Spin-liquid phases in two-dimensional frustrated XY models

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In this paper we consider the J_1-J_2-J_3 classical and quantum two-dimensional XY model. Spin-wave calculations show that a spin-liquid phase still exists in the quantum case as for Heisenberg models. We formulate a semiclassical approach of these models based on spin-wave action and use a variational method to study the role played by vortices. Liquid and crystal phases of vortex could emerge in this description. These phases seem to be directly correlated with the spin-liquid phase and to its crystalline interpretation. [S0163-1829(97)00838-2]

I. INTRODUCTION

Low-dimensional magnetic systems have been extensively studied these last years since they are known to exhibit nontrivial quantum behaviors. This comes in particular from the fact that classical Heisenberg two-dimensional (2D) antiferromagnets can order only at zero temperature so that quantum fluctuations cannot *a priori* be neglected. In fact, it is now well known that in antiferromagnets, the quantum fluctuations can ($S<S_{\rm critical}$) or cannot ($S>S_{\rm critical}$) induce a transition at $T=0$ from a Néel ordered ground state to a quantum disordered one, characterized by short-range order.[1] The enhancement of quantum fluctuations has naturally been looked for in frustrated spin systems. The combination of frustration and quantum fluctuations can possibly lead to a spin-liquid state (for a review about spin-liquid states, see for example, Ref. 2). For the 2D Heisenberg model with nearest-neighbor (NN) and next-nearest-neighbors (NNN) antiferromagnetic (AF) interactions, Chandra and Douçot have shown by spin-wave calculations that it can be disordered at $T=0$.[3] This has been confirmed by exact diagonalizations on finite lattices,[4,2] by series expansions,[5] and by a renormalization-group analysis of the associated nonlinear σ models.[6–8] The latter study has shown that the couplings of the models flow under renormalization-group transformations towards a strong-coupling regime when this liquid phase is approached.[8] Einarsson and Johannesson[6] have shown that precisely close to this liquid state, there is a proliferation of tological excitations in the path-integral representation of the frustrated Heisenberg model.[9] They have suggested from these instanton considerations possible realizations of this disordered liquid phase in analogy with works of Sachdev *et al.*[10]

In this article, we want to emphasize that there exists a system where it is likely that this relationship between topological excitations and this liquid state is much more direct, namely the quantum frustrated 2D XY model. We have indeed shown in a preceding paper that a 2D classical XY model with NN and NNN interactions has a point in the $(J_2/J_1,T)$ plane, where $T_{\rm KT}=0$.[11] This result can be extended by adding a next-next-nearest-neighbor (NNNN) interaction with a whole line of Kosterlitz-Thouless transitions at $T=0$, in the parameter space $(J_2/J_1,J_3/J_1)$. This line is precisely that around which, quantically, a spin-liquid phase is found. We hypothesize in this article, that while quantum fluctuations can surely not be *a priori* neglected in this case, they are, together with vortices, responsible for a spin-liquid state in this frustrated XY model.

The paper is organized as follows: In Sec. II, we consider the classical XY model with competing interactions. The classical phase diagram at $T=0$ is recalled. It contains no spin-liquid phase. Then, we pay special attention to the role played by vortices and show that there is a critical line where vortices are allowed at $T=0$. This proves the need to incorporate quantum fluctuations. Indeed, in Sec. III, we show that linear quantum spin-wave computations predict a spin-liquid phase (around this critical line) in the quantum frustrated XY model contrary to the classical model. Then, in Sec. IV, we formulate a general Ginzburg-Landau-Wilson (GLW) semiclassical action from the quantum spin-wave action in order to discuss the nature of this liquid phase. When approaching this phase, the spin stiffness, $\kappa(S)$ gets very small and we are *a priori* forced to take into account the effects of all quartic terms. We pay particular attention to the nonperturbative sector of this action. We find a phase where pairs of vortices can be confined at a short finite lattice distance. The phase described here is some kind of liquid of vortex. This liquid state is favored when spin waves become softer [when $\kappa(S)\to 0$] i.e., when the usual spin-liquid phase is predicted. Finally, Sec. V contains a summary of the results and some concluding remarks.

II. CLASSICAL FRUSTRATED XY MODELS

The purpose of this section is to study the role played by vortices in classical frustrated XY models, especially in the weakly frustrated phase. Let us first consider the 2D classical J_1-J_2, XY model

$$H=-J_1\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j)+J_2\sum_{\langle\langle k,l\rangle\rangle}\cos(\theta_k-\theta_l), \quad (2.1)$$

with $J_1,J_2>0$ and θ_i the angles associated with the classical $O(2)$ spin $\vec{S}_i$. $\langle\ \rangle$ is for nearest neighbors (NN) and $\langle\langle\ \rangle\rangle$ for next-nearest neighbors (NNN). Two ground states are possible in the model. When $|J_1|>2J_2$ the ground state is ferromagnetic, whereas, when $|J_1|<2J_2$, the ground state consists of two independent diagonal sublattices with antiferromagnetic order.[14]

From a spin-wave approximation, an effective action can be derived in the ferromagnetic phase as

$$\mathcal{A}_1=\frac{1}{2T}\int d^2x\{(J_1-2J_2)(\nabla\theta)^2+J_2(\nabla_x\nabla_y\theta)^2\}. \tag{2.2}$$

The classical vacuum is the standard ferromagnetic one, and therefore the term $(\nabla_x\nabla_y\theta)^2$ is irrelevant according to standard perturbative arguments. In this case, the system can be well approximated by an XY model with an effective NN coupling constant J_1-2J_2, so that the action associated with a neutral pair of vortices is

$$S_0=4\pi(J_1-2J_2)\ln\left(\frac{\rho}{a}\right), \tag{2.3}$$

with ρ the distance separating the two vortices. Therefore, there is a Kosterlitz-Thouless transition at the temperature $T_{\mathrm{KT}}=\pi/2(J_1-2J_2)$. This result can be proved more rigorously by a Coulomb gas treatment of the original Hamiltonian (2.1).[11] It has to be noticed that the pairs of vortices play the role of the instantons in the nonlinear σ model.[12] When $J_1=2J_2, T_{\mathrm{KT}}=0$, a result which indicates that vortex solutions are allowed in the classical vacuum at $T=0$. This suggests that the model can have a nontrivial behavior around this point when quantum fluctuations are considered. When $J_1<2J_2$, the behavior is drastically different; the action (2.2) is somewhat meaningless in this ''antiferromagnetic'' phase and a new effective theory must be found.[11] Nevertheless, if we suppose that the notion of a pair of vortices still makes sense in this case, the action associated with this pair becomes negative so that it becomes energetically favorable to fill the vacuum with pairs of vortices. Indeed, it has to be noticed that the antiferromagnetic ground state defined above (for $J_1<\frac{1}{2}J_2$) can be interpreted as a lattice of vortex antivortex (for more details, see Ref. 11). To summarize, we have seen that the action associated with a pair of vortices indicates the changes in the classical vacuum and especially that its value equals zero at the Lifshitz point.

This analysis can be extended when a new coupling constant $J_3>0$, corresponding to a (NNNN) AF interaction, is added to the action (2.1). When $J_1-2J_2-4J_3>0$, the classical vacuum is a standard ferromagnetic one. Let us consider the isotropic case ($J_2=2J_3$) for convenience, the physics along the whole line $J_1-2J_2-4J_3=0$ being the same.[8] The associated action, again using a spin-wave approximation, can be written as

$$\mathcal{A}_2=\int d^2x\frac{1}{2T}\{A(\nabla\theta)^2+B(\nabla^2\theta)^2\}, \tag{2.4}$$

where $A=(J_1-8J_3), B=J_3$. In the nonisotropic case, the extra term $(J_2-2J_3)(\nabla_x\nabla_y\theta)^2$ has to be added to the action (2.4) and does not change qualitatively the results. The saddle-point equation is

$$A\Delta\theta-B\Delta^2\theta=0, \tag{2.5}$$

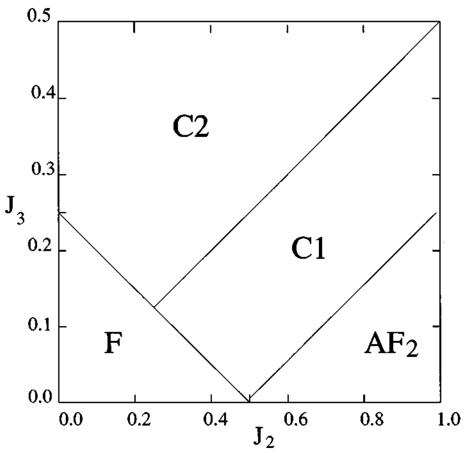

FIG. 1. The classical phase diagram for the $J_1-J_2-J_3$ XY model on a square lattice. F corresponds to a ferromagnetic ground state, AF_2 to two decoupled sublattices with independent AF order and C_1, C_2 two incommensurate chiral phases.

with $\Delta=\nabla^2$ the Laplacian. The neutral pair of vortices is a solution of Eq. (2.5) and its associated action reads $S_0=4\pi(J_1-8J_3)\ln(\rho/a)$. There is no dependence in B, so this result indicates that we are in a situation similar to the previous model. Since $S_0=0$ at the critical line $J_1-2J_2-4J_3=0$, vortex excitations are present at $T=0$ along the whole line suggesting that the quantum fluctuations have to be included. Moreover, along this line, quadratic terms vanish. The propagator is then governed by quartic terms and so is of short-range order favoring a disordered state at $T=0$. When $J_1-2J_2-4J_3<0$, one has to know which wave vector minimizes the spin-wave action (2.4) in order to obtain the whole phase diagram. By this method, we obtain the same phase diagram at the classical level as for Heisenberg spins (see Ref. 15). This has been reported for completeness in Fig. 1. At low J_3 and $J_2>J_1/2$, we recover the phase with independent AF order on each sublattice (noted AF_2) as in the J_1-J_2 model. The critical line separates the ferromagnetic phase from two helical incommensurate phases with respective wave vectors $(\pi,\pm Q_1)$, $(\pm Q_1,\pi)$ (phase C_1) and $(\pm Q_2,\pm Q_2)$ (phase C_2), where Q_1, Q_2 are defined by $\cos(Q_1)=(2J_2-J_1)/4J_3$ and $\cos(Q_2)=-J_1/(2J_2+4J_3)$ (see Fig. 1).

We have seen that vortex excitations are allowed on the classical critical line $J_1-2J_2-4J_3=0$. Nevertheless, no liquid phase has been found around it. Consequently, frustration is not sufficient at the classical level to induce a spin-liquid state and quantum fluctuations must be included.

III. LINEAR SPIN-WAVE THEORY IN QUANTUM FRUSTRATED XY MODELS

We consider now the quantum version of these models. We present in this section spin-wave results that are well known for the Heisenberg model[3,15] but not, to my knowledge, for XY spins. The Hamiltonian can be written as

$$H=\sum_{i,j} J_{ij}(S_i^xS_j^x+S_i^yS_j^y), \tag{3.1}$$

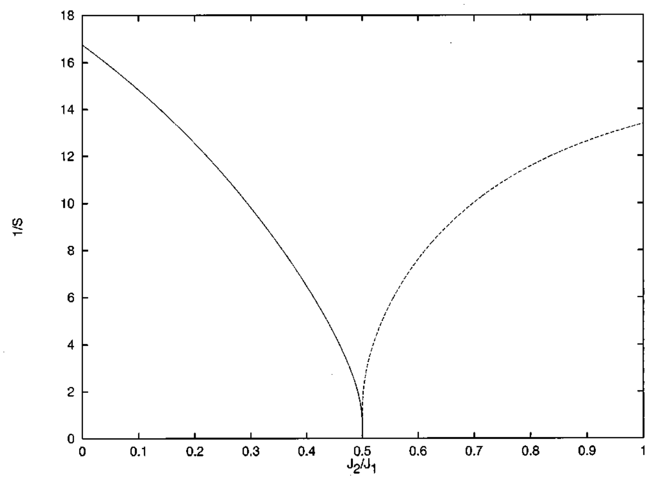

FIG. 2. First quantum corrections to the lattice magnetization in the J_1-J_2 XY model. The $O(1/S)$ spin-wave theory predicts an intermediary region where the ground state is nonmagnetic and thus can be a spin-liquid phase.

where $J_{ij}=J_1$ for a NN interaction, $J_{ij}=J_2$ for a NNN interaction, and $J_{ij}=J_3$ for a NNNN interaction. The zero-point fluctuations around different ordered states can be computed in the large S limit by different methods: a standard one with Holstein-Primakov bosons (see, for example, Ref. 15), and a second one with ''a semipolar'' representation of spin operators introduced by Villain.[16] Both give the same results though the latter seems more appropriate for XY models. Let us first consider the J_1-J_2, XY model (with $J_1<0, J_2>0$).

When $J_1>\frac{1}{2}J_2$ the ground state is ferromagnetic. The magnetization $\langle S^z\rangle$ reads

$$\langle S^z\rangle=S+\frac{1}{2}-\frac{1}{8\pi^2}\int d^2k \times\frac{2-\eta_k-\alpha_2(2-\eta_k')}{\{[2-\eta_k-\alpha_2(2-\eta_k')]^2-(\eta_k-\alpha_2\eta_k')^2\}^{1/2}}. \tag{3.2}$$

We have defined $\alpha_2=J_2/J_1$, $\eta_k=\frac{1}{2}[\cos(k_x)+\cos(k_y)]$, and $\eta_k'=\cos(k_x)\cos(k_y)$. The integral over k runs over the first Brillouin zone $[-\pi,\pi]^2$ of the two-dimensional lattice. We see that the mean-field value of S^z is decreased by including the first corrections in $1/S$.

When $J_1<\frac{1}{2}J_2$ a collinear ground state is selected by spin waves.[14] The same kind of calculation can be generalized, paying attention to the anisotropy, and leads to

$$\langle S^z\rangle=S+\frac{1}{2}-\frac{1}{8\pi^2}\int d^2k \times\frac{|2-\cos(k_x)\cos(k_y)-\lambda_2[\cos(k_x)-\cos(k_y)]|}{2\{1-\cos(k_x)\cos(k_y)-\lambda_2[\cos(k_x)-\cos(k_y)]\}^{1/2}}, \tag{3.3}$$

where $\lambda_2=|J_1|/2J_2$.

In Fig. 2, we represent spin-wave corrections as a function of J_2. We can draw conclusions analogous to those found in the study of Heisenberg spins,[3,15] namely linear spin-wave theory, which predicts at any S a finite region around the Lifshitz point where the ground state is disordered (see Fig. 2). This seems to indicate the presence of a spin-liquid state.

Furthermore, the quantum fluctuations diverge when $\alpha_2=J_2/|J_1|\rightarrow 1/2$ as

$$\begin{aligned}\langle S^z\rangle&\sim S+\frac{1}{2}-\gamma\ln^2(1-2\alpha_2), \quad \alpha_2<\frac{1}{2}\\ \langle S^z\rangle&\sim S+\frac{1}{2}-\gamma'(2\alpha_2-1)^{-1/2}, \quad \alpha_2>\frac{1}{2},\end{aligned} \tag{3.4}$$

where γ and γ' are two unimportant real numbers.

We find asymptotic behaviors similar to Heisenberg spins. This is not in fact so surprising since the Lifshitz point is the same. The validity of this first-order approximation compared to other methods like Schwinger bosons mean-field theory has been discussed in Ref. 8. Second-order spin-wave calculations go along the same line and confirm the first-order calculations (it also leads to a renormalization of the spin stiffness). Moreover, some exact diagonalizations on finite lattices agree qualitatively with the existence of a spin-liquid state around the Lifshitz point.[4,2]

A similar analysis can be performed for the $J_1-J_2-J_3$, XY model. We are interested in the region around the classical critical line $J_1-2J_2-4J_3=0$ which separates the ferromagnetic phase from two helical phases as we have seen in Sec. II. The Villain method can be easily generalized for chiral phases.[16] The pitch wave vector $\vec{Q}_{0,cl}$ is defined by $\partial_Q J_{Q_0,cl}=0$ with J_Q being the Fourier transform associated with J_{ij}. The spin-wave frequency and the staggered magnetization are in this case

$$\omega_k=2\sqrt{J(Q_0)[J(Q_0)-J(Q_0+k)+J(Q_0-k)/2]}, \tag{3.5}$$

$$\langle S^z\rangle=S+\frac{1}{2}-\frac{1}{8\pi^2}\int d^2k \times\frac{|-J(Q_0)+J(Q_0+k)+J(Q_0-k)/4|}{\sqrt{J(Q_0)[J(Q_0)-J(Q_0+k)+J(Q_0-k)/2]}}. \tag{3.6}$$

Along the critical line $\kappa_{cl}=J_1-2J_2-4J_3$, the staggered magnetization has the same kind of divergences as in the J_1-J_2 model. For example, we evaluate at the point $J_2=0$ the leading divergence

$$\langle S^z\rangle\sim S+\frac{1}{2}-\alpha\ln(J_1-4J_3). \tag{3.7}$$

This behavior is similar for the whole critical line (apart from the Lifshitz point). We notice that we recover directly spin-wave calculations in the ferromagnetic phase by taking $Q_0=0$. It supports the fact that we can go continuously from the ferromagnetic phase to the chiral phases as is already the case classically. These states have the same symmetries contrary to the case of the ferromagnetic and collinear phases separated by a Lifshitz point, and thus both states can be described by Eqs. (3.5), (3.6). Since results for XY and

Heisenberg spins are very similar at this order, it is reasonable to think that higher-order corrections will have similar effects like a renormalization of the spin stiffness $\kappa_R(S)$. It can be used to define the quantum critical line as $\kappa_R(S)=0$.[8] Therefore, to summarize this section, we have shown that spin waves predict the presence of a spin-liquid phase around $\kappa_R(S)=0$ generated by the combination of quantum fluctuations and frustration.

IV. A SEMICLASSICAL TREATMENT OF QUANTUM FRUSTRATED *XY* MODELS

In this section, we want to find an effective action able to describe the quantum behavior of the 2D, $J_1-J_2-J_3$, XY model especially near the critical line. The most rigorous way to find an effective action for spin models is to decompose the pure spin Hamiltonian on a basis of coherent states, which enables a path-integral description of a single spin.[9,6] Another possibility is to formulate directly the most general effective action from the spin-wave action and symmetry considerations. The latter strategy has been already used by Ferrer to study the scaling properties of the 2D $J_1-J_2-J_3$ quantum Heisenberg model.[8] We follow a similar strategy.

In our case, we have seen from the spin-wave analysis that quartic terms coming from NNN and NNNN interactions have to be included near the critical line since quadratic terms vanish on this line. Therefore, the spin waves have a propagator of the following form: $\mathcal{P}_{\mathrm{SW}}^{-1}=a\vec{k}^2+b\vec{k}^4$ (in the isotropic case). Following Amit *et al.*,[13] we wonder what physical systems could be described by a theory having the anomalous propagator k^2+k^4 and how it could be relevant for the understanding of the quantum phase diagram of the 2D $J_1-J_2-J_3$ XY model. Therefore, we are interested in the sequel in the most general Landau-Ginzburg-Wilson action for an XY spin $\vec{S}$, with such an anomalous propagator

$$\mathcal{A}=\int d^2x[A|\nabla\vec{S}|^2+B|\nabla^2\vec{S}|^2+B'|\nabla\vec{S}|^4+\mathcal{V}(\vec{S})],\qquad(4.1)$$

where $\mathcal{V}(\vec{S})=r_0|\vec{S}|^2+\lambda|\vec{S}|^4$. This effective action has a purely classical origin. Indeed, $A=(J_1-2J_2-4J_3)$, $B=J_3$, and $B'=0$ correspond to the classical spin-wave action (2.4). We will now make an apparently ''crude'' approximation, namely we suppose that the quantum action has a form similar to Eq. (4.1) except an extra dimension and a renormalization of coupling constants, such that the values given above for A,B,B' are no longer valid. Therefore, the quantum model takes *a priori* different values in the space of coupling constants.

Nevertheless, we have an important constraint from spin-wave calculations: the quantum critical line is always characterized by $A\sim\kappa_r(S)\rightarrow 0$. Under the above hypothesis, this implies that the main differences between the classical and quantum situations rely on the behavior of higher derivative terms. It has to be noticed that similar effective actions have been derived in Refs. 7 and 8 for frustrated Heisenberg models. We do not claim that this simple effective action contains all the physics associated with the spin-liquid phase but nevertheless, we will see that it can be useful to understand the role played by instantonic sector when the critical line corresponding to $A\sim\kappa(S)\rightarrow 0$ is approached.

We suppose the spin S to be large and analyze phase fluctuations as it is common for XY spins. Therefore, we write $S(r)=S_0e^{i\theta(r)}$ where S_0 is spatially uniform and depends on the parameters of the model (see spin-wave calculations). In fact, it just corresponds to the large-S version of the Villain semipolar representation.[16] Fluctuations around S_0 will be discussed later in the article. In this case, the action (4.1) reads

$$\mathcal{A}=\int d^2x[AS_0^2(\nabla\theta)^2+BS_0^2a^2(\nabla^2\theta)^2+CS_0^2a^2(\nabla\theta)^4],\qquad(4.2)$$

with $C=B+B'S_0^2$. The lattice spacing a is introduced in order to have dimensionless coupling constants. The potential part has been omitted because the main effects we will study involve the derivative part. Normally, the operator $|\partial_\mu\vec{S}\partial_\nu\vec{S}|^2$ should have been included in the original action (4.1) but, since it gives a similar contribution as $|\nabla\vec{S}|^4$, it has also been omitted. To study this action, it seems useless to use perturbative arguments, because, first the critical line $[A\sim\kappa(S)\rightarrow 0]$ is characterized by a strong-coupling regime,[8] and second the higher operators are irrelevant according to the usual perturbation scheme. Because of the nonlinear term $(\nabla\theta)^4$, it is difficult to find the saddle points analytically. Nevertheless, in Sec. II, we have noticed that the action associated with a vortex-antivortex pair, characterized by the relative distance ρ, indicates the changes of the ground state when approaching the critical line (moreover, it is a genuine solution when $C=0$). Therefore, we use a variational approach based on a wave function of pairs of vortices defined by

$$\theta_0=\arctan\frac{y-y_1}{x-x_1}-\arctan\frac{y-y_2}{x-x_2},\qquad(4.3)$$

with $\vec{r}_1=(x_1,y_1),\vec{r}_2=(x_2,y_2)$ the position of both vortices $(\rho^2=|\vec{r}_1-\vec{r}_2|^2)$. Introducing θ_0 in Eq. (4.2), we have to compute

$$I=\int\int dr(\nabla\theta_0)^4=\int\int d^2\vec{r}\frac{\rho^4a^2}{|\vec{r}-\vec{r}_1|^4|\vec{r}-\vec{r}_2|^4}.\qquad(4.4)$$

Of course, we must regularize the integrals. The lattice constant a is the most natural regulator in our case. The computation is made in the Appendix. We obtain the following action for the pair of vortices:

$$\mathcal{A}_V(\rho)=4\pi S_0^2\left\{A\ln\left(\frac{\rho}{a}\right)+C\left[4\ln\left(\frac{\rho}{a}\right)\frac{a^2}{\rho^2}-\alpha_2\frac{a^2}{\rho^2}\right]\right\},\qquad(4.5)$$

with $\alpha_2>0$ a real number. When $A>0$ and $C>0$, this action is minimum for $\rho=a$ (our cutoff). Namely, we have a standard attractive increasing potential and charges tend to form dipoles and so do not contribute. The classical situation corresponds to $A=\kappa_{\mathrm{clas}}>0C=J_3>0$, hence in the weakly frustrated region, the vortices do not contribute except on the classical critical line. The above conclusions are not altered in the anisotropic case. Nevertheless, when $C/A<C_0/A<0$, the situation becomes totally different. We can see in Fig. 3 that the potential has now a nontrivial minimum ρ_0. This corresponds to a liquid phase of vortices. Moreover, when $C/A<C_1/A<C_0/A$ the action associated with the pair of vortices can be negative, meaning that it becomes energetically favorable to fill the semiclassical vacuum with pairs of

vortices separated by a size ρ_0 of order $2a$ (see Fig. 3). This result does not hardly depend on the ratio C/A. It corresponds to a crystalline phase of vortices. Note that the possible role of some higher-order derivative terms in the nonperturbative sector has been already studied in lattice gauge theories in Ref. 17 where some rather similar conclusions have been drawn. Before commenting on these results, let us first perform one-loop calculations, in order to see the contributions of such vortex effects to the path integral.

In this perspective, we use the semiclassical approximation and expand the field θ around θ_0 as $\theta=\theta_0+\eta$ and keep terms at most quadratic in η in the action (4.1). The partition function reads

$$Z=\int \mathcal{D}\eta \exp(-S_0(\rho))\exp\left(-\int dx[A(\nabla\eta)^2 + B(\nabla^2\eta)^2+2C(\nabla\eta)^2(\nabla\theta_0)^2]\right). \tag{4.6}$$

The action in terms of η after integration by parts reads

$$\mathcal{A}(\eta)=A\eta\Box\eta-B\eta\Box^2\eta+2C\eta[(\nabla\theta_0)^2\Box-(\partial_\mu\partial^\nu\theta_0)^2]\eta. \tag{4.7}$$

After the Gaussian integration, it remains to compute the determinant associated with the differential operator

$$\mathcal{O}=A\Box-B\Box^2+2C\Box(\nabla\theta_0)^2-2C(\partial_\mu\partial^\nu\theta_0)^2. \tag{4.8}$$

Of course, it seems impossible to obtain the spectrum associated with this operator. Nevertheless, we can use the fact that $(\partial_\mu\partial^\nu\theta_0)^2$ and $(\nabla\theta_0)^2$ are ultralocal functions of $\vec{r}$ ($\rho_0\sim a$) whose asymptotic behaviors are, respectively, in $1/r^6$ and $1/r^4$. Therefore, one can approximate the fluctuation determinant by $\det[A\Box-B\Box^2]$, which just corresponds to the determinant associated with quantum spin waves. To be more explicit, we write

$$\det[\mathcal{O}]=\det[\mathcal{P}]\det[1-\mathcal{P}^{-1}(-2C(\partial_\mu\partial^\nu\theta_0)^2)], \tag{4.9}$$

with $\mathcal{P}=A\Box-B\Box^2+2C\Box(\nabla\theta_0)^2$. The second determinant can be treated perturbatively. Indeed, asymptotically

$$\mathcal{P}^{-1}(-2C(\partial_\mu\partial^\nu\theta_0)^2(\vec{r}))\sim\int d^2\vec{r}_1\frac{\ln|\vec{r}-\vec{r}_1|}{r_1^6},$$

and so does not contribute. Similar work can be performed on $\mathcal{P}$ using the ultralocality of $(\nabla\theta_0)^2$ justifying the approximation. (The treatment of $\mathcal{P}$ is equivalent to solving a Schrödinger equation in a regularized attractive potential in $1/r^4$. There will be just a few bound states plus the continuum spectrum similar to the case $C=0$.[18]) In the partition function, we now have to integrate over ρ, ρ playing now the role of the collective coordinate.[19] The only scale and translationally invariant measure is $d^2r d\rho\rho^{-3}$.[20] In that case, the partition function reads

$$Z=V\int_0^{+\infty}\frac{d\rho}{\rho^3}e^{-S_0(\rho)}[\det\mathcal{O}(\rho)]^{-1/2}, \tag{4.10}$$

where a is the lattice cutoff. An interesting quantity often used is the ratio of the partition function in the unit and zero winding number. In fact, this ratio measures the weight of

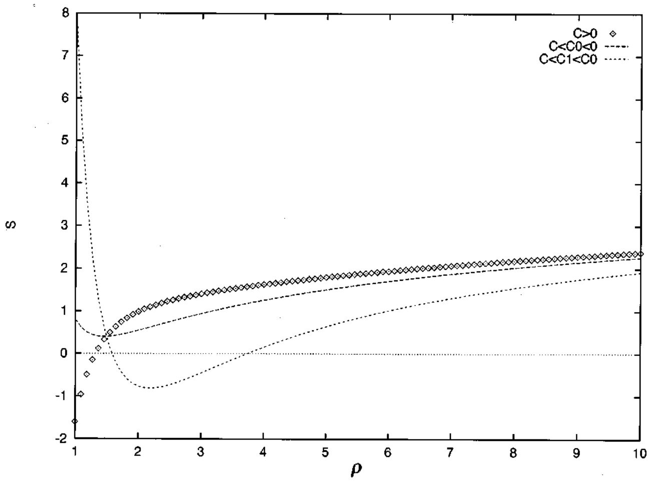

FIG. 3. The action associated with a pair of vortices separated by a distance ρ. Three cases have been represented: a positive action without any extremum, a positive action with a minimum in $\rho=\rho_0$ and a negative action with a similar minimum around $\rho=2a$.

singular solutions in the path-integral compared to spin waves. It will be clearly dominated by the tree sector, in so far as $\det[\mathcal{O}(\rho)]\sim\det[A\Box-B\Box]$.

The last integral defined in Eq. (4.10) is carried out in the saddle-point approximation around $\rho=\rho_0$. It yields

$$Z=Ve^{-S_0(\rho_0)}\int_0^{+\infty}\frac{d\rho}{\rho^2}\times e^{-2\pi S_0^2 g(A,C,\rho_0)(1/\rho^2-1/\rho_0^2)^2}[\det\mathcal{O}(\rho_0)]^{-1/2}, \tag{4.11}$$

where $g(A,C,\rho_0)$ is a positive function defined by the second derivative of the vortex action at $\rho=\rho_0$. By performing the Gaussian integration over ρ, the result is

$$Z=\lambda Ve^{-S_0(\rho_0)}[\det\mathcal{O}(\rho_0)2\pi S_0^2 g(A,C,\rho_0)]^{-1/2}, \tag{4.12}$$

where λ is an unimportant numerical constant. This expression goes one step further than the vortex action (4.5). We find a competition between the exponential tree-level factor and the fluctuation determinant representing quantum spin-wave effects that are long-range ordered.

We are now obliged to wonder whether and when this variational approach makes sense. It is clear that in the weakly frustrated phase, far from the quantum critical line, spin waves dominate the path integral. In this region, the studies based on quantum spin-wave calculations, omitting instanton configurations, have proved to capture the essential of the infrared behavior.[3,6,8] Nevertheless, when we are close to the quantum critical line, when A gets very small compared to B or C, spin waves become softer and weaker and short-range interactions (the quartic terms) are enhanced compared to spin waves. In that case, our variational method based on topological defects can be applied. Moreover, as it was already mentioned, the approach of the quantum critical line is characterized by a strong-coupling regime where topological excitations proliferate.[6]

Let us now summarize and comment on the results obtained so far. We have found that when spin waves fall down close to the quantum critical line, there is a range of parameters in the action (4.2) where pairs of vortices can prolifer-

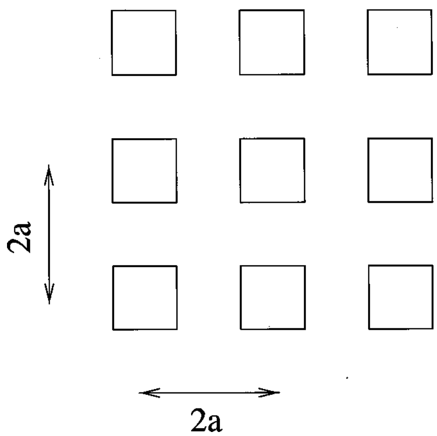

FIG. 4. A schematic representation of a lattice crystal of vortex separated by $\rho=2a$.

ate and stabilize in a liquid phase or a crystal phase of vortices. It has to be noticed that the liquid phase of vortices corresponds to a small area in the parameter space. When $C/A<C_1/A<0$, we found a crystalline phase where pairs of vortices are separated by a distance of order $2a$ (independently of the ratio C/A). Such a crystalline ground state is represented in Fig. 4. This scenario supposes that in the quantum situation, there is a strong renormalization of the coupling constants able to change the classical behavior. This is our main hypothesis. We cannot prove it, but such a possibility cannot be ruled out *a priori*. Moreover, the scenario we describe seems *a posteriori* consistent with the proliferation of vortices able to induce a crystalline ground state discussed by Einarsson *et al.*

Before concluding, a few remarks are in order. In our analysis, the higher-order term $(\nabla\theta)^4$ plays an important role. Amit *et al.*[13] have already wondered about the role of such ''dangerous irrelevant operators'' (see also Ref. 21). In our study, we have shown that its importance relies essentially in the nonperturbative sector. In the usual perturbation scheme, power counting arguments eliminate these kinds of operators because they are irrelevant in the infrared limit. Nevertheless, the spin-liquid phase is characterized by a strong-coupling regime where renormalization-group techniques fail. So, the usual arguments used to eliminate such operators cannot be applied here. Near the critical line, the semiclassical vacuum becomes disordered and dominated by short-range order operators (higher gradient ones). In that case, it is not so surprising that such an operator can play a role. For more justifications, we refer the reader to Ref. 17 where this question is largely addressed.

In our analysis, the operator $(\nabla\theta)^4$ has emerged because we have decomposed $\vec{S}(x)=S_0e^{i\theta(x)}$. If we consider amplitude fluctuations, namely if we write $\vec{S}(x)=[\vec{S}_0+\Delta\vec{S}'(x)]e^{i\theta(x)}$ and integrate out the fluctuations of the order parameter (in a potential) we would generate a series of operators like $(\partial_\mu\vec{S}\partial_\nu\vec{S})^n$ as described in Ref. 22. We hope that these higher corrections do not change qualitatively the physics presented in this section. Moreover, we will always find a range of parameters where the scenario we have described should apply.

V. CONCLUSION

In this paper, we have studied the 2D classical and quantum $J_1-J_2-J_3$, XY model on a square lattice. We have shown that there is a classical critical line where vortices are present even at $T=0$. It justifies why quantum fluctuations have to be included near this critical line. Quantum spin-wave calculations predicts a spin-liquid phase around this critical line. To study the role played by vortices when approaching this spin-liquid phase, we have considered a general GLW action deduced from spin-wave calculations. We have found nonperturbatively a range of parameters where a crystal phase of vortices can take place around this critical line. This phase seems to be directly correlated with the spin-liquid phase predicted by spin-wave calculations. It is difficult to compare this crystalline phase with the possible disordered ground states of Heisenberg spins proposed by Sachdev *et al.*[10] except that both descriptions have a short-range crystalline order. Yet, this study has the advantage of showing qualitatively how a nontrivial liquid phase can emerge nonperturbatively in two dimensions at $T=0$. It should be very interesting to investigate numerically this model. To test the validity of the scenario described in this paper, a possibility could be to look at the nature of the transition between the spin-liquid phase and the chiral phases. It may correspond to a melting of this crystal of the vortex phase as in Ref. 23, induced by chiral spin waves. In that case, the transition could be of the KT type.

Finally, this analysis suggests that we may build classical spin models that can also be described by an effective action similar to Eq. (4.1). Such spin models would include multibody interactions. A general study of 3D classical spin systems without long-range order has been done by Alcaraz *et al.* (Ref. 24 and references therein). They have made a general classification of these spin systems from symmetry considerations. These kinds of models (especially their strange symmetry) could be useful in the description of certain aspects of disordered phases[24] in statistical systems. The link between such classical models and spin-liquid phases is not clear and will be the subject of future work.

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APPENDIX

In this appendix, we give the main steps of the computation of the integral (4.4). We place one vortex charge at the center $O(0,0)$ and the other in $A(\rho,0)$. In the polar coordinate, the integral I Eq. (4.4) reads

$$I=\int r dr d\theta \frac{\rho^4 a^2}{r^4[r^2+\rho^2-2r\rho\cos(\theta)]^2}$$
$$=\int 2\pi dr \frac{(r^2+\rho^2)\mathrm{sgn}(r^2-\rho^2)}{r^3(r-\rho)^3(r+\rho)^3}. \tag{A1}$$

The integration of this rational function over r gives

$$I=2\pi a^2\Bigg[\Bigg(-4\frac{\ln(r/a)}{\rho^2}+\frac{1}{2r^2}+2\frac{\ln(|r+\rho|/a)}{\rho^2}-\frac{7}{8\rho(r+\rho)}-\frac{1}{8(r+\rho)^2}+2\frac{\ln(|r-\rho|/a)}{\rho^2}-\frac{7}{8\rho(r-\rho)}-\frac{1}{8(r-\rho)^2}\Bigg)\mathrm{sgn}(r^2-\rho^2)\Bigg]_0^{+\infty}. \quad \text{(A2)}$$

The contribution at $r=0$ and $r=\rho$ must be taken with our lattice regularization. We finally obtain

$$I=4\pi a^2\left[\frac{4\ln(r/a)}{\rho^2}-\frac{\alpha}{\rho^2}\right] \quad \text{(A3)}$$

with $\alpha=2\ln 2+5/32$. Notice that with this method, we recover the known result of $\int dx(\nabla\theta_0)^2=4\pi\ln(\rho/a)$ (where θ_0 represents a pair of vortices separated by a distance ρ).

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