Laughlin liquid to charge-density-wave transition at high Landau levels

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We compare the energies of the Laughlin liquid and a charge density wave in a weak magnetic field for the upper Landau-level filling factors $\nu_N=\frac{1}{3}$ and $\frac{1}{5}$. The charge-density-wave period has been optimized and was found to be $\simeq 3.3R_c$, where R_c is the cyclotron radius. We conclude that the optimal charge density wave is lower in energy than the Laughlin liquid for the Landau-level numbers $N\geqslant 2$ at $\nu_N=\frac{1}{3}$ and for $N\geqslant 3$ at $\nu_N=\frac{1}{5}$. This implies that the $\frac{1}{3}$ quantum Hall plateaus cannot be observed for $N\geqslant 2$, in agreement with the experiment. [S0163-1829(97)03916-7]

The fractional quantum Hall effect (FQHE) was first discovered at the lowest Landau level (LL).[1,2] This remarkable phenomenon occurring at certain unique values of the filling factor $\nu=\frac{1}{3},\frac{1}{5},\ldots$ has been associated with the formation of a uniform incompressible quantum state, or the Laughlin liquid.[3] The traditional alternative to the Laughlin liquid is a charge density wave (CDW), which does not exhibit the FQHE. The FQHE occurs because the Laughlin liquid is lower in energy than the optimal CDW, which at the lowest LL (LL index $N=0$) has the same spacial periodicity as the Wigner crystal.[4]

Later, the FQHE was observed at the next LL (LL index $N=1$) and then studied theoretically.[5–9] Except for the first of these works, the exact diagonalization of small systems (typically with six particles) was used. Since already at $N=1$ the FQHE energy gap is rather small, it has been suggested in these works that at $N>1$ the FQHE is absent.[10] To prove this statement, however, it is imperative to increase the system size to keep it larger than the cyclotron radius R_c, which is the characteristic spread of electron wave functions. Such a calculation does not appear to be feasible at present. Another method, used by MacDonald and Girvin,[5] is to compare the energies of trial liquid and crystalline states. MacDonald and Girvin proposed the following wave function for the liquid state at the Nth LL:

$$|\Psi_L^N\rangle=\prod_i \frac{(a_i^\dagger)^N}{\sqrt{N!}}|\Psi_L^0\rangle. \tag{1}$$

Here $a_i^\dagger$ is the *inter*-LL ladder operator, raising the ith electron to the next LL, and $|\Psi_L^0\rangle$ is the Laughlin state at the lowest LL.

Although the failure of the Laughlin liquid to be the ground state at $N>1$ has been conjectured,[5–9] until recently no alternative candidate has been proposed. Indeed, at $N=2$ the $\frac{1}{3}$ Laughlin liquid is still lower in energy than the conventional Wigner crystal.[5] Recently such a candidate was identified as the ''bubble'' phase,[11] or the state where the guiding centers of the cyclotron orbits fill the large domains (bubbles) forming a triangular lattice [Fig. 1(a)]. A quasiclassical image of a single bubble is shown in Fig. 1(b). The optimal number of electrons in a bubble was estimated to be[11]

$$\widetilde{M}\simeq 3\nu_N N, \tag{2}$$

which corresponds to the separation $\simeq 3.3R_c$ between nearest bubbles.

Our goal is to identify the LL index N at which the transition from the Laughlin liquid to the CDW occurs. We show that for, e.g., the $\frac{1}{3}$ FQHE state, this transition indeed takes place at $N=2$.

At small N the quasiclassical picture is too crude, and we have to describe the bubble phase by a trial wave function. To elucidate its structure, we will construct such a wave function in several steps. First, we define the wave function of a single bubble with M electrons at the *lowest* LL:

$$\Psi_0\{\boldsymbol{r}_k\}=\prod_{i<j}(z_i-z_j)\times\exp\left(-\sum_{i=1}^{M}\frac{|z_i|^2}{4l^2}\right). \tag{3}$$

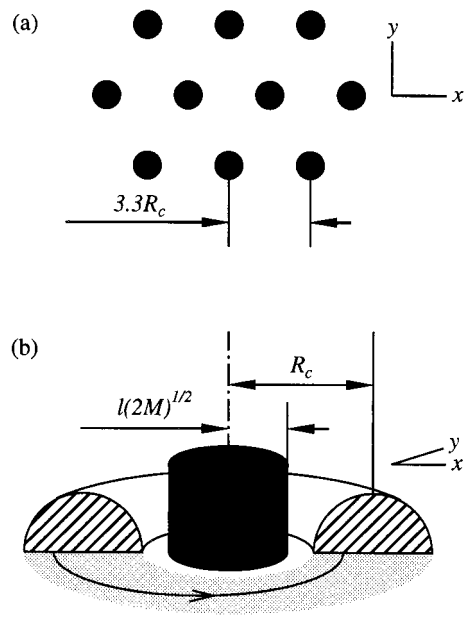

FIG. 1. The quasiclassical image of the bubble phase. (a) Top view. The bubbles (dark circles) are the places where the accumulation of the guiding centers occurs. (b) The enlarged view of one bubble. The dark region shows the guiding center density $\nu(x,y)/2\pi l^2$, while the toroidal figure illustrates the charge density distribution $\rho_N(x,y)$ around the bubble (half of the charge density is removed). This charge density is created by electrons moving in the cyclotron orbits centered inside the bubble. One such cyclotron orbit is shown by the arc.

Here $z_j = x_j + iy_j$ is the complex coordinate of the jth electron, and l is the magnetic length. Second, we construct the wave function of a bubble at the Nth LL centered at point R. This is achieved with the help of the magnetic translation and inter-LL ladder operators,

$$\Psi\{\boldsymbol{r}_k\} = \prod_{i=1}^{M} \frac{(a_i^\dagger)^N}{\sqrt{N!}} \exp\left(\frac{b_i^\dagger \overline{R} - b_i R}{l\sqrt{2}} \right) \Psi_0\{\boldsymbol{r}_k\}, \tag{4}$$

where b_i is an *intra*-LL ladder operator. To finally obtain the wave function of the CDW, we build an antisymmetric combination of the bubbles centered at the triangular lattice sites $\boldsymbol{R}_l$,

$$\Psi_{\mathrm{CDW}} = \sum_P \mathrm{sgn}(P) \prod_l \Psi_l\{P(\boldsymbol{r}_k)\}. \tag{5}$$

Here P's are the permutations of electrons between bubbles. For the case $M=1$ this trial state coincides with the Wigner crystal wave function.[13] It can be easily seen that Ψ_{CDW} is of the Fock type, and that the overlap between the wave functions of different bubbles (for $M = 3\nu_N N$) is negligible.

It proves to be useful for further calculations to introduce the guiding center density

$$\hat{\nu}(\boldsymbol{r}) = 2\pi l^2 \sum_i \delta(\boldsymbol{r} - \hat{\boldsymbol{R}}_i). \tag{6}$$

The summation here is carried over the electrons at the considered LL, and $\hat{\boldsymbol{R}}_i = \boldsymbol{r}_i + (1/\omega_c)[\hat{\boldsymbol{z}} \times \hat{\boldsymbol{v}}_i]$ is the guiding center operator, with $\hat{\boldsymbol{v}}_i$ and ω_c being the velocity of the ith particle and the cyclotron frequency, respectively.[14] It can be shown that for the state defined by Eq. (5),

$$\langle \hat{\nu}(q) \rangle = \frac{\nu_N A}{M} F_{MM-1}(q), \quad F_{MK} = L_K^{M-K}\left(\frac{q^2 l^2}{2} \right) e^{-q^2 l^2/4}, \tag{7}$$

where A is the area of the system, and $L_M^K(x)$ is the Laguerre polynomial.

Now we would like to find the cohesive energy $E_{\mathrm{coh}}^{\mathrm{CDW}}$ of our trial state. The calculation is quite similar to the case of the lowest LL,[4]

$$E_{\mathrm{coh}}^{\mathrm{CDW}} = \frac{1}{2\nu_N} \sum_{q \neq 0} u_{\mathrm{HF}}(q) \left| \frac{\langle \hat{\nu}(q) \rangle}{A} \right|^2 \tag{8}$$

(see Ref. 11 for details). The summation in Eq. (8) is carried over the reciprocal vectors of the triangular lattice. The Hartree-Fock interaction potential $u_{\mathrm{HF}}(q)$ is defined as follows (cf. Ref. 11):

$$u_{\mathrm{HF}}(q) = u_{\mathrm{H}}(q) - u_{\mathrm{ex}}(q),$$

$$u_{\mathrm{H}}(q) = \frac{v(q)}{2\pi l^2 \epsilon(q)} F_{NN}(q),$$

$$u_{\mathrm{ex}}(q) = 2\pi l^2 \int \frac{d^2 q'}{(2\pi)^2} e^{i q q' l^2} u_{\mathrm{H}}(q').$$

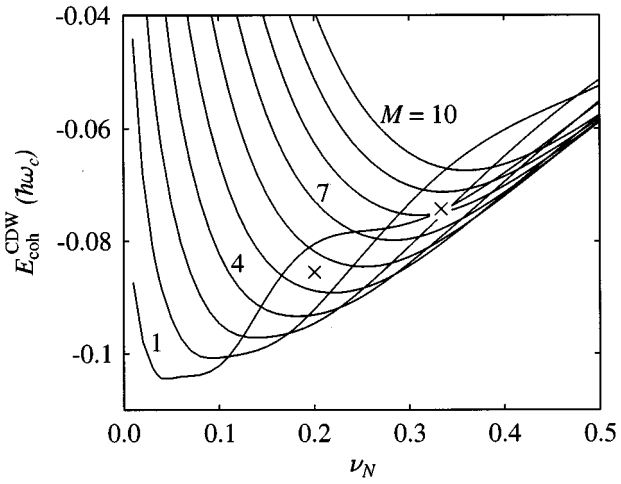

FIG. 2. The cohesive energy of the CDW as a function of ν_N for different numbers of electrons in a bubble M. The calculations are made for $N=5$ and $r_s = \sqrt{2}$. The crosses mark the Laughlin liquid energies.

Here $\epsilon(q)$ takes into account the screening of the Coulomb interaction among the electrons at the upper LL by lower LL's. It is given by[12]

$$\epsilon(q) = 1 + v(q)\Pi(q), \tag{9}$$

$$\Pi(q) = \frac{2}{\pi l^2} \sum_{m < N \leq n} \frac{(-1)^{n-m}}{\hbar\omega_c (n-m)} F_{nm}(q) F_{mn}(q), \tag{10}$$

where $v(q) = 2\pi e^2/\kappa q$ is the Coulomb potential. This dielectric function tends to unity in the limits $q \to 0$ and $q \to \infty$, and reaches its largest value of $1 + \sqrt{2} N r_s$ at $q \sim R_c^{-1}$. Here $r_s = \sqrt{2} e^2/\kappa \hbar v_F$ is the gas parameter. Accounting for the LL mixing by means of the dielectric function was shown to be accurate provided $r_s \ll 1$ and $N r_s \gg 1$ (Ref. 12). Moreover, the results obtained within the framework of this model remain correct to the leading order in r_s even for $N r_s \ll 1$. In the latter limit $\epsilon(q) \simeq 1$, which is consistent with the fact that the LL mixing can be ignored completely.

Using Eqs. (7) and (8), the cohesive energy for any given ν_N can be calculated numerically. The result is, of course, different for different values of M (see Fig. 2). Therefore, one has to find $\widetilde{M}$ corresponding to the lowest energy. The energies of the CDW optimized in this way are summarized in Tables I and II. Notice that the energy unit used in the tables, $r_s \hbar\omega_c$, differs from $e^2/\kappa l$ traditionally used in the case of the lowest LL. The reason for this difference is that, at high LL's, the relevant length scale is not the magnetic length l but the cyclotron radius $R_c = \sqrt{2N+1}\, l$. Hence the natural energy scale is $e^2/\kappa R_c$, which coincides with $r_s \hbar\omega_c$ up to a numerical factor.[11]

In Table I we present the results for the case $N r_s \ll 1$, when the LL mixing can be ignored completely, i.e., $\epsilon(q) \equiv 1$. Table II contains the results for the case $r_s = \sqrt{2}$, which corresponds to the practical range of electron densities in GaAs heterostructures. One can see that the optimal number of electrons per bubble is the same both with and without the screening, and is in perfect agreement with Eq. (2).

TABLE I. The cohesive energies of the Laughlin liquid $E_{\rm coh}^{\rm L}$ and the CDW $E_{\rm coh}^{\rm CDW}$ in the limit $r_s \to 0$. $\tilde{M}$ is the optimal number of electrons per bubble. The energy unit is $r_s \hbar \omega_c$. The energy per electron in the uniform uncorrelated state $E^{\rm UEL}$ is provided for reference. Correlated crystal $E_{\rm coh}^{\rm CDW} = -0.2473 r_s \hbar \omega_c$.

			$\nu_N = \frac{1}{3}$		
N	$\tilde{M}$	$E^{\rm UEL}$	$E_{\rm coh}^{\rm L}$	$E_{\rm coh}^{\rm CDW}$	$\delta E/E_{\rm coh}^{\rm CDW}$
0	1	−0.0853	−0.0820(1)	−0.0733	−11.9%
1	1	−0.1692	−0.1831(3)	−0.1726	−6.1%
2	2	−0.1970	−0.1925(4)	−0.2163	11.0%
3	3	−0.2135	−0.2073(5)	−0.2433	14.8%
4	4	−0.2251	−0.2226(6)	−0.2480	10.3%
5	5	−0.2341	−0.2340(7)	−0.2767	15.4%
			$\nu_N = \frac{1}{5}$		
N	$\tilde{M}$	$E^{\rm UEL}$	$E_{\rm coh}^{\rm L}$	$E_{\rm coh}^{\rm CDW}$	$\delta E/E_{\rm coh}^{\rm CDW}$
0	1	−0.0396	−0.0638(1)	−0.0622	−2.7%
1	1	−0.0986	−0.2110(4)	−0.2043	−3.3%
2	1	−0.1164	−0.2473(6)	$-0.2454^{\dagger}$	−0.75%
3	2	−0.1267	−0.2458(8)	−0.2811	12.6%
4	2	−0.1340	−0.2481(9)	−0.2990	17.0%
5	3	−0.1395	−0.2569(9)	−0.3187	19.4%

The above results have been tested by the self-consistent Hartree-Fock procedure, similar to that described in Ref. 4. Starting from the initial approximation given by wave function (5), this procedure finds the optimal set of $\langle \hat{\nu}(q) \rangle$ for a given periodicity of the CDW. The obtained corrections are of the order of $10^{-5} r_s \hbar \omega_c$, and do not affect the significant digits displayed in Tables I and II. We associate the corrections with a slight nonorthogonality of the wave functions of different bubbles.

Let us now discuss the Laughlin liquid at high LL's. As Haldane pointed out in Ref. 1, the interaction of electrons confined to a single LL is described by means of a discrete set of pseudopotentials V_m defined by

$$V_m = \frac{1}{2\pi} \int d^2 q u_{\rm H}(q) F_{mm}(\sqrt{2} q). \tag{11}$$

In particular, the cohesive energy of the Laughlin liquid can be written in the following form:

$$E_{\rm coh}^{\rm L} = \frac{\nu_N}{\pi} \sum_{m=1}^{\infty} c_m V_m . \tag{12}$$

In this formula c_m are the coefficients in the expansion[15]

$$h(r) = 2 \sum_{m=1}^{\infty} \frac{c_m}{m!} \left(\frac{r^2}{4l^2} \right)^{2m} e^{-r^2/4l^2} \tag{13}$$

for the density-density correlation function $h(r)$ of the Laughlin liquid at the *lowest* LL,

$$h(r) \equiv \frac{\langle \hat{\rho}(r) \hat{\rho}(0) \rangle - \langle \hat{\rho} \rangle^2}{\langle \hat{\rho} \rangle^2}. \tag{14}$$

The coefficients c_m have been found from a fit to our Monte Carlo data on $h(r)$. Although the problem of finding such a fit is a nontrivial one, we are unable to give the details here because of space limitations. The results are summarized in Tables I and II.

At this point we can compare the energies of the Laughlin liquid and the CDW. As one can see, at $N=0$ and 1, the Laughlin liquid is lower in energy. At large N, however, the CDW wins. The transition to the bubble state both with and without screening takes place at $N=2$ for $\nu_N = \frac{1}{3}$ and at $N=3$ for $\nu_N = \frac{1}{5}$. The difference in the energies of these two states at $N=2$ and $\nu_N = \frac{1}{5}$ is very small. For this reason we attempted to improve the trial state (5) further by introducing the magnetophonon correlations.[16] The optimal CDW at this point has only one electron per unit cell, and so only few modifications to the original method of Lam and Girvin[16] were necessary here. Our computations show that the difference between the energies of the *correlated* Wigner crystal and the Laughlin liquid is smaller than the numerical error (see Table I). Hence it is still unclear which phase is the ground state at $\nu = 4\frac{1}{5}$.

So far, we have considered a somewhat idealized system. In order to make contact with the experimental practice, we will briefly discuss the effects of disorder and the finite-z extent of the wave functions. In regard to the former, it is rather clear that an external impurity potential will favor the CDW state. Indeed, the CDW can lower its energy by adjusting to the external potential, while the incompressible $\nu_N = \frac{1}{5}$ liquid state cannot.[17] We have also studied the effect of the finite z extent of the wave functions. To this end we used the Fang-Howard form factor[18] with the thickness parameter b as large as $b = 2k_{\rm F}^{-1}$. Qualitatively, the result is that the finite z extent diminishes the energy difference between the CDW and the Laughlin liquid. However, the transition point remains the same.

Finally, another comment is in order here. Although the FQHE at the lowest LL has been unambiguously identified with the formation of the Laughlin liquid, at higher LL's

TABLE II. Same as Table I, but $r_s = \sqrt{2}$, which corresponds to the electron density of 1.6×10^{11} cm^{-2} in GaAs-Al$_x$Ga$_{1-x}$As heterostructures. The energies are now given in the units of $\hbar \omega_c$.

			$\nu_N = \frac{1}{3}$		
N	$\tilde{M}$	$E^{\rm UEL}$	$E_{\rm coh}^{\rm L}$	$E_{\rm coh}^{\rm CDW}$	$\delta E/E_{\rm coh}^{\rm CDW}$
0	1	−0.1206	−0.1159(1)	−0.1037	−11.8%
1	1	−0.1297	−0.1519(3)	−0.1424	−6.7%
2	2	−0.1136	−0.1141(3)	−0.1188	4.0%
3	3	−0.1034	−0.0946(3)	−0.1018	7.1%
4	4	−0.0965	−0.0824(3)	−0.0896	8.0%
5	5	−0.0914	−0.0733(3)	−0.0805	8.9%
			$\nu_N = \frac{1}{5}$		
N	$\tilde{M}$	$E^{\rm UEL}$	$E_{\rm coh}^{\rm L}$	$E_{\rm coh}^{\rm CDW}$	$\delta E/E_{\rm coh}^{\rm CDW}$
0	1	−0.0560	−0.0903(2)	−0.0880	−2.6%
1	1	−0.0765	−0.1727(7)	−0.1692	−2.1%
2	1	−0.0677	−0.1420(9)	−0.1396	−1.7%
3	2	−0.0618	−0.1139(9)	−0.1202	5.2%
4	2	−0.0577	−0.0963(9)	−0.1050	8.3%
5	3	−0.0547	−0.0849(9)	−0.0946	10.3%

some other liquid state may exist. One can speculate, for example, that the tendency to form many electron groups is generic for high LL's, and instead of the Laughlin liquid one has a liquid of bubbles. A similar idea was put forward by Halperin[19] to explain the FQHE at $\nu=\frac{2}{5},\frac{2}{7}$, etc., at the lowest LL. Despite the attractiveness of such an idea, we doubt the existence of either Laughlin or even more sophisticated liquid states at high LL's for the following reason. The liquid state can be thought of as a CDW melted by the zero-point vibrations. For such a melting to occur, the amplitude of these vibrations must be comparable to the lattice constant (the Lindemann criterion). This amplitude is determined by the magnetic barrier, since the interaction energy per electron is smaller than the cyclotron gap. Hence it does not exceed the magnetic length l. Now an important difference between the low and high LL's becomes clear. At low LL's the CDW contains only one electron per unit cell, and the lattice constant decreases with increasing filling factor. At some value of ν_N it becomes of the order of l, and the crystal melts into the Laughlin liquid. At high LL's, however, the lattice constant does not change much as one increases the LL filling, but remains of the order of $R_c \gg l$. Hence it is highly unlikely that the CDW would be melted by the quantum fluctuations at high LL's.

In conclusion, we have compared the energies of the Laughlin liquid and the CDW with the optimized period ($\sim R_c$) at the upper LL filling factors $\nu_N=\frac{1}{3}$ and $\frac{1}{5}$. We found that the $\frac{1}{3}$ liquid state is unstable for $N\geqslant 2$, while the $\frac{1}{5}$ state loses to the CDW at $N\geqslant 3$. Our result implies that the $\frac{1}{3}$ quantum Hall plateaus cannot be observed at filling factors $\nu>4$. This conclusion is in agreement with the existing experimental data.[2] The difference between the energies of the CDW and the Laughlin liquid at $N=2$ and $\nu_N=\frac{1}{5}$ is so small that more work is needed to distinguish them unambiguously.

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