

Integrabilities of the long-range t - J models with twisted boundary conditions

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The integrability of the one-dimensional long-range supersymmetric t - J model has previously been established for both open systems and those closed by periodic boundary conditions through explicit construction of its integrals of motion. Recently the system has been extended to include the effect of magnetic flux, which gives rise to a closed chain with twisted boundary conditions. While the t - J model with twisted boundary conditions has been solved for the ground state and full energy spectrum, proof of its integrability has so far been lacking. In this paper we extend the proof of integrability of the long-range supersymmetric t - J model and its $SU(m|n)$ generalization to include the case of twisted boundary conditions. [S0163-1829(97)51606-7]

Solvable models have attracted attention from both the high-energy and condensed-matter communities. These models provide important examples where it is possible to deal with many degrees of freedom without having to resort to perturbation theory. Interesting models in condensed-matter physics that have been solved include the short-range spin model,¹ delta-function Bose gas,² delta-function electron gas,^{3,4} the Hubbard model,⁵ the Luttinger model,⁶ the magnetic impurity model and the Anderson impurity model.^{7,8}

In one dimension, ever since Haldane and Shastry independently introduced the $1/r^2$ spin model,^{9,10} there has been considerable interest in the model and its generalizations, such as the long range supersymmetric t - J model¹¹⁻¹⁴ and its multicomponent $SU(m|n)$ version (where m and n denote the number of bosonic and fermionic species, respectively). All of these models are characterized by having a ground-state wave function which takes on a Jastrow product form, and by having quasiparticle scattering matrices of a very simple form, as in the continuous Calogero-Sutherland systems describing nonrelativistic quantum particles.¹⁵ In particular, the Haldane-Shastry spin model can also be identified as a free system composed of identical particles obeying Haldane's generalized Pauli principle,¹⁶ and obeying a generalized statistical distribution function at finite temperature.¹⁷ In 1992, Gebhard and Ruckenstein introduced the long-range Hubbard model, in which the electrons are described by the $1/r$ Hubbard model. It is noteworthy that this $1/r$ Hubbard model is integrable for any on-site energy; the full energy spectrum and thermodynamics have been solved explicitly.¹⁸ At half-filling and in the limit of large interaction, this model reduces to the $SU(2)$ Haldane-Shastry spin chain. For less than half-filling, but still in the limit of $U = \infty$, the system remains characterized by eigenfunctions of a Jastrow product form.

Recently there has been considerable interest in adding magnetic flux to the Haldane-Shastry-type models. For a one-dimensional ring threaded by flux, this reduces to the problem of incorporating twisted boundary conditions. A twisted version of the long-range integrable Haldane-Shastry spin chain has been introduced, and was solved in the ratio-

nal flux case.¹⁹ Subsequently, this was generalized to the case of the long-range t - J model with twisted boundary conditions.^{20,21} In particular, it was shown that the irrational flux case can be treated identically,²¹ indicating that there is no essential difference between rational and irrational flux. Based on the exact solutions, it is natural to expect that the long-range models remain integrable despite the twisted boundary conditions. However, until now this has remained an open problem. In this paper, we provide a proof of the integrability of the long-range t - J model and its $SU(m|n)$ generalization with twisted boundary conditions by explicitly constructing an infinite number of simultaneous constants of motion. This construction is a straightforward extension of the methods used in the absence of flux,²²⁻²⁷ and is motivated by the mapping of the closed ring onto an equivalent open system where the flux is manifested in twisted boundary conditions. A further consequence of this mapping is that it yields a unified treatment of the integrability of both the open and closed chains.

Because of the subtleties involved with introducing magnetic flux into a model with long-range interactions, we follow the procedure of Ref. 19, and start with an open chain which is subsequently closed through appropriate boundary conditions to form a ring of N sites. The Hamiltonian of the $SU(1|2)$ t - J model on this open lattice takes the form

$$H_0 = \frac{1}{2} P_G \sum_{\alpha \neq \beta} \frac{1}{(q_\alpha - q_\beta)^2} \left[- \sum_{\sigma} (c_{\alpha\sigma}^\dagger c_{\beta\sigma} + \text{H.c.}) + \hat{P}_{\alpha,\beta} - (1 - n_\alpha)(1 - n_\beta) \right] P_G, \quad (1)$$

where P_G is the Gutzwiller projection onto single occupancy and the operators c and c^\dagger are the usual fermion operators that satisfy standard fermionic anticommutation relations. The summation over σ is to sum over all the spin components of the electrons, $\sigma = \uparrow, \downarrow$ for $SU(1|2)$. The spin exchange operator $\hat{P}_{\alpha,\beta}$ is given by

$$\hat{P}_{\alpha,\beta} = \sum_{\sigma} \sum_{\sigma'} c_{\alpha\sigma}^\dagger c_{\alpha\sigma'} c_{\beta\sigma'}^\dagger c_{\beta\sigma}, \quad (2)$$

and the electron density operator is $n_\alpha = \sum_{\sigma} c_{\alpha\sigma}^\dagger c_{\alpha\sigma}$. The lattice permutation form of this Hamiltonian may be made explicit by introducing the *graded* permutation operator $\Pi_{\alpha,\beta}^{\nu,\nu'}$ which exchanges particles of species ν and ν' at locations α and β (where $\nu, \nu' = 0, \uparrow, \downarrow$ with 0 denoting a hole). Written in terms of Π , the Hamiltonian becomes

$$H_0 = -\frac{1}{2} \sum_{\alpha \neq \beta} \sum_{\nu, \nu'} \frac{\Pi_{\alpha,\beta}^{\nu,\nu'}}{(q_\alpha - q_\beta)^2}, \quad (3)$$

where the Gutzwiller projection is no longer necessary since the graded permutation operator does not generate doubly occupied states. In this form, the $SU(m|n)$ generalization is immediately obvious.

For a finite open chain with L sites, integrability is achieved when the lattice positions q_α lie at the roots of the L th Hermite polynomial.²² In the limit $L \rightarrow \infty$ these roots become equally spaced, and translational invariance is restored. It is precisely in this limit that it is possible to close the chain by demanding twisted boundary conditions. We allow a separate twist ϕ_ν for each independent species ν , so that the twisted boundary conditions for a ring of N sites may be encoded by the requirement that

$$\Pi_{\alpha,\beta+lN}^{\nu,\nu'} = z^{lN(\phi_\nu - \phi_{\nu'})} \Pi_{\alpha,\beta}^{\nu,\nu'}, \quad (4)$$

where z is the primitive N th root of unity. Note that the basic period for the spectral flow corresponds to ϕ_ν running from 0 to 1. Since the resulting closed system is translationally invariant, we single out one period and reexpress the Hamiltonian, after gauge transformation, as $H_0 = E_0 + H$ where¹⁹

$$\begin{aligned} H &= -\frac{1}{2} \sum_{1 \leq i \neq j \leq N} \sum_{\nu, \nu'} \sum_{l=-\infty}^{\infty} \frac{z^{(i-j-lN)(\phi_\nu - \phi_{\nu'})} \Pi_{i,j}^{\nu,\nu'}}{(i-j-lN)^2} \\ &= -\frac{1}{2} \sum_{1 \leq i \neq j \leq N} \sum_{\nu, \nu'} J_{\phi_\nu - \phi_{\nu'}}(i-j) \Pi_{i,j}^{\nu,\nu'} \end{aligned} \quad (5)$$

now defines the long-range supersymmetric t - J model on a periodic ring. The offset $E_0 = \pi^2/3N$ accounts for exchanges lN units apart (which is present in H_0 but not in the periodic H), and may be interpreted as a shift in the ground-state energy from finite-size effects. The sum over l ensures the appropriate periodicity of the ring under translations, and yields the inverse trigonometric potential²¹

$$\begin{aligned} J_\phi(n) &= \sum_l \frac{z^{(n+lN)\phi}}{(n+lN)^2} \\ &= \left(\frac{\pi}{N}\right)^2 \frac{z^{[\phi]n} [1 + (\phi - [\phi])(z^n - 1)]}{\sin^2(n\pi/N)}. \end{aligned} \quad (6)$$

This expression is piecewise linear and continuous in ϕ , leading to many remarkable features of this model.¹⁹⁻²¹ In the case of periodic boundary conditions ($\phi=0$), the physical properties of the supersymmetric t - J model on a uniform closed chain have been studied previously.¹¹⁻¹⁴

To provide a proof for the long-range t - J model with twisted boundary conditions, we are motivated by the previous results on the integrabilities of the uniform long-range t - J model with periodic boundary conditions, and the non-

uniform long-range t - J model with open boundary conditions,^{26,27} which were generalizations of the spin chain results.²²⁻²⁵ Proof of the integrability proceeds by first mapping the species-exchange Hamiltonian, Eq. (5), to a lattice permutation equivalent using slave-boson techniques. The resulting Hamiltonian then acts on wave functions ψ written in the form $\psi(q_1\nu_1, q_2\nu_2, \dots, q_N\nu_N)$, where q_i and ν_i label the position and $SU(m|n)$ ‘‘spin’’ of particle i . Acting on such wave functions, and using the fact that $\{q_i\}$ span the lattice due to single occupancy, the Hamiltonian becomes

$$H = -\frac{1}{2} \sum_{1 \leq i \neq j \leq N} J_{\phi_{\nu_i} - \phi_{\nu_j}}(q_i - q_j) M_{ij}, \quad (7)$$

where the particle exchange operator M_{ij} is defined by

$$\begin{aligned} M_{ij} \psi(\dots, q_i \sigma_i, \dots, q_j \sigma_j, \dots) \\ = \psi(\dots, q_j \sigma_i, \dots, q_i \sigma_j, \dots). \end{aligned} \quad (8)$$

Note that the fermionic and bosonic nature of the individual species is fully encoded in the wavefunctions; $\psi \rightarrow \pm \psi$ under simultaneous interchange of position and spin. This independence of the exchange operator from the particle statistics ensures that the proof of integrability holds for *all* $SU(m|n)$ extended t - J models, and not just for the $SU(1|2)$ case.

Based on the integrability proof for the open chain and for the ring closed by periodic boundary conditions studied in Refs. 22 and 23, we introduce the generalized operators

$$\pi_j = i \sum_{k \neq j} u_{\phi_{\nu_j} - \phi_{\nu_k}}(q_j - q_k) M_{jk}, \quad (9)$$

where $u_\phi(n)$ is the (twisted) periodic version of $1/r$

$$u_\phi(n) = \sum_l \frac{z^{(n+lN)\phi}}{n+lN}. \quad (10)$$

As in the case for $J_\phi(n)$, this sum may be performed, yielding

$$u_\phi(n) = \frac{2\pi i}{N} \frac{z^{[\phi]n}}{1 - z^{-n}} \quad (11)$$

(for nonintegral ϕ). Note that ϕ enters discontinuously, with a jump in $u_\phi(n)$ at integral values of ϕ . A careful treatment of convergence issues for integral ϕ indicates that the actual value of the infinite sum in Eq. (10) is the average of the values of $u_\phi(n)$ before and after the discontinuity. Nevertheless, for a consistent treatment of the invariants, we take Eq. (11) as the definition of $u_\phi(n)$ for *all* values of ϕ . A consequence of this asymmetry is to pick a preferred ordering, thus breaking the parity symmetry $u_{-\phi}(-n) = -u_\phi(n)$, which otherwise holds for nonintegral ϕ . Nevertheless, this particular choice of ordering gives

$$u_0(n) = \frac{2\pi i}{N} \frac{1}{1 - z^{-n}}, \quad (12)$$

in agreement with previous results in the absence of flux.^{22,23} Overall, this subtle treatment of integral twists indicates that, surprisingly enough, it is actually the *zero* flux case that is

exceptional; this case corresponds to working on top of the locations of the cusps in the spectral flow itself.

Explicit evaluation of the commutators brings out a distinction between integral and nonintegral twists:

$$[\pi_i, \pi_j] = \begin{cases} \frac{2\pi}{N} z^{\phi_{ij} q_{ij}} M_{ij}(\pi_i - \pi_j) & \text{for } \phi_{ij} \in Z \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

where $\phi_{ij} = \phi_{v_i} - \phi_{v_j}$ and $q_{ij} = q_i - q_j$. Using the relation $(z^{\phi_{ij} q_{ij}} M_{ij}) \pi_j = \pi_i (z^{\phi_{ij} q_{ij}} M_{ij})$, valid whenever $\phi_{ij} \in Z$, the above commutator may be reexpressed in a form similar to that of Ref. 23,

$$[\pi_i, \pi_j] = \begin{cases} \frac{2\pi}{N} [z^{\phi_{ij} q_{ij}} M_{ij}, \pi_i] & \text{for } \phi_{ij} \in Z \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

From here it is obvious that the commutation results of Fowler and Minahan²³ are easily generalized to the present case. Therefore the infinite set of Hermitian operators

$$I_M = \sum_i \pi_i^M \quad (15)$$

(where $M=0,1,2,\dots$), provides a set of mutually commuting operators, $[I_M, I_N]=0$, regardless of the individual species twists. Note that the commutation is trivial for nonintegral relative twists, and is basically a consequence of the simple open chain result, $[\pi_{0\alpha}, \pi_{0\beta}]=0$, where

$$\pi_{0\alpha} = i \sum_{\beta \neq \alpha} \frac{1}{q_\alpha - q_\beta} M_{\alpha\beta} \quad (16)$$

is the corresponding open chain operator [compare with Eq. (9)].

For a finite open chain, it is known that the π_0 operators do not commute with the Hamiltonian,²²

$$[H_0, \pi_{0\alpha}] = 2i \sum_{\beta \neq \alpha} \frac{1}{(q_\beta - q_\alpha)^3}. \quad (17)$$

However, in making the system periodic, the odd exponent in Eq. (17) allows the cancellation of terms exchanging to the left and right. As a result, we find

$$[H, \pi_i] = 0 \quad (18)$$

for the periodic case, regardless of the relative twist angles. It is now clear that the mutually commuting set of operators, $\{I_M\}$ for $M=0,1,2,\dots$, provide explicit constants of motion of the Hamiltonian H , and hence proves the integrability of the long-range $SU(m|n)$ t - J model on a ring (for either twisted or untwisted boundary conditions).

In conclusion, we have provided a proof for the integrability of the long-range t - J models with twisted boundary conditions by explicitly constructing an infinite set of mutually commuting constants of motion. This proof generalizes previous results for rings without flux, and makes use of the viewpoint that the closed chain is simply a periodic version of the open system. A consequence of this similar treatment for both closed and open chains is the demonstration that the key property behind the integrability of these models is simply the permutation nature of the system. These results have filled a gap in that the integrability condition for the twisted t - J model was as yet unknown, in spite of the fact that several thorough studies of the long-range model in the presence of flux have been provided.

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- ¹H. Bethe, Z. Phys. **75**, 205 (1931).
²E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); E. H. Lieb, *ibid.* **130**, 1616 (1963).
³F. Flicker and E. H. Lieb, Phys. Rev. **161**, 179 (1967).
⁴C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); M. Gaudin, Phys. Lett. A **24**, 55 (1967).
⁵E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968); C. N. Yang, *ibid.* **63**, 2144 (1989); C. N. Yang and S. C. Zhang, Mod. Phys. Lett. B **4**, 759 (1990).
⁶D. C. Mattis and E. H. Lieb, J. Math. Phys. (N.Y.) **6**, 304 (1965).
⁷N. Andrei, K. Furuya, and J. H. Lowenstein, Rev. Mod. Phys. **55**, 331 (1983), and references therein.
⁸P. B. Wiegmann, J. Phys. A **14**, 1463 (1981).
⁹F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
¹⁰B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
¹¹Y. Kuramoto and H. Yokoyama, Phys. Rev. Lett. **67**, 1338 (1991).
¹²N. Kawakami, Phys. Rev. B **46**, 1005 (1992).
¹³D. F. Wang, J. T. Liu, and P. Coleman, Phys. Rev. B **46**, 6639 (1992).
¹⁴Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. B **46**, 9359 (1992).
¹⁵B. Sutherland, J. Math. Phys. (N.Y.) **12**, 246 (1971); **12**, 251 (1971); Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972); F. Calogero, J. Math. Phys. (N.Y.) **10**, 2191 (1969); **10**, 2197 (1969).
¹⁶F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).
¹⁷Y. S. Wu, Phys. Rev. Lett. **73**, 922 (1994).
¹⁸F. Gebhard and A. Ruckenstein, Phys. Rev. Lett. **68**, 244 (1992).
¹⁹T. Fukui and N. Kawakami, Phys. Rev. Lett. **76**, 4242 (1996).
²⁰T. Fukui and N. Kawakami, Phys. Rev. B **54**, 5346 (1996).
²¹James T. Liu and D. F. Wang (unpublished).
²²A. P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992); **70**, 2329 (1993).
²³M. Fowler and J. A. Minahan, Phys. Rev. Lett. **70**, 2325 (1993).
²⁴L. Brink, T. H. Hansson, and M. A. Vasiliev, Phys. Lett. B **286**, 109 (1992).
²⁵V. I. Inozemtsev, J. Stat. Phys. **59**, 1143 (1990).
²⁶D. F. Wang and C. Gruber, Phys. Rev. B **49**, 15 712 (1994).
²⁷C. Gruber and D. F. Wang, Phys. Rev. B **50**, 3103 (1994).