (C1d)$$

$$RV(6k+4)-\kappa\widetilde{\Lambda}V(6k+5)+\mu V(6k+6)=0, \quad (C1e)$$

$$\mu V(6k+5)-\frac{\widetilde{\Lambda}}{\kappa}V(6k+6)+R^{f_{3k+3}}V(6k+7)=0. \quad (C1f)$$

Equations (C1b)–(C1e) can be used to write

$$V(6k+2)=\frac{c\kappa\widetilde{\Lambda}}{e\mu}V(6k+1)+\frac{R\mu^2}{e}V(6k+6),$$

$$V(6k+5)=\frac{R\mu^2}{e}V(6k+1)+\frac{d\widetilde{\Lambda}}{e\kappa\mu}V(6k+6),$$

$$c=\mu^2(\widetilde{\Lambda}^2-\mu^2-R^2), \quad d=\mu^2(\widetilde{\Lambda}^2-\mu^2-1),$$

$$e=(\widetilde{\Lambda}^2-1)(\widetilde{\Lambda}^2-R^2)-\mu^2\widetilde{\Lambda}^2, \quad (C2)$$

which, inserted into Eqs. (C1a) and (C1f), lead to the renormalized equations

$$R^{f_{3k}}V(6k)-\kappa\widetilde{\Lambda}\left(1-\frac{c}{e}\right)V(6k+1)+\frac{R\mu^3}{e}V(6k+6)=0,$$

$$\frac{R\mu^3}{e}V(6k+1)-\frac{\widetilde{\Lambda}}{\kappa}\left(1-\frac{d}{e}\right)V(6k+6)+R^{f_{3k+3}}V(6k+7)$$

$$=0. \quad (C3)$$

Since $f_{3l}=f_l$ according to Eqs. (6.2), these equations take the form

$$R^{f_k}V'(2k)-\kappa'\widetilde{\Lambda}'V'(2k+1)+\mu'V'(2k+2)=0,$$

$$\mu'V'(2k+1)-\frac{\widetilde{\Lambda}'}{\kappa'}V'(2k+2)+R^{f_{k+1}}V'(2k+3)=0, \quad (C4)$$

with the renormalized variables given in Eqs. (6.3)

At the surface, Eqs. (C1c)–(C1f) with $k=0$ have to be supplemented by

$$-\kappa\widetilde{\Lambda}V(1)+\theta\mu_s V(2)=0,$$

$$\frac{\mu_s}{\theta}V(1)-\frac{\widetilde{\Lambda}}{\kappa}V(2)+V(3)=0. \quad (C5)$$

Rewriting $V(2)$ and $V(5)$ as functions of $V(1)$ and $V(6)$ in the first and last equations of the surface block, one obtains the renormalized equations

$$-\kappa\widetilde{\Lambda}\left(1-\frac{c}{e}\right)V(1)+\frac{\theta\mu_s\mu^2 R}{e}\frac{c-e}{c(\mu_s/\mu)^2-e}V(6)=0,$$

$$\frac{\mu_s\mu^2 R}{\theta e}V(1)-\frac{\widetilde{\Lambda}}{\kappa}\left(1-\frac{d}{e}\right)V(6)+V(7)=0. \quad (C6)$$

A comparison with Eqs. (C5) leads to the renormalized parameters given in Eqs. (6.9).

APPENDIX D: RENORMALIZATION OF THE PAPER-FOLDING SEQUENCE

The following blocks have to be considered:

$$R^{f_{4k}}V(8k)-\kappa_\alpha\kappa\widetilde{\Lambda}V(8k+1)+\mu_\alpha V(8k+2)=0, \quad (D1a)$$

$$\mu_\alpha V(8k+1)-\frac{\widetilde{\Lambda}}{\kappa}V(8k+2)+V(8k+3)=0, \quad (D1b)$$

$$V(8k+2)-\kappa\widetilde{\Lambda}V(8k+3)+\mu_\beta V(8k+4)=0, \quad (D1c)$$

$$\mu_\beta V(8k+3)-\frac{\kappa_\beta\widetilde{\Lambda}}{\kappa}V(8k+4)+R^{f_{4k+2}}V(8k+5)=0, \quad (D1d)$$

when the central interaction is J and

$$R^{f_{4k+2}}V(8k+4)-\kappa_\alpha\kappa\widetilde{\Lambda}V(8k+5)+\mu_\alpha V(8k+6)=0, \quad (D2a)$$

$$\mu_\alpha V(8k+5)-\frac{\widetilde{\Lambda}}{\kappa}V(8k+6)+RV(8k+7)=0, \quad (D2b)$$

$$RV(8k+6)-\kappa\widetilde{\Lambda}V(8k+7)+\mu_\beta V(8k+8)=0, \quad (D2c)$$

$$\mu_\beta V(8k+7) - \frac{\kappa_\beta\widetilde{\Lambda}}{\kappa} V(8k+8) + R^{f_{4k+4}}V(8k+9) = 0,$$
$$\text{(D2d)}$$

when the central interaction is RJ. $\mu_\alpha = h_\alpha/J$ and $\mu_\beta = h_\beta/J$ are reduced temperaturelike parameters and $\widetilde{\Lambda}$ the reduced eigenvalue defined before. Since in each block two sites out of four are eliminated, lengths are rescaled by a factor $b=2$.

The two intermediate equations in Eqs. (D1) and (D2) give

$$V(8k+2) = \frac{\kappa\mu_\alpha\widetilde{\Lambda}}{\widetilde{\Lambda}^2-1} V(8k+1) + \frac{\mu_\beta}{\widetilde{\Lambda}^2-1} V(8k+4),$$

$$V(8k+3) = \frac{\mu_\alpha}{\widetilde{\Lambda}^2-1} V(8k+1) + \frac{\kappa^{-1}\mu_\beta\widetilde{\Lambda}}{\widetilde{\Lambda}^2-1} V(8k+4),$$

$$V(8k+6) = \frac{\kappa\mu_\alpha\widetilde{\Lambda}}{\widetilde{\Lambda}^2-R^2} V(8k+5) + \frac{\mu_\beta R}{\widetilde{\Lambda}^2-R^2} V(8k+8),$$

$$V(8k+7) = \frac{\mu_\alpha R}{\widetilde{\Lambda}^2-R^2} V(8k+5) + \frac{\kappa^{-1}\mu_\beta\widetilde{\Lambda}}{\widetilde{\Lambda}^2-R^2} V(8k+8),$$
$$\text{(D3)}$$

which can be used in the first and last lines of Eqs. (D1) and (D2), together with the first relation in Eqs. (7.2), to write the renormalized equations

$$R^{f_{2k}}V'(4k) - \kappa'_\alpha\kappa'\widetilde{\Lambda}'V'(4k+1) + \mu'_\alpha V'(4k+2) = 0,$$

$$\mu'_\alpha V'(4k+1) - \frac{\widetilde{\Lambda}'}{\kappa'} V'(4k+2) + R^{f_{2k+1}}V'(4k+3) = 0,$$

$$R^{f_{2k+1}}V'(4k+2) - \kappa'\widetilde{\Lambda}'V'(4k+3) + \mu'_\beta V'(4k+4) = 0,$$

$$\mu'_\beta V'(4k+3) - \frac{\kappa'_\beta\widetilde{\Lambda}'}{\kappa'} V'(4k+4) + R^{f_{2k+2}}V'(4k+5) = 0.$$
$$\text{(D4)}$$

Here the components of the eigenvectors transform according to

$$V'(4k) = V(8k), \quad V'(4k+1) = V(8k+1),$$

$$V'(4k+2) = V(8k+4), \quad V'(4k+3) = V(8k+5).$$
$$\text{(D5)}$$

The renormalized parameters are given in Eqs. (7.3).

The surface field $h_s = J\zeta_s$ introduces a supplementary equation in the surface block which now begins with

$$-\frac{\kappa_\beta\widetilde{\Lambda}}{\kappa} V(0) + \zeta_s V(1) = 0,$$

$$\zeta_s V(0) - \kappa_\alpha\kappa\widetilde{\Lambda}V(1) + \theta\mu_s V(2) = 0,$$

$$\frac{\mu_s}{\theta} V(1) - \frac{\widetilde{\Lambda}}{\kappa} V(2) + V(3) = 0, \qquad \text{(D6)}$$

where, as before, $\mu_s = h_1/J$ is a temperaturelike surface variable and θ an auxiliary variable.

The first equation in Eqs. (D6) gives the value of $V(0)$ which can be used in the second equation to obtain a surface block in its standard form:

$$-\kappa\widetilde{\Lambda}\left(\kappa_\alpha - \frac{\zeta_s^2}{\kappa_\beta\widetilde{\Lambda}^2}\right)V(1) + \theta\mu_s V(2) = 0,$$

$$\frac{\mu_s}{\theta} V(1) - \frac{\widetilde{\Lambda}}{\kappa} V(2) + V(3) = 0. \qquad \text{(D7)}$$

The two remaining equations are given by (D1c) and (D1d) with $k=0$. As usual, writing $V(2)$ and $V(3)$ as functions of $V(1)$ and $V(4)$, one obtains the renormalized equations

$$-d_\alpha\kappa\widetilde{\Lambda}\left(1 - \frac{\zeta_s^2}{d_s\kappa_\beta\widetilde{\Lambda}^2}\right)V(1) + \frac{\theta\mu_s d_\alpha\mu_\beta}{d_s(\widetilde{\Lambda}^2-1)} V(4) = 0,$$

$$\frac{\mu_s\mu_\beta}{\theta(\widetilde{\Lambda}^2-1)} V(1) - \frac{d_\beta\widetilde{\Lambda}}{\kappa} V(4) + V(5) = 0, \qquad \text{(D8)}$$

where $d_s = \kappa_\alpha - \mu_s^2/(\widetilde{\Lambda}^2-1)$. When compared to Eqs. (D7) they lead to the renormalized parameters given in Eqs. (7.7).

APPENDIX E: RENORMALIZATION OF THE FREDHOLM SEQUENCE

We have to consider the following block of equations:

$$\lambda R^{f_{2k}}V(4k) - \hat{\Lambda}V(4k+1) + V(4k+2) = 0, \quad \text{(E1a)}$$

$$V(4k+1) - \hat{\Lambda}V(4k+2) + \lambda V(4k+3) = 0, \quad \text{(E1b)}$$

$$\lambda V(4k+2) - \hat{\Lambda}V(4k+3) + V(4k+4) = 0, \quad \text{(E1c)}$$

$$V(4k+3) - \hat{\Lambda}V(4k+4) + \lambda R^{f_{2k+2}}V(4k+5) = 0. \qquad \text{(E1d)}$$

Equations (E1b) and (E1c) give the eigenvector components

$$V(4k+2) = \frac{\hat{\Lambda}}{\hat{\Lambda}^2-\lambda^2} V(4k+1) + \frac{\lambda}{\hat{\Lambda}^2-\lambda^2} V(4k+4),$$

$$V(4k+3) = \frac{\lambda}{\hat{\Lambda}^2-\lambda^2} V(4k+1) + \frac{\hat{\Lambda}}{\hat{\Lambda}^2-\lambda^2} V(4k+4),$$
$$\text{(E2)}$$

which can be used to rewrite the first and last equations as

$$(\hat{\Lambda}^2-\lambda^2)R^{f_{2k}}V(4k) - \frac{\hat{\Lambda}}{\lambda}(\hat{\Lambda}^2-\lambda^2-1)V(4k+1)$$

$$+ V(4k+4) = 0,$$

$$V(4k+1) - \frac{\hat{\Lambda}}{\lambda}(\hat{\Lambda}^2-\lambda^2-1)V(4k+4)$$

$$+ (\hat{\Lambda}^2-\lambda^2)R^{f_{2k+2}}V(4k+5) = 0. \qquad \text{(E3)}$$

Since $f_{2k} = f_k$ and $f_{2k+2} = f_{k+1}$, the renormalized equations keep their original form with the transformed parameters given by Eqs. (8.3).

The surface block reads

$$-\hat{\Lambda}V(1) + t_s V(2) = 0, \qquad (E4a)$$

$$t_s V(1) - \kappa\hat{\Lambda}V(2) + \lambda R V(3) = 0, \qquad (E4b)$$

$$\lambda R V(2) - \hat{\Lambda}V(3) + V(4) = 0, \qquad (E4c)$$

$$V(3) - \hat{\Lambda}V(4) + \lambda R V(5) = 0. \qquad (E4d)$$

where the auxiliary variable κ takes into account the change of the intermediate interaction which is now λR instead of λ for the bulk equations in Eqs. (E1). From Eqs. (E4b) and (E1c) we deduce the eigenvector components

$$V(2) = \frac{t_s \hat{\Lambda}}{\kappa\hat{\Lambda}^2 - \lambda^2 R^2} V(1) + \frac{\lambda R}{\kappa\hat{\Lambda}^2 - \lambda^2 R^2} V(4),$$

$$V(3) = \frac{t_s \lambda R}{\kappa\hat{\Lambda}^2 - \lambda^2 R^2} V(1) + \frac{\kappa\hat{\Lambda}}{\kappa\hat{\Lambda}^2 - \lambda^2 R^2} V(4), \qquad (E5)$$

which are used to rewrite Eqs. (E4a) and (E4d) as

$$-\hat{\Lambda}'\theta V(1) + t_s' V(4) = 0,$$

$$t_s'\theta V(1) - \kappa'\hat{\Lambda}'V(4) + \lambda' R V(5) = 0. \qquad (E6)$$

In these renormalized equations θ can be interpreted as a renormalization factor for $V(1)$ which transforms according

to Eqs. (8.6). The RG equations for the other parameters are given in Eqs. (8.7).

In the bulk, the block of eigenvalue equations to be renormalized, corresponding to the center of the defect, is the following:

$$\lambda R V(-2) - \hat{\Lambda}V(-1) + V(0) = 0, \qquad (E7a)$$

$$V(-1) - \hat{\Lambda}V(0) + \lambda R V(1) = 0, \qquad (E7b)$$

$$\lambda R V(0) - \kappa\hat{\Lambda}V(1) + \tau_d V(2) = 0, \qquad (E7c)$$

$$\tau_d V(1) - \kappa\hat{\Lambda}V(2) + \lambda R V(3) = 0, \qquad (E7d)$$

$$\lambda R V(2) - \hat{\Lambda}V(3) + V(4) = 0, \qquad (E7e)$$

$$V(3) - \hat{\Lambda}V(4) + \lambda R V(5) = 0. \qquad (E7f)$$

With $\lambda = 0$ in Eq. (E7c), the three last equations differ from the surface equations in Eqs. (E4) only through the auxiliary factor κ in the first one which is necessary to preserve the symmetry of the block.

Expressing $V(0)$ and $V(3)$ in terms of $V(-1)$ and $V(4)$, the first and last equations of the block take the same form as the two central ones,

$$\lambda' R V(-2) - \kappa'\hat{\Lambda}'V(-1) + \tau_d' V(4) = 0,$$

$$\tau_d' V(-1) - \kappa'\hat{\Lambda}'V(4) + \lambda' R V(5) = 0, \qquad (E8)$$

with the renormalized local parameters given by Eqs. (8.16).

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