

Exact calculation of multifractal exponents of the critical wave function of Dirac fermions in a random magnetic field

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(Received 10 June 1997)

The multifractal scaling exponents are calculated for the critical wave function of a two-dimensional Dirac fermion in the presence of a random magnetic field. It is shown that the problem of calculating the multifractal spectrum maps into the thermodynamics of a static particle in a random potential. The multifractal exponents are simply given in terms of thermodynamic functions, such as free energy and entropy, which are argued to be self-averaging in the thermodynamic limit. These thermodynamic functions are shown to coincide exactly with those of a generalized random energy model, in agreement with previous results obtained using Gaussian field theories in an ultrametric space. [S0163-1829(97)05040-6]

I. INTRODUCTION

In recent years it has become clear that the wave functions of noninteracting disordered systems at a continuous metal-insulator transition have multifractal scaling properties. As opposed to a simple fractal, these statistical self-similar wave functions cannot be described by a single fractal dimension, but instead an infinite set of scaling exponents is needed. Indeed, such families of scaling exponents have been obtained for the critical wave function at a localization-delocalization transition within several different frameworks, including perturbative renormalization-group treatments on replicated¹ and supersymmetric² nonlinear sigma models, as well as numerical simulations.³ Although these results seem to provide sufficient evidence for multifractality at the metal-insulator transition, none of them allows one to understand the full spectrum of multifractal exponents. This is so because nonperturbative techniques are needed to probe the full spectrum.

Given the present state of affairs of this problem it would be highly desirable to have an exactly solvable system that exhibits a multifractal wave function. In recent years it has been shown that the Dirac equation in random fields in two-spatial dimensions (2D) is actually an example of such a system. Moreover, it has also become clear that the random Dirac Hamiltonian in 2D describes the universality class of the metal-insulator transition at a half-filled Landau level of disordered *nonrelativistic* noninteracting electrons in a very high magnetic field.^{4,5} This surprising result has been recently established by direct derivation⁶ of the random Dirac

equation from the Chalker-Coddington model of percolating edge states.⁷

Recently, *exact* results for the *full* multifractal spectrum have been obtained for Dirac fermions interacting with a random magnetic field.⁸ These results were obtained by mapping the problem to a Gaussian field theory in an ultrametric space (a Cayley tree). The calculation of multifractal scaling properties was then reduced to the computation of thermodynamic functions of a special generalized random energy model (GREM), which are known exactly.⁹⁻¹² The field theory in the localization problem is defined in Euclidean space. Nevertheless, it was argued that as the exponents are a measure of a global property, the difference between the ultrametric and the Euclidean metric would not alter the results. Moreover, a phase transition that occurred in the ultrametric model¹¹ also seems to be present in the original Euclidean problem, as evidenced by analytical arguments and by Monte Carlo simulations.⁸

It has also been pointed out that there exists a deep connection between the multifractal spectrum for the critical wave function of a Dirac fermion in a random magnetic field and the spectrum of primary fields in nonunitary conformal field theories with vanishing Virasoro conformal charge^{13,14} on the one hand, and Liouville field theory¹⁵ on the other. It is then tempting to speculate that there must be a counterpart to the freezing transition characterizing GREM in these quantum field theories.

In the present paper we show that the exact results of Ref. 8 on the multifractal spectrum and, in particular, the existence of a phase transition, can be obtained completely within the framework of the Gaussian field theory in two-

dimensional Euclidean space. The key idea of our approach is the introduction of a quantity $\Omega(E)$ that counts the number of points where the critical wave function has amplitude (or height) $\Psi \propto e^{-E/2}$ in a disk of radius L (suitably discretized as a lattice of spacing a). We show that this quantity is directly related to the microcanonical density of states of the GREM and which we will also call the density of states. The advantage of the approach that we present in this work is that it is direct and it does not rely on the use of either replicas or supersymmetry. Our results are essentially rigorous, except for a conjecture, which we believe to be true, about the self-averaging property of the probability distribution of the entropy $\ln \Omega(E)$ in the thermodynamic limit $N \rightarrow \infty$. We also explain in detail why the Gaussian field theories in both Euclidean and ultrametric spaces give exactly the same results.

This paper is organized as follows. In Sec. II we introduce the model and the quantities of main interest: the multifractal exponents $\tau(q)$ of the critical wave function, its Legendre transform $f(\alpha)$, and the density of states $\Omega(E)$. Here we draw an analogy between the computation of the inverse participation ratio of order q of the wave function and the partition function for all spatial configurations of a static particle in a random potential $\phi(\mathbf{x}) = -\ln|\Psi(\mathbf{x})|$. $\Omega(E)$ is the microcanonical density of states of this equivalent problem. In Sec. III we derive the thermodynamic properties of the equivalent system. Here we calculate the average density of states and use it to show that the equivalent problem has a phase transition at a critical ‘‘energy’’ determined by the width g of the probability distribution of the random vector potentials. We show that the thermodynamic functions of the equivalent problem are exactly those of the GREM. In particular, we show that, in terms of the critical wave function, the phase transition of the GREM represents the onset of the regime where the probability distribution of the wave function is *undersampled* in a given discretization of the plane. In Sec. IV we use these results to derive the exact form of the functions $f(\alpha)$ and $\tau(q)$. Our results agree completely with the analysis given in Ref. 8. Section V is devoted to the conclusions. Technical details of our calculations are given in the appendices.

II. MODEL

We consider the problem of a massless Dirac fermion moving on a plane and interacting with a static random magnetic field normal to the plane.^{4,5,16,13,6,17} In this model, the wave functions are localized for all energies other than the critical energy $E=0$, at which the wave function is multifractal.^{5,13} This model thus describes a metal-insulator transition in two dimensions.

The Dirac Hamiltonian in random and static *vector potentials* in two space dimensions is

$$H = \sigma_\mu [i v_F \partial_\mu - A_\mu(\mathbf{x})]. \quad (2.1)$$

For convenience we will set the Fermi velocity v_F to unity from now on. This operator acts on the space of normalizable (in a finite area) two-component spinor states $\Psi_\alpha(\mathbf{x})$, with $\alpha=1,2$. In Eq. (2.1) σ denotes a two-component vector of two 2×2 Pauli matrices, which we take to be σ_1 and σ_2 , respectively, with $\mu=1,2$ being the two orthogonal directions on the plane. The probability distribution of the random

vector potential $\mathbf{A}(\mathbf{x})$ will be specified below. In principle, other sorts of random fields, such as random mass and random chemical potential, are also allowed in a general situation.⁴

If only random vector potentials are allowed, $E=0$ is an exact eigenenergy for all realizations of the disorder and the corresponding wave functions can be determined exactly.¹⁸ Indeed, let Ψ_0 be a two-component spinor with energy $E=0$, i.e., $H\Psi_0=0$. Then by means of a combination of chiral and gauge transformations, parametrized by the fields $\phi(\mathbf{x})$ and $\chi(\mathbf{x})$, respectively,

$$\Psi_0(\mathbf{x}) = e^{-\phi(\mathbf{x})\sigma_3 - i\chi(\mathbf{x})} \eta(\mathbf{x}); \quad (2.2)$$

the eigenvalue equation reduces to the requirement that the two component spinor $\eta(\mathbf{x})$ satisfies the nonrandom Dirac equation

$$i\sigma_\mu \partial_\mu \eta(\mathbf{x}) = 0. \quad (2.3)$$

Here, the chiral ‘‘angle’’ $\phi(\mathbf{x})$ and the gauge transformation $\chi(\mathbf{x})$ must be chosen to solve Eq. (2.5) below.

In this work we will only be interested in the multifractal properties of the amplitudes of the wave functions. These properties involve only the magnitude of the wave functions and are independent of their phases, and hence are gauge-invariant properties. However, it should be stressed that the decomposition of Eq. (2.5) is only valid if the total magnetic flux threading the disk always vanishes, an assumption that we will implement by an appropriate choice for the probability distribution of the vector potential $\mathbf{A}(\mathbf{x})$.¹⁹ The remaining degrees of freedom carried by the spinor $\eta(\mathbf{x})$ then span a two-dimensional Hilbert space. Here we will *choose* the spinor $\eta(\mathbf{x}) = (1,0)$ for convenience. It is worth noting that any choice of spinor breaks the chiral symmetry generated by σ_3 . This procedure makes sense if we think of switching on an average uniform magnetic field and then taking it to zero. Indeed, a magnetic field selects a state with unique chirality determined by the sign of the magnetic field. In the context of the Chalker-Coddington model the choice of spinor is thus the equivalent of the choice of the chirality of the edge current for a system on an open geometry such as a disk.

With the above considerations we write the $E=0$ wave function as

$$\psi(\mathbf{x}) = e^{-\phi(\mathbf{x})}, \quad (2.4)$$

and drop the constant spinor $\eta(\mathbf{x}) = (1,0)$ altogether. The random vector potential and magnetic field are given by

$$A_\nu(\mathbf{x}) = \epsilon_{\nu\rho} \partial_\rho \phi(\mathbf{x}) + \partial_\nu \chi(\mathbf{x}), \quad (2.5)$$

$$B(\mathbf{x}) = -\nabla^2 \phi(\mathbf{x}). \quad (2.6)$$

Notice that the gauge degrees of freedom enter through χ and not ϕ , and any phase in the wave function can be eliminated by a gauge transformation. Finally, we assume a Gaussian distribution of magnetic fields²⁰ as follows:

$$P[\phi(\mathbf{x})] \propto e^{-1/2g \int d^2\mathbf{x} (\nabla\phi(\mathbf{x}))^2}. \quad (2.7)$$

where g is the width of the probability distribution and plays the role of a coupling constant in this problem. One verifies

that, in the thermodynamic limit, the uniform magnetic field does indeed vanish for all realizations of the disorder. As a technical point, it is understood that, among all fields $\phi(\mathbf{x})$ that differ from each other by a uniform value, only one representative is counted in the disorder average over ϕ [for example, of all possible uniform fields, only the field $\phi(\mathbf{x})=0$ is counted]. In this way, the ambiguity in the many to one relation between $\phi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ in Eq. (2.5) is removed (see Appendix A).

The multifractal nature of the wave function can be probed through the moments of the probabilities $p_{\mathbf{x}}$ obtained from the normalized wave function $\Psi(\mathbf{x})$. As we anticipated above, we shall use a lattice regularization with a microscopic cutoff distance a (box size), and macroscopic system size L . There are thus L^2/a^2 sites. The moments of $p_{\mathbf{x}}$ are the inverse participation ratios $\mathcal{P}(q, a/L)$:

$$\begin{aligned} \mathcal{P}\left(q, \frac{a}{L}\right) &\equiv \sum_{\mathbf{x}} p_{\mathbf{x}}^q \equiv \sum_{\mathbf{x}} |\Psi(\mathbf{x})|^{2q} \equiv \frac{\sum_{\mathbf{x}} |\psi(\mathbf{x})|^{2q}}{\left(\sum_{\mathbf{x}} |\psi(\mathbf{x})|^2\right)^q} \\ &= \frac{\sum_{\mathbf{x}} e^{-2q\phi(\mathbf{x})}}{\left\{\sum_{\mathbf{x}} e^{-2\phi(\mathbf{x})}\right\}^q}. \end{aligned} \quad (2.8)$$

The multifractal exponents $\tau(q)$ for a given wave function are then defined by^{21,22}

$$\begin{aligned} \tau(q) &\equiv D(q)(q-1) \equiv \lim_{a/L \rightarrow 0} \frac{1}{\ln\left(\frac{a}{L}\right)} \ln\left(\sum_{\mathbf{x}} p_{\mathbf{x}}^q\right) \\ &= \lim_{a/L \rightarrow 0} \frac{\ln \mathcal{P}(q, a/L)}{\ln\left(\frac{a}{L}\right)}. \end{aligned} \quad (2.9)$$

We will show later that $\tau(q)$ is self-averaging, i.e., that for any realization of the disorder $\tau(q) = \langle \tau(q) \rangle$, where $\langle \dots \rangle$ denotes the average over the distribution of random magnetic fields of Eq. (2.7).

An equivalent way of describing the multifractal properties of a *given* wave function is through the scaling exponents $\alpha_{\mathbf{x}}$, defined by²³⁻²⁵

$$p_{\mathbf{x}} \sim \left(\frac{a}{L}\right)^{\alpha_{\mathbf{x}}}. \quad (2.10)$$

The number of lattice points at which the exponent $\alpha_{\mathbf{x}}$ takes values between α' and $\alpha' + d\alpha'$ defines the functions ρ and f :

$$d\alpha' \rho(\alpha') \left(\frac{a}{L}\right)^{-f(\alpha')}. \quad (2.11)$$

As is well known,^{24,25} the two sets of exponents $\tau(q)$ and $f(\alpha)$ are related by a Legendre transformation:

$$\alpha = \frac{d\tau(q)}{dq}, \quad (2.12)$$

$$f(\alpha) = \alpha q - \tau(q). \quad (2.13)$$

It is possible in general to map the quantities that describe a given multifractal wave function into thermodynamic quantities:²⁶ q maps into an inverse temperature β , $\tau(q)$ into a free energy $f_0(\beta)$, α into an internal energy e , and $f(\alpha)$ into an entropy $s(e)$. In our case, however, the equivalence is more evident due to the particular form of the wave function of Eq. (2.4). We map our problem into the statistical mechanics of a single particle in a lattice of spacing a and size L with a random site potential $V(\mathbf{x}) = 2\phi(\mathbf{x})$, in the static limit (i.e., for the hopping matrix element equal to zero). For this model, the random canonical partition function for a particular realization of the disorder reads

$$Z(\beta) \equiv \sum_{\mathbf{x}} e^{-\beta V(\mathbf{x})}, \quad (2.14)$$

where the role of the random energies is played by the values that the disorder potential $V(\mathbf{x}) = 2\phi(\mathbf{x})$ takes. For each disorder realization, the free energy of the system is

$$F(\beta) \equiv -\frac{1}{\beta} \ln Z(\beta). \quad (2.15)$$

In this system, the number of thermodynamic degrees of freedom is

$$N = \ln\left(\frac{L}{a}\right)^2, \quad (2.16)$$

and the number of energy levels is

$$e^N = \left(\frac{L}{a}\right)^2. \quad (2.17)$$

This is similar to a system of N spins with $S = \frac{1}{2}$, where one has 2^N states. Thus, the intensive free energy is defined as

$$f_0(\beta) \equiv \frac{F(\beta)}{N}. \quad (2.18)$$

[The subscript 0 is introduced to avoid confusion with the spectrum $f(\alpha)$.]

In this problem it turns out that it is also useful to count states directly, which naturally leads us to define a microcanonical partition function (or density of states) for each random field configuration:

$$\Omega(E) \equiv \sum_{\mathbf{x}} \delta_W[E - V(\mathbf{x})]. \quad (2.19)$$

Here $\delta_W(E)$ counts the number of states in a region of width W around E . One can choose for it, e.g., either a top hat (i.e., a product of step functions) $\delta_W(E) = \theta(E - W/2)\theta(W/2 - E)$, or a smooth function such as

$$\delta_W(E) \equiv e^{-E^2/2W^2}. \quad (2.20)$$

The microcanonical and canonical partition functions are then related by Laplace transformation:

$$Z(\beta) = \int \frac{dE}{W} \Omega(E) e^{-\beta E}. \quad (2.21)$$

Here, as is usual in statistical mechanics, W has to be taken as small as possible but still larger than the average level spacing Δ . As we show later, in the present case this translates into the condition

$$1 \gg W \gg \Delta = 4 \sqrt{\frac{g}{2\pi}} \frac{\ln\left(\frac{L}{a}\right)^2}{\left(\frac{L}{a}\right)^2}. \quad (2.22)$$

From the microcanonical partition function one can obtain the total entropy:

$$S(E) \equiv \ln \Omega(E), \quad (2.23)$$

and again it will be convenient to define the entropy and energy per thermodynamic degree of freedom:

$$s \equiv S/N, \quad e \equiv E/N. \quad (2.24)$$

Having established the thermodynamic dictionary, the normalization factor for the wave function and the inverse participation ratios $\mathcal{P}(q, a/L)$ can be written in terms of the partition function:

$$\Psi(\mathbf{x}) = \frac{e^{-\phi(\mathbf{x})}}{\sqrt{Z(1)}}, \quad (2.25)$$

$$\mathcal{P}\left(q, \frac{a}{L}\right) = \frac{Z(q)}{Z(1)^q}. \quad (2.26)$$

In turn, the function $\tau(q)$ can be simply expressed in terms of the free energy at the inverse temperatures $\beta = q$ and $\beta = 1$:

$$\tau(q) = -q \lim_{a/L \rightarrow 0} \frac{F(q) - F(1)}{\ln\left(\frac{a}{L}\right)} = 2q \lim_{a/L \rightarrow 0} [f_0(q) - f_0(1)]. \quad (2.27)$$

Similarly, from the definition of $\alpha_{\mathbf{x}}$ [see Eq. (2.10)] and the value of the wave function [Eq. (2.25)], we see that

$$\alpha_{\mathbf{x}} = 2 \left[\frac{V(\mathbf{x})}{\ln\left(\frac{L}{a}\right)^2} - f_0(1) \right]. \quad (2.28)$$

Thus, comparing the definitions of $f(\alpha)$ [Eq. (2.11)] and the entropy, one verifies that

$$\alpha = 2[e - f_0(1)], \quad (2.29)$$

$$f(\alpha) = 2s(e).$$

By using the relation between $\tau(q)$ and $f_0(\beta)$, and between $f(\alpha)$ and $s(e)$ one recovers the well-known result that the Legendre transformation in the language of multifractal exponents [Eq. (2.13)] is equivalent to the thermodynamic relation:

$$f_0(\beta) = e - \frac{s(e)}{\beta}. \quad (2.30)$$

III. THERMODYNAMICS OF THE MODEL

In this section, we are going to show that the probability to find energy levels outside a particular window of energy $[-e_c, e_c]$ vanishes in the thermodynamic limit. This is the most important result of our work.

The key step in our argument is our estimate for the disorder average of the microcanonical partition function, which relies crucially on the choice of a distribution for the disorder with a variance depending logarithmically on spatial separation.

The simplest way to study the thermodynamic properties of the system is to obtain the disorder average of the microcanonical partition function:

$$\begin{aligned} \langle \Omega(E) \rangle &= \int \mathcal{D}V(\mathbf{x}) P[V(\mathbf{x})] \sum_{\mathbf{x}} \delta_w[E - V(\mathbf{x})] \\ &= \int \mathcal{D}\phi(\mathbf{x}) P[\phi(\mathbf{x})] \sum_{\mathbf{x}} \delta_w[E - 2\phi(\mathbf{x})] \\ &= W \sum_{\mathbf{x}} \frac{\exp\{-E^2/2[W^2 + 4G(\mathbf{x}, \mathbf{x})]\}}{\sqrt{W^2 + 4G(\mathbf{x}, \mathbf{x})}} \quad (3.1) \\ &\approx W \left(\frac{L}{a}\right)^2 \frac{\exp\left\{-E^2/\left[\frac{4g}{\pi} \ln\left(\frac{L}{a}\right)\right]\right\}}{\sqrt{\frac{2g}{\pi} \ln\left(\frac{L}{a}\right)}}. \quad (3.2) \end{aligned}$$

Here, $G(\mathbf{x}, \mathbf{y})$ is the Green function²⁰ for the field theory defined by the action of Eq. (2.7), with a short distance cut-off length given by a . Its form in the short distance limit is

$$G(\mathbf{x}, \mathbf{y}) = -\frac{g}{4\pi} \ln\left(\frac{|\mathbf{x} - \mathbf{y}|^2 + a^2}{L^2}\right). \quad (3.3)$$

On the last line of Eq. (3.2), we have neglected the width W as compared to terms of order $\ln(L/a)$.

Alternatively, in terms of intensive quantities, the disorder averaged number of states in an energy interval of width $w = W/N$ around $e = E/N$ is

$$\begin{aligned} \langle \Omega(e) \rangle &\approx w \sqrt{\frac{\ln\left(\frac{L}{a}\right)}{g}} \exp\left[-2 \ln\left(\frac{L}{a}\right) \left(\frac{e^2}{4(g/2\pi)} - 1\right)\right] \\ &= w \sqrt{\frac{N}{2g}} \exp\left[N \left(1 - \frac{e^2}{4(g/2\pi)}\right)\right]. \quad (3.4) \end{aligned}$$

This means that in the thermodynamic limit ($N \rightarrow \infty$), $\langle \Omega(e) \rangle$ goes to zero for $|e| > 2\sqrt{g/2\pi}$ and diverges exponentially with the number of degrees of freedom for

$|e| < 2\sqrt{g/2\pi}$. This behavior for the average number of states $\langle \Omega(e) \rangle$ indicates that some sort of transition should occur at the critical energy $|e| = e_c$ where

$$e_c \equiv 2\sqrt{\frac{g}{2\pi}}. \quad (3.5)$$

To show that, indeed, this is a phase transition, we must consider not the average $\langle \Omega(e) \rangle$, but the number of states $\Omega(e)$ for a *given realization*. Indeed, as shown in Appendix B, $\Omega(E)$ is a random variable with very strong fluctuations. Hence, we cannot rely *a priori* on any single moment of $\Omega(E)$ to identify a phase transition. Rather, we must use stronger probabilistic arguments to show that e_c is indeed *the critical energy of interest*. We do so below by considering the two regimes $|e| > e_c$ and $|e| < e_c$ separately.

A. $|e| \geq e_c$

In this region the thermodynamic limit of the microcanonical partition function is $\langle \Omega(e) \rangle \rightarrow 0$. Alternatively, we may say that for a large enough system size N , the average number of states $\langle \Omega(e) \rangle \ll 1$ in this region. If we naively try to obtain the entropy as $\ln \langle \Omega(e) \rangle$, we would find that it becomes negative for $|e| > e_c$. This is *not* correct; the entropy must be defined for *each and every* realization of the disorder. That means we must focus on $S(e) = \ln \Omega(e)$ for separate realizations.

We are going to show now that the probability of finding *any* state at all with $|e| > e_c$ vanishes in the thermodynamic limit. To do that, we define the random variable $\Omega_{>}(e)$ that counts the number of states with energies e' such that $|e'| > e$:

$$\Omega_{>}(e) \equiv \int_{-e}^{-\infty} de' \frac{\Omega(e')}{w} + \int_e^{\infty} de' \frac{\Omega(e')}{w}. \quad (3.6)$$

Due to the fact that $\Omega_{>}(e)$ is either positive or zero, we can bound the probability $P\{\Omega_{>}(e) \geq 1\}$ of finding at least one state with $|e'| > e$ by the average of $\Omega_{>}(e)$:²⁷

$$P\{\Omega_{>}(e) \geq 1\} \leq \langle \Omega_{>}(e) \rangle. \quad (3.7)$$

Equation (3.7) is very general since it applies to any random microcanonical partition function. However, this inequality becomes very powerful when combined with our estimate Eq. (3.4) for $\langle \Omega(e) \rangle$. Indeed, with the help of

$$\begin{aligned} \langle \Omega_{>}(e) \rangle &= \frac{2}{w} \int_e^{\infty} de' \langle \Omega(e') \rangle \\ &= \frac{1}{\sqrt{\pi N}(e/e_c)} e^{-N[(e/e_c)^2 - 1]} \left[1 - \mathcal{O}\left(\frac{e_c}{Ne}\right) \right], \end{aligned} \quad (3.8)$$

and our upper bound Eq. (3.7), we see that $P\{\Omega_{>}(e_c) \geq 1\}$ vanishes in the thermodynamic limit $N \rightarrow \infty$. We conclude that, *for a given realization*, the energy levels will fall within the interval $|e| \leq e_c$ with probability 1 in the thermodynamic limit.

It is now clear how to estimate the average level spacing Δ in Eq. (2.22). We simply multiply $2e_c$ by the ratio of the number N of thermodynamic degrees of freedom to the number e^N of energy levels:

$$\Delta = \frac{2Ne_c}{e^N} = 4\sqrt{\frac{g}{2\pi}} \frac{\ln\left(\frac{L}{a}\right)^2}{\left(\frac{L}{a}\right)^2}. \quad (3.9)$$

Finally, we can bound from above the probability for the density of states $\Omega(e)$ to be nonzero. For a given realization, $\Omega(e)$ counts states in an energy interval $w = W/N$ around e . Therefore $\Omega(e)$ must be a positive integer,²⁸ in contrast with the average $\langle \Omega(e) \rangle$. In the same way as before, the probability of having a nonzero microcanonical partition function is bounded from above by a number that goes to zero in the thermodynamic limit:²⁷

$$P\{\Omega(e) > 0\} = P\{\Omega(e) \geq 1\} \leq \langle \Omega(e) \rangle. \quad (3.10)$$

The entropy is therefore *not* defined for $|e| > e_c$.

One can get some intuition for these results with one word: *undersampling*. Consider $\Omega(e)$ for an individual realization as the histogram of the number of states per energy interval. One can think of $\Omega(e)$ as the product of two factors: one is a constant that counts the total number of states in the system, and the other is the probability (normalized to 1) for *one* state to fall within a given energy interval. In the present case the first factor has the value e^N and the second is Gaussian with a width that grows linearly with N . In other words, the number of data points we have is e^N , which is not enough to sample the tails of the Gaussian distribution. If the number of data points were $e^{N\gamma}$, with $\gamma > 1$, by taking N large enough we could sample the whole distribution, and all bins in the histogram would contain a positive number of points.

B. $|e| < e_c$

In this case we can define the entropy for a given realization $S(e) = \ln \Omega(e)$ [recall Eq. (2.23)]. We will now show that the entropy, for all realizations of the disorder, has a common upper bound that scales linearly with system size. Furthermore, we will also show that the probability to find a realization with entropy *less or equal* than $\ln \langle \Omega(e) \rangle$ is *equal to one* in the $N \rightarrow \infty$ limit. The arguments go as follows.

According to our definition Eq. (2.19), $\Omega(e)$ counts the number of energy levels in some energy window. That number cannot be larger than the total number of energy levels $e^N = L^2/a^2$. Consequently, $\ln \Omega(e)$ is bounded from above by N . Incidentally, no such bound holds for $\ln Z(\beta)$ for individual realizations of the disorder.

Next, we introduce the probability to find the intensive entropy $s = (1/N) \ln \Omega(e)$ in a given interval (s_1, s_2) by

$$P_N(s_1 \leq s \leq s_2) \equiv \int_{s_1}^{s_2} d\mu_N(s). \quad (3.11)$$

Here, $d\mu_N(s)$ is the measure of the entropy for N degrees of freedom. We will make the following assumptions:

(1) The probability measure $d\mu_N(s)$ has a well-defined thermodynamic limit:

$$\lim_{N \rightarrow \infty} \int_{s_1}^{s_2} d\mu_N(s) = \int_{s_1}^{s_2} d\mu_\infty(s) \quad (3.12)$$

for all $s_1 < s_2$.

(2) There exist two positive constants A and B such that

$$\langle \Omega(e) \rangle \equiv \int_{-\infty}^{+\infty} d\mu_N(s) e^{Ns} = \sqrt{N} e^{AN+B} \quad (3.13)$$

for N sufficiently large.

Notice that we have proved the validity of the second hypothesis and that A and B can both be read off from Eq. (3.4). In particular,

$$A = 1 - \frac{e^2}{e_c^2} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \Omega(e) \rangle. \quad (3.14)$$

We now choose $s_+ > A$. By assumption, $s_+ > 0$ and

$$\begin{aligned} \langle \Omega(e) \rangle &\geq \int_{s_+}^{+\infty} d\mu_N(s) e^{Ns} \geq \int_{s_+}^{+\infty} d\mu_N(s) e^{Ns_+} \\ &\equiv e^{Ns_+} P_N(s_+ \leq s). \end{aligned} \quad (3.15)$$

We have thus established the upper bound,

$$P_N(s_+ \leq s) \leq \sqrt{N} e^{(A-s_+)N+B}, \quad (3.16)$$

for all $s_+ > A$ and for all sufficiently large N . Since the thermodynamic limit is assumed to be well defined, we conclude that

$$P_\infty(A < s) = 0. \quad (3.17)$$

In other words, the probability to find the entropy density $(1/N) \ln \Omega(e) > (1/N) \ln \langle \Omega(e) \rangle$ vanishes in the thermodynamic limit $N \rightarrow \infty$.

If we could prove that the probability to find an intensive entropy lower than $(1/N) \ln \langle \Omega(e) \rangle$ also vanishes in the thermodynamic limit, we would have shown that the entropy density is indeed self-averaging (i.e., almost surely takes one and only one value), and given by $\lim_{N \rightarrow \infty} (1/N) \ln \langle \Omega(e) \rangle$. However, without further knowledge of the probability distribution for the entropy, it is not possible to prove the existence of a similar lower bound. In fact it is possible to construct a sequence of measures which obey Eqs. (3.12) and (3.13) but is not self-averaging in the thermodynamic limit. We nevertheless expect the entropy density to be self-averaging for two reasons. First and in addition to our upper bound, it is possible to estimate the ratio of the n th moment of the partition function $Z(\beta)$ to its mean raised to the power n . This is done in Appendix B. This estimate suggests that the field theory computations of the multifractal dimensions (with replicas, supersymmetry, and Liouville field theory) are reliable⁸ in the regime $|e| < e_c$. Second, the entropy density for any GREM is known to be self-averaging in the thermodynamic limit⁹⁻¹² and in view of the close connection between GREM and our problem we expect this to also be true here.

Thus, although we do not have a rigorous proof, we conjecture that $(1/N) \ln \Omega(e)$ is a random variable with vanishing variance in the thermodynamic limit $N \rightarrow \infty$, i.e., self-averaging, and which converges to $\lim_{N \rightarrow \infty} (1/N) \ln \langle \Omega(e) \rangle$. This conjecture is supported by the fact that, in addition to our bound, it is in this regime that the field theory computations of the multifractal dimensions (with replicas, supersymmetry, and Liouville field theory) are reliable. In other words, the entropy should be self-averaging in the thermodynamic limit and it is given by

$$S(e) = \ln \Omega(e) = \langle \ln \Omega(e) \rangle = \ln \langle \Omega(e) \rangle \quad (3.18)$$

$$= N \left(1 - \frac{e^2}{e_c^2} \right) \quad (3.19)$$

for any realization of disorder. Hence, for this range of energies $|e| < e_c$ we may commute the order of \ln and $\langle \dots \rangle$, i.e., the ‘‘quenched’’ and ‘‘annealed’’ entropies coincide.²⁹

C. Temperature and free energy

Up to this point we have proven that the thermodynamics of this model is identical to that of the random energy model⁹ for $|e| \geq e_c$ and argued that this is also so for $|e| < e_c$. In order to obtain the free energy as a function of temperature, we follow Derrida.⁹ In the region that contains states ($|e| < e_c$), we obtain the temperature from

$$T = \left(\frac{ds}{de} \right)^{-1} = - \frac{e_c^2}{2e}. \quad (3.20)$$

In this region the temperature will be in the range $|T| > e_c/2$, and the free energy, computed from the relation $df_0/dT = -s$, is given by

$$f_0(T) = -T - \frac{e_c^2}{4T}. \quad (3.21)$$

In the limit when $T \rightarrow T_c \equiv e_c/2$, we obtain $e \rightarrow -e_c$, $s \rightarrow 0$ and $f_0 \rightarrow -e_c$. Below this temperature, the system cannot lower its energy because there are no states for $e < -e_c$. The system remains frozen at $e = -e_c$, with entropy $s = 0$ and free energy $f_0(T) = -e_c$. In other words, for $|T|$ lower than the freezing temperature T_c , the energy, the entropy, and the free energy remain at their accumulation points. In summary,

$$\beta f_0(\beta) = \begin{cases} - \left(1 + \frac{\beta^2}{q_c^2} \right) & \text{if } |\beta| \leq q_c \\ -2 \frac{|\beta|}{q_c} & \text{if } |\beta| > q_c, \end{cases} \quad (3.22)$$

where $q_c \equiv \sqrt{2\pi/g}$.

The behavior in the low temperature regime becomes clear if we consider the $|\beta| \rightarrow \infty$ limit. For any configuration and for a given value of L/a , there is an absolute minimum E_{\min} and an absolute maximum E_{\max} for the values of the $V(\mathbf{x})$. For large enough $|\beta|$ these extreme values will dominate the partition function, and one obtains, in the case of positive sign for β ,

$$Z(\beta) \sim e^{-\beta E_{\min}}. \quad (3.23)$$

This means that

$$\lim_{N \rightarrow \infty} f_0(\beta) = \lim_{N \rightarrow \infty} \frac{E_{\min}}{N} = -e_c, \quad (3.24)$$

because we know that for a large enough system $E_{\min}/N = -e_c$ with probability one. This implies that $f_0(\beta)$ is self-averaging and equal to $-e_c = -2/q_c$, as shown above.

Notice, however, that in the low temperature regime quenched and annealed values for the free energy are different: $\langle \ln Z(\beta) \rangle \neq \ln \langle Z(\beta) \rangle$. Indeed, combining Eqs. (2.21) and (3.2) one obtains for all β the so-called lognormal spectrum

$$\ln \langle Z(\beta) \rangle = N \left(1 + \frac{\beta^2}{q_c^2} \right), \quad (3.25)$$

which shows no obvious sign of the phase transition.

On the other hand, in the high-temperature regime, the quenched and annealed free energies $\langle \ln Z(\beta) \rangle$ and $\ln \langle Z(\beta) \rangle$ do coincide. This can be understood by computing $\ln Z(\beta)$ for one particular realization of the disorder [see Eq. (2.21)]:

$$Z(\beta) = N \int_{E_{\min}/N}^{E_{\max}/N} de e^{-N[\beta e - s(e)]}. \quad (3.26)$$

If one accepts our conjecture that the entropy density is self-averaging in the thermodynamic limit, then the saddle-point approximation on the integrand in Eq. (3.26) probes the parabolic part of the entropy, and one obtains for $\ln Z(\beta)$ the result of Eq. (3.25). In other words, not only is $\ln Z(\beta)$ self-averaging but in this case \ln and $\langle \dots \rangle$ commute. In the language of multifractality, this is the region in which the parabolic approximation is exact. Besides, since at $|\beta| = q_c$ the minimum of $\beta e - s(e)$ falls at the edge of the populated region, there is a singularity in the derivative of $\ln Z(\beta)$, and one recovers the phase transition point.

D. Equivalence with random energy models

We have calculated the thermodynamic functions of our model and they are identical to those of a special generalized random energy model.⁸

It is interesting to see how this comes about. All the results we obtained really follow from our calculation of $\langle \Omega(E) \rangle$ in Eq. (3.1). The analogous calculation for the generalized random energy model of Derrida and Spohn¹¹ gives the same result, except that positions \mathbf{x} in the lattice are replaced by directed paths \mathcal{P} in a Cayley tree, and the value $G(\mathbf{x}, \mathbf{x}) = (g/2\pi) \ln(L/a)$ is replaced by $G(\mathcal{P}, \mathcal{P}) = (g/\ln K) \ln d(\mathcal{P}, \mathcal{P})$. Then, the two average partition functions must correspond if the parameters are chosen properly. Note that the requirement that $G(\mathcal{P}, \mathcal{P})$ depends logarithmically on the ultrametric distance $d(\mathcal{P}, \mathcal{P})$ on the tree uniquely defines the GREM in the thermodynamic limit.

One can take this analysis further by studying moments of the microcanonical partition function. Although at first sight the expressions for the two models appear different, if one changes variables into logarithms of the distances, one can actually see that the expressions for the GREM actually give the Riemann sums that correspond to the integrals in the case of our model.

IV. CONSEQUENCES FOR THE MULTIFRACTAL SPECTRUM

Now that we have the values of the thermodynamic functions $s(e)$ and $f_0(\beta)$, we can translate them into the language of multifractality, using Eqs. (2.27) and (2.29) as our dictionary.

The results are as follows: starting by the spectral weight function $f(\alpha)$, we find that it is defined only in the interval $d_- \leq \alpha \leq d_+$ (corresponding to the entropy being defined only in the interval $-e_c \leq e \leq e_c$), and has the value

$$f(\alpha) = 8 \frac{(d_+ - \alpha)(\alpha - d_-)}{(d_+ - d_-)^2}. \quad (4.1)$$

However, the values of d_- and d_+ will change with the strength of the disorder.

Depending on the strength of the disorder, there are two regimes: in the weak disorder regime, which corresponds to $g < 2\pi$, the quenched and annealed averages for the logarithm of the wave-function normalization factor $Z(1)$ are coincident, while in the strong disorder regime, which corresponds to $g > 2\pi$, they are not equal anymore.

In the weak disorder regime, the extremal dimensions d_- and d_+ are both positive:

$$d_{\pm} = 2 \left(1 \pm \sqrt{\frac{g}{2\pi}} \right)^2, \quad (4.2)$$

and $\tau(q)$ has the form

$$\tau(q) = \begin{cases} 2(q-1) \left(1 - \frac{q}{q_c} \right) & \text{if } |q| \leq q_c \\ 2q \left(1 - \frac{\text{sgn}(q)}{q_c} \right)^2 & \text{if } |q| > q_c. \end{cases} \quad (4.3)$$

On the other hand, in the strong disorder regime, the lower extremal dimension is zero:

$$d_- = 0, \\ d_+ = 8 \sqrt{\frac{g}{2\pi}}, \quad (4.4)$$

and $\tau(q)$ has the form

$$\tau(q) = \begin{cases} -2q \left(1 - \frac{q}{q_c} \right)^2 & \text{if } |q| \leq q_c \\ \frac{4}{q_c} (q - |q|) & \text{if } |q| > q_c. \end{cases} \quad (4.5)$$

Notice that in this regime we find $\tau(q) = 0$ for $q > q_c$, meaning that for all integer moments the inverse participation ratio does not scale with system size. This is usually interpreted as characteristic of a localized wave function.

V. CONCLUSIONS

We have calculated the multifractal scaling exponents of the critical wave function for two-dimensional Dirac fermions in the presence of a random magnetic field. There is a transition in the multifractal spectrum, which is interpreted

as a freezing transition common to glassy systems. This freezing transition is a rigorous result of the present work and had been previously conjectured on the basis of a comparison with GREM.⁸

We have proven that a previously proposed mapping⁸ between these multifractal properties and the thermodynamics of a generalized random energy model describing directed polymers on a Cayley tree^{10,11} is indeed exact in the glassy regime. Our proof generalizes entropy considerations on GREM (Ref. 9) to a two-dimensional Gaussian field theory.

Derrida has also shown that a direct computation of the quenched free energy was possible on GREM. This suggests that the same could be done on the field theory. In fact, it can be shown that the generalization of Derrida's calculation for GREM naturally leads to estimating the partition function in Liouville field theory. We thus believe that there exists a counterpart to the freezing transition of GREM in Liouville field theory. It is an interesting question to probe this issue further.

Another open issue is the fate of replica symmetry if the replica approach is used to calculate the multifractal scaling exponents. Indeed, it is known that the freezing transition in GREM is associated to replica symmetry breaking.^{9,30} It would be interesting to see how this replica symmetry breaking manifests itself in a replicated version of our Gaussian field theory.

ACKNOWLEDGMENTS

We are indebted to J. Chalker, L. Balents, A. W. W. Ludwig, M. P. A. Fisher, and X.-G. Wen for many useful comments and penetrating questions. This work was completed while three of us (C.C.C., C.M., and E.F.) were participants at the Program on Quantum Field Theory in Low Dimensions at the Institute for Theoretical Physics of the University of California at Santa Barbara. We are very grateful to the organizers of the program and particularly to Professor Jim Hartle, Director of ITP, for his warm hospitality. This work was supported in part by the NSF through the Grants Nos. NSF DMR94-24511 and NSF DMR-89-20538 at the University of Illinois at Urbana-Champaign (C.C.C., E.F., H.C., and P.G.), NSF DMR94-11574 at Massachusetts Institute of Technology (C.M.), and by NSF PHY94-07194 at the Institute for Theoretical Physics of the University of California at Santa Barbara.

APPENDIX A: SYMMETRY AND GREEN FUNCTIONS

The probability distribution for the disorder, Eq. (2.7), allows for an exact symmetry under a constant real shift of the field configuration $\phi(\mathbf{x})$:

$$\phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}) + \zeta, \quad (\text{A1})$$

$$\psi(\mathbf{x}) \rightarrow e^{-\zeta} \psi(\mathbf{x}), \quad (\text{A2})$$

which leaves both $P[\phi(\mathbf{x})]$ and $\Psi(\mathbf{x})$ unchanged. However, neither $Z(q)$ nor $\Omega(E)$ is invariant:

$$\Omega(E) \rightarrow \Omega(E - 2\zeta), \quad (\text{A3})$$

$$Z(q) \rightarrow e^{-2q\zeta} Z(q), \quad (\text{A4})$$

$$\langle \ln Z(q) \rangle \rightarrow -2q\zeta + \langle \ln Z(q) \rangle. \quad (\text{A5})$$

This means that, although $\langle \ln Z(q) \rangle - q \langle \ln Z(1) \rangle$ is well defined and, in principle, one can compute $\tau(q)$, in a naive calculation $\langle \ln Z(q) \rangle$ would be ill defined.

To perform an actual calculation it is convenient to break this symmetry in a controlled way. The simplest approach is to add a mass term $m = 1/L$ to the action. This penalizes configurations for which $\phi \neq 0$. Another possibility is to impose Dirichlet boundary conditions on $\phi(\mathbf{x})$. Although Green functions for $\phi(\mathbf{x})$ in each of these cases will be different, the only value we need for our purposes is their short-distance limit, which is the same in all cases, namely,

$$G(\mathbf{x}, \mathbf{y}) \approx -\frac{g}{4\pi} \ln \left(\frac{|\mathbf{x} - \mathbf{y}|^2 + a^2}{L^2} \right). \quad (\text{A6})$$

APPENDIX B: MOMENTS OF $\Omega(E)$ AND $Z(\beta)$

In this appendix, we estimate moments of the density of states $\Omega(E)$ and of the partition function $Z(\beta)$ which, we recall, are related by Eq. (2.21). Such moments are needed to decide if quenched and annealed averages are equal. We begin with $\Omega(E)$. Let n be an integer larger than one. By definition,

$$\begin{aligned} \langle \Omega^n(E) \rangle &\equiv \lim_{W \rightarrow 0} \int D[\phi(\mathbf{x})] P[\phi(\mathbf{x})] \\ &\times \sum_{\mathbf{x}_1} \cdots \sum_{\mathbf{x}_n} \int \frac{d\lambda_1}{2\pi} \cdots \int \frac{d\lambda_n}{2\pi} \\ &\times \exp \left[-\frac{W^2}{2} \sum_{k=1}^n \lambda_k^2 + iE \right. \\ &\left. \times \sum_{k=1}^n \lambda_k - 2i \sum_{k=1}^n \lambda_k \phi(\mathbf{x}_k) \right]. \end{aligned} \quad (\text{B1})$$

We assume that all sums and integrals can be freely interchanged. Averaging over disorder is a Gaussian integral yielding

$$\begin{aligned} \langle \Omega^n(E) \rangle &= \lim_{W \rightarrow 0} \sum_{\mathbf{x}_1} \cdots \sum_{\mathbf{x}_n} \int \frac{d\lambda_1}{2\pi} \cdots \int \frac{d\lambda_n}{2\pi} \\ &\times \exp \left[-\frac{1}{2} \sum_{k,l=1}^n \lambda_k (W^2 \delta_{kl} + 4G_{kl}) \lambda_l \right. \\ &\left. + iE \sum_{k=1}^n \lambda_k \right]. \end{aligned} \quad (\text{B2})$$

Here, G_{kl} is a shorthand notation for the Green function in Eq. (A6) with arguments \mathbf{x}_k and \mathbf{x}_l . We notice that the integrand on the right-hand side of Eq. (B2) does not depend on $\mathbf{x}_1, \dots, \mathbf{x}_n$ for $n=1$ but does for $n>1$. For higher moments than $n=1$, the statistical correlations encoded by $G_{kl} \neq 0$ for $k \neq l$, imply that, in a finite system, $\langle \Omega^n(E) \rangle \neq \langle \Omega(E) \rangle^n$. In the thermodynamic limit, the difference between $\langle \Omega^n(E) \rangle$

and $\langle \Omega(E) \rangle^n$ disappears if the statistical correlations are short range, as is the case in the original random energy model (REM) of Derrida, where all energies are assumed to be identically and independently distributed random variables.⁹ This is not so, however, if G_{kl} encodes long range statistical correlations as is the case here. For any moment $n \geq 1$, there will be a critical energy density $e_c(n) \leq e_c(n-1)$ such that the ratio $\langle \Omega^n(E) \rangle / \langle \Omega(E) \rangle^n$ diverges in the thermodynamic limit for $|e| > e_c(n)$. In other words, $\Omega(e)$ is broadly distributed in the thermodynamic limit. Our claim is that it is the limit $e_c \equiv \lim_{n \rightarrow 1} e_c(n)$ that controls the freezing transition, and not the naive replica limit $\lim_{n \rightarrow 0} e_c(n) = \infty$.

Instead of calculating the sequence $e_c(n)$ explicitly, we calculate the ratio

$$R_n(\beta) \equiv \frac{\langle Z^n(\beta) \rangle}{\langle Z(\beta) \rangle^n}. \quad (\text{B3})$$

We find that

$$R_n(\beta) \sim \int \frac{d^2 x_1}{L^2} \cdots \frac{d^2 x_n}{L^2} \prod_{i < j} \left| \frac{\mathbf{x}_i - \mathbf{x}_j}{L} \right|^{-2g\beta^2/\pi} \\ \sim c_n(g\beta^2) + \left(\frac{a}{L} \right)^{2(n-1)(1-ng\beta^2/2\pi)}. \quad (\text{B4})$$

In the thermodynamic limit, the right-hand side is a finite number for $n \leq (2\pi)/(g\beta^2)$ (assuming $n > 1$). In this case $Z^n(\beta)$ fluctuates weakly. But for $n > (2\pi)/(g\beta^2)$, $R_n(\beta)$ diverges, and thus $Z^n(\beta)$ fluctuates strongly. There are important consequences that follow from Eq. (B4).

(1) There exists a *sequence of critical* β_n given by

$$\beta_n^2 = \frac{2\pi}{ng} \equiv \frac{\beta_c^2}{n}, \quad (\text{B5})$$

below which $\ln \langle Z^n(\beta) \rangle / N = \ln \langle Z(\beta) \rangle^n / N$ in the thermodynamic limit $N \rightarrow \infty$. Remarkably, the *same* sequence of critical moments is shared by the random energy models studied in Refs. 9–11.

(2) Caution is needed when using the replica trick

$$\langle \ln Z(\beta) \rangle = \lim_{n \rightarrow 0} \frac{\langle Z^n(\beta) \rangle - 1}{n}. \quad (\text{B6})$$

Indeed, $\langle Z^n(\beta) \rangle$ is not an analytic function of n due to singularities at $\beta = 2\pi/\beta_c^2 g$, and $n = 1$, and caution must be used when using the replica trick to calculate $\ln Z(\beta)$.

For GREM, β_c indicates a phase transition between the regime $\beta \leq \beta_c$ and the regime $\beta > \beta_c$. For all GREM, the quenched and annealed free energy are equal if $\beta \leq \beta_c$. However, quenched and annealed free energy are not equal if $\beta > \beta_c$, and need not obey the same functional dependency on β for different GREM in this regime of temperatures. There are essentially two Gaussian GREM's whose quenched free energy obey the same functional dependency on β : the REM with uncorrelated energies⁹ and the GREM with logarithmic correlated energies.^{10,11} Both undergo a sharp freezing transition characterized by a discontinuous one step specific heat (as opposed to continuous). Since we have proven that the same freezing transition characterizes our multifractal scaling exponents, we conclude that they must be self-averaging and given by an annealed average below the critical moment q_c .

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¹⁹In contrast, if there is a nonvanishing total flux of the magnetic field, new effects need to be considered. In the presence of such a flux, the number of zero modes of the Dirac operator are determined by the index theorem, which holds regardless of the existence of a topologically trivial random component. Consequently, the fields ϕ and χ acquire topologically nontrivial components associated with large gauge transformations and the allowed states become chiral eigenstates with chiral eigenvalue determined by the sign of the magnetic field.

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$$\begin{aligned}
 KP\{X \geq K\} &= K \int_K^\infty P\{X=y\} dy \leq \int_K^\infty y P\{X=y\} dy \\
 &\leq \int_0^\infty y P\{X=y\} dy \equiv \langle X \rangle.
 \end{aligned}$$

The first inequality holds simply using $K \leq y$, and the last equality is just the definition of average value. By taking $K=1$ and $X=\Omega_{>}(e)$ we obtain Eq. (3.7), and by taking $K=1$ and $X=\Omega(e)$ we obtain Eq. (3.10).

²⁸To simplify the argument, we are assuming here a Heaviside function regularization for $\delta_W(E)$.

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