Transition-temperature features of layered superconductors

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A model theory for the superconducting transition temperature T_c is derived that is based on a generalized version of the Eliashberg equations taking into account an energy dependence in the electronic density of states $N(\epsilon)$. We treat an electronic structure with characteristic two-dimensional character. An analytical expression for T_c obtained for intermediate coupling establishes features of the T_c in the logarithmic van Hove scenario. Results describe the behavior of T_c vs x observed in well-annealed $La_{2-x}Sr_xCuO_4$ oxides. [S0163-1829(97)03714-4]

INTRODUCTION

A characteristic feature of nonconventional superconductors is the presence of low-dimensional structural elements such as weakly coupled planes or chains of atoms. Because of this marked anisotropy it is possible that $N(\epsilon)$, the electronic density of states (EDOS), in cuprate superconductors, for example, is not smooth but exhibits appreciable variations with energy ϵ over a significant scale. In particular, the singularities in a van Hove scenario, necessarily present in periodic potentials, may play an important role.[1–8] Recent experimental,[9–14] as well as theoretical results[15–17] in nonconventional superconducting materials suggesting a Fermi level pinned very close to a van Hove saddle point in $N(\epsilon)$, raise important questions about the possible role of a nonsmooth EDOS and its subsequent effect on T_c in these materials.

In conventional superconductivity the $N(\epsilon)$ is assumed not to vary around the Fermi energy, ϵ_F, over the characteristic energy range corresponding to the maximum phonon energy, ω_0. The EDOS is then replaced by its value at ϵ_F. On the other hand, if ϵ_F is located near a singularity any infinitesimal departure of ϵ from ϵ_F gives rise to sharp variations in $N(\epsilon)$ that must be taken into account in a consistent theory.

The formalism with nonconstant $N(\epsilon)$ was initially developed in anticipation of possible applications to ''classical'' high-temperature superconductors such as the $A15$ compounds, and were carried out within the framework of BCS, as well as of Eliashberg theory which has since been extended by many authors.[18–22] With the discovery of high-T_c cuprate superconductors, interest in this area has resurfaced as a possible explanation of the T_c enhancement observed in the cuprates. Nevertheless, in recent work[23] within the framework of the Fermi-surface-restricted Eliashberg theory,[18,24] it is concluded that a van Hove scenario (VHS) to enhance T_c operates *only in weak coupling*, and that inclusion of strong coupling deactivates the T_c-enhancing effects of the VHS. In the pioneering work of Pickett[18] it is argued that for strongly coupled superconductors peaks in the EDOS are quite ineffective in raising T_c. By assuming that ϵ_F lies near the center of a very narrow peak, these authors estimated *a reduction of* T_c by 15% in $A15$ compounds. However, in more recent studies by Mansor and Carbotte[25] a T_c greatly enhanced over the value when van Hove singularities are not present is obtained which drops rapidly by shifting ϵ_F away from the center of the EDOS peak, as expected. Clearly, additional work is required in order to clarify the situation.

Eliashberg theory determines the critical temperature as a functional of the electron-phonon interaction spectrum, $\alpha^2F(\omega)$, so that obtaining a universal T_c describing all superconducting materials is, strictly speaking, impossible. Fortunately, however, T_c depends mainly on the characteristic phonon frequency $\omega_{\rm ph}$ as well as on the value of the mass renormalization factor λ. This circumstance admits various analytical approaches based on physical assumptions about the actual electron-phonon interaction. Specifically, the familiar expressions for T_c of McMillan,[26] of Allen and Dynes,[27] of Leavens and Carbotte,[28] as well as the more recent formula of Kresin,[29] all follow after making reasonable approximations of the gap function and of the effective electronic mass. These solutions for T_c succeed in describing the behavior of the exact solution of the ordinary Eliashberg equations for appropriately chosen ranges of the interaction parameter λ. But for modified Eliashberg equations with a nonconstant EDOS analytical solutions for T_c have not been reported. Results are obtainable either from numerical analyses or for weak coupling, $\lambda \to 0$.[16,17] Meanwhile, a simple analytical representation based on specifically physical assumptions would shed light on the dependence of T_c on the relevant parameters of the underlying strongly correlated electron-phonon system, and perhaps suggest nontrivial information about such a complex many-particle system.

The purpose of this paper is to present the results of analytic calculations for T_c for values of $\lambda < 1.5$ and for

$\omega_{\rm ph} \gg \pi T_c$. For such parameter restrictions the method of solving the standard Eliashberg equations was developed in Refs. 30 and 31 (see below) according to which the solution for the gap in the quasiparticle excitation spectrum is represented in a form reflecting features of the $\alpha^2F(\omega)$ without specifying its explicit shape. The final expression for the critical temperature follows as the eigenvalue of an integral equation for the gap function and is a functional of $\alpha^2F(\omega)$. Naturally, one looks for a solution of the modified Eliashberg equations by the methods of the usual theory. Since for cuprates one deals with conducting planes, we shall use the general form of $N(\epsilon)$ which is known for low-dimensional periodic structures.

I. BASIC EQUATIONS AND METHOD

For $T=T_c$ the Eliashberg equations written in the imaginary-axis representation, valid for general $N(\epsilon)$,[18,22] are

$$\Delta(i\omega_n)\widetilde{Z}(i\omega_n)=\pi T\sum_{m=-\infty}^{+\infty}\{\lambda I(m-n)-\mu(\omega_c)\}\times\frac{\Delta(i\omega_m)}{|\omega_m|}\widetilde{N}(|\widetilde{\omega}_m|), \tag{1}$$

and

$$\widetilde{Z}(i\omega_n)=1+\frac{\pi T\lambda}{\omega_n}\sum_{m=-\infty}^{+\infty} I(m-n)\,\mathrm{sgn}(\omega_m)\widetilde{N}(|\widetilde{\omega}_m|), \tag{2}$$

$$\widetilde{\omega}_n=\omega_n\widetilde{Z}(i\omega_n),$$

where $i\omega_n=i\pi T_c(2n-1)$ are the Matsubara frequencies. Equations (1) and (2) determine the critical temperature T_c resulting from a specific electron-phonon spectral density $\alpha^2F(\omega)$ which enters through

$$I(n-m)=\frac{2}{\lambda}\int_0^\infty\frac{\omega}{\omega^2+(\omega_n-\omega_m)^2}\alpha^2F(\omega)\,d\omega, \tag{3}$$

where $\lambda=2\int_0^\infty(d\omega/\omega)\alpha^2F(\omega)$, and the Coulomb potential $\mu(\omega_c)$ appropriate to the cutoff ω_c in Eq. (1). Finally $\widetilde{N}(\widetilde{\omega}_n)$ is given by

$$\widetilde{N}(|\widetilde{\omega}_n|)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\epsilon\,\frac{|\widetilde{\omega}_n|}{\epsilon^2+\widetilde{\omega}_n^2}\frac{N(\epsilon)}{N(\epsilon_F)}, \tag{4}$$

where the energy integral in Eq. (4) can be performed once the density of states $N(\epsilon)$ is specified. In the following we assume that the standard electron-phonon spectral density $\alpha^2F(\omega)$, namely wherein all characteristic phonon frequencies $\omega_{\rm ph}$ are of the same order, and such that the unequalities $\omega_{\rm ph}\gg\Delta(0)$ or $\omega_{\rm ph}\gg\pi T_c$ hold. Equation (1) can, in principle, then be solved by iteration. For the case of constant $N(\epsilon)$ the regularization procedure (that is, elimination of the singularity at $\omega\to0$) and the iteration solution for the Eliashberg equation at $T=T_c$ is carried out in the real-axis representation for intermediate coupling ($\lambda<1.5$), in Refs. 30 and 31.

Let us define the function

$$\phi(i\omega_n)=\frac{\Delta(i\omega_n)\widetilde{Z}(i\omega_n)}{\Delta(0)\widetilde{Z}(0)}, \tag{5}$$

where $\Delta(0)$ and $\widetilde{Z}(0)$ are understood in the limit of $\omega\to0$. Then Eq. (1) can be rewritten as

$$\phi(i\omega_n)=\phi^o(i\omega_n)+\pi T\lambda\sum_{m=-\infty}^{+\infty}[I(n,m)-I(n,0)I(0,m)]\times\frac{1}{|\omega_m|}\frac{\phi(i\omega_m)}{\widetilde{Z}(i\omega_m)}\widetilde{N}(|\widetilde{\omega}_m|), \tag{6}$$

where in the last expression the identity $\phi(0)=1$ was used, namely,

$$\pi T\sum_{m=-\infty}^{+\infty}\{\lambda I(0,m)-\mu(\omega_c)\}\frac{1}{|\omega_m|}\frac{\phi(i\omega_m)}{\widetilde{Z}(i\omega_m)}\widetilde{N}(|\widetilde{\omega}_m|)=1, \tag{7}$$

and the functions, $\phi^o(i\omega)$ and κ are introduced

$$\phi^o(i\omega_n)=I(n,0)+\kappa[I(n,0)-1], \tag{8}$$

$$\kappa=\mu(\omega_c)\pi T\sum_{m=-\infty}^{+\infty}\frac{1}{|\omega_m|}\frac{\phi(i\omega_m)}{\widetilde{Z}(i\omega_m)}\widetilde{N}(|\widetilde{\omega}_m|). \tag{9}$$

Note, however, that κ does not depend on frequency but only on temperature. By definition $I(0,0)=1$. Moreover, for values $n\to\infty$ (and also for $m\to\infty$) the function $I(\omega_n,\omega_m)$ goes to zero. The latter circumstance ensures a vanishing kernel in Eq. (6) for small and large ω_n. Therefore, for small and large values of ω_n the free term $\phi^o(i\omega_n)$ is equivalent to an exact solution of Eq. (6) $\phi(i\omega_n)$. Because of the factor $|\omega_m|$ in the denominator in Eq. (7) as well as the rapid decrease of $I(\omega_n,\omega_m)$ with increasing ω_n, it is not difficult to see that low frequencies ω_n dominate in Eq. (7), which in turn is an eigenvalue equation. But for such frequencies the difference between $\phi^o(i\omega_n)$ and the exact solution $\phi(i\omega_n)$ is negligible. That is why using $\phi^o(i\omega_n)$ instead of $\phi(i\omega)$ in Eq. (7) leads to an equation for determining T_c whose solution differs negligibly from the exact eigenvalue of Eq. (6). It is also of interest that the analytical continuation of the free term $\phi^o(i\omega_n)$ on the real axis ω has the characteristic features at $\omega_i<\omega_0$, repeating the features of $\alpha^2F(\omega)$ (which stem from the interaction of electrons with individual groups of phonons with frequency ω_i), while at $\omega\gg\omega_0$ its behavior is determined by the Coulomb interaction. Note that ω_0 is just $\max\{\omega_i\}$. Following the results of Refs. 30,31 we construct an iteration series for Eq. (6) by starting from the trial function $\phi^o(i\omega_n)$ expressed in terms of $I(n,0)$ and with κ as the zero-order approximation. The kernel of Eq. (6) possesses no singularities so that the iteration procedure converges rapidly. Subsequent iteration corrections are small and do not significantly affect the accuracy of results obtained by using $\phi^o(\omega_n)$. In this scheme Eq. (7) is effectively clearly an eigenvalue equation.

As to the renormalization factor for the electronic mass Eq. (2), it is convenient to write it as

$$\widetilde{Z}(i\omega_n)=1+2\int_0^\infty d\omega^2 S(\omega)\sum_{m=0}^{\infty}\frac{\pi T_c}{\omega_m}\frac{\widetilde{N}(|\widetilde{\omega}_m|)}{\omega^2+\omega_n^2+\omega_m^2}\times\sum_{p=0}^{\infty}\left(\frac{2\omega_n\omega_m}{\omega^2+\omega_n^2+\omega_m^2}\right)^{2p}. \quad (10)$$

[In the above equation and from now on the electron-phonon interaction spectrum $\alpha^2F(\omega)$ will be denoted as $S(\omega)$.] Expression (10) differs from the standard one only by the factor $\widetilde{N}(\widetilde{\omega}_m)$. The convergence of the sum over p in Eq. (10) is sufficiently rapid. Specifically, for the case of a constant EDOS only the first term needed being retained. Furthermore, the variation of $\widetilde{Z}(i\omega_n)$ with n is not large, so that in the integrals $\widetilde{Z}(i\omega_n)$ can be replaced by a constant value at $n=1$.[32] In determining $\widetilde{Z}(i\omega_n)$ and T_c we use for $\widetilde{N}(\widetilde{\omega}_m)$ the value $\widetilde{\omega}_n=\omega_n Z(0)$, where $Z(0)=1+\lambda$ is a renormalization factor corresponding to $N(\epsilon)=$const.

II. THE MODEL

We retain only nearest-neighbor terms in the dispersion relation

$$\epsilon(k)=-2B[\cos(k_x a)+\cos(k_y a)]+4B. \quad (11)$$

The EDOS per spin associated with the plane band Eq. (11) is given for $\epsilon<4B$ by[33]

$$N(\epsilon)=\frac{4N}{\pi^2}\frac{1}{4B-\epsilon}K\left(\frac{\epsilon}{8B-\epsilon}\right), \quad (12)$$

where K is an elliptic integral[34] and N is the number of atoms per unit two-dimensional lattice cell. In the vicinity of the middle of the band $\epsilon\sim 4B$, $N(\epsilon)$ is well approximated by

$$N(\epsilon)=N_0\ln\left(\frac{2W}{|\epsilon-4B|}\right). \quad (13)$$

In expression (13) $N_0=N/2\pi^2B$ where $W=8B$ is the full bandwidth. Let us define the filling factor $s=4B-\epsilon_F$ which describes the position of ϵ_F with respect to the saddle point in $N(\epsilon)$. As usual, put $\epsilon\rightarrow\epsilon-\epsilon_F$. We then have

$$N(\epsilon)=N_0\ln\frac{2W}{|\epsilon-s|}. \quad (14)$$

Since the variation of the EDOS is important only near ϵ_F, one may represent $N(\epsilon)$ over the entire energy range as

$$N(\epsilon)=N_0[1+\delta n(\epsilon)], \quad (15)$$

where the background value N_0 is chosen so as to yield the critical temperature $T_c^{(0)}$. This corresponds to the case of $N(\epsilon)=$const, and the function $\delta n(\epsilon)$ (whose implicit form is postulated according to the physical situation) modulates the variation of $N(\epsilon)$ near ϵ_F. By definition, $\delta n(\epsilon)$ goes to zero over a characteristic energy scale describing the damping of the rapidly varying portion of $N(\epsilon)$. If the EDOS is modeled, for example, by a Lorentzian factor of width a and superimposed on a background N_0, then $\delta n(\epsilon)$ vanishes for energies of the order of a. But for a logarithmic variation of $\delta n(\epsilon)$, if we wish to represent a varying EDOS in the form (15) one must utilize some prefactor for $\delta n(\epsilon)$ in order to suppress the logarithm at large energies ($\delta n\rightarrow 0$ only at $\epsilon\rightarrow\infty$). Including an energy-dependent damping factor (as done, for example, in Ref. 25) only causes additional difficulties in the analytic calculations due to the additional complication of an explicit form of $\widetilde{N}(\widetilde{\omega}_n)$. These difficulties are avoidable by employing the direct definition (4) for $\widetilde{N}(\widetilde{\omega}_n)$ since as emphasized in Ref. 4, Eq. (13) is a very good approximation to Eq. (12). Specifically, the values of $N(\epsilon_F)$ (as well as magnitudes of EDOS integrated over the entire range of ϵ), calculated, respectively, by using Eq. (13) as well as the exact expression (12), differed only slightly from each other. Therefore one can approximate the exact EDOS (12) by the simple form (13) over the entire range of ϵ—and this guarantees analytic results. Inserting expression (13) for $N(\epsilon)$ into Eq. (4) immediately leads to

$$\widetilde{N}(\widetilde{\omega}_n)=\frac{1}{N(s)}\ln\frac{2W}{\sqrt{s^2+|\widetilde{\omega}_n^2|}}, \quad (16)$$

where $N(s)=\ln(2W/s)$. It should be noted that the EDOS at ϵ_F is given by

$$N(\epsilon_F)=N_0\ln\frac{2W}{s}, \quad (17)$$

and depends on the filling factor s. To obtain a physical value for $N(\epsilon_F)$ at $s\rightarrow 0$ we must include, e.g., the transition matrix elements for charge carriers between conducting layers. Henceforth we deal only with the case of $s\gg\pi T_c$ rather than $s=0$.

III. SUPERCONDUCTING TRANSITION TEMPERATURE

At temperatures near T_c the electron mass renormalization factor $\widetilde{Z}(\omega_n)$ does not involve the gap function $\Delta(\omega_n)$ so that it may calculated independently. Due to rapid convergence of the sum over p in Eq. (10) one can accurately represent $\widetilde{Z}(\omega_n)$ as

$$\widetilde{Z}(\omega_n)=1+\int_0^\infty d\omega^2 S(\omega)\times\left\{\frac{1}{\omega^2+\omega_n^2}+\frac{1}{\ln(2W/s)}\frac{1}{[s/Z(0)]^2-\omega^2-\omega_n^2}\times\left[\frac{s^2}{\omega^2+\omega_n^2}\frac{\ln Z(0)}{Z^2(0)}+\ln\frac{\sqrt{\omega^2+\omega_n^2}}{s}\right]\right\}. \quad (18)$$

In Eq. (18) we have neglected terms proportional to $(\pi T_c)^2/\overline{\omega_{\rm ph}^2}$. Expression (18) differs from the analogous result for $Z(\omega_n)$ corresponding to constant EDOS only by the second term in the curly brackets. As a rule, because of the small variation with ω_n, the $Z(\omega_n)$ can be replaced by its constant value at $n=1$ which thus allows analytic calculation (see, for example in Ref. 32). It is not difficult to realize that such is the case here; namely, as ω_n increases up to the maximum phonon frequency ω_0, $\widetilde{Z}(\omega_n)$ changes only by a few percent. Therefore, to the same accuracy as in the usual

case, one can substitute $\widetilde{Z}(\omega_1)$ for $\widetilde{Z}(\omega_n)$ to be designated in the following as $\widetilde{Z}(0)$. In accordance with Eq. (18) we have

$$\widetilde{Z}(0)=Z(0)[1+\nu\rho(s)], \tag{19}$$

where $\nu=\lambda/(1+\lambda)$ and $Z(0)=1+\lambda$ is the renormalization factor corresponding to constant EDOS; $\rho(s)$ is given by

$$\rho(s)=\frac{1}{\ln(2W/s)}\,\frac{\int_0^\infty d\omega^2 S(\omega)[1/(t^2-\omega^2)][(t^2/\omega^2)\ln Z(0)+\ln(\omega/s)]}{\int_0^\infty (d\omega^2/\omega^2)S(\omega)}, \quad t=s/Z(0). \tag{20}$$

Consider the eigenvalue equation (7) where we put $\phi(i\omega_n)=\phi^o(i\omega_n)$. Substituting $\widetilde{Z}(0)$ for $\widetilde{Z}(\omega_m)$ Eq. (7) can be rewritten as

$$\frac{1}{g}=I_1-\frac{\kappa^o}{1+\kappa^o}I_2, \tag{21}$$

where the following notation was used:

$$g=\frac{\lambda}{\widetilde{Z}(0)}, \tag{22}$$

$$I_1=\frac{1}{\lambda^2}\int_0^\infty d\omega_1^2 S(\omega_1) \times\int_0^\infty d\omega_2^2 S(\omega_2)\frac{1}{\omega_2^2-\omega_1^2}[P(\omega_1)-P(\omega_2)], \tag{23}$$

$$I_2=\frac{1}{\lambda}\int_0^\infty d\omega^2 S(\omega)P(\omega), \tag{24}$$

$$P(\omega)=\sum_{m=0}^{\infty}\frac{2\pi T_c}{|\omega_m|}\,\frac{1}{\omega^2+\omega_m^2}\widetilde{N}(|\widetilde{\omega}_m|), \tag{25}$$

$$\kappa^o=\mu_\star I_2, \quad \mu_\star=\frac{\mu(\omega_c)/Z(0)}{1+[\mu(\omega_c)/Z(0)]\ln(2W/\omega_{\rm ph})}. \tag{26}$$

Note that the weak Coulomb pseudopotential $\mu_\star$ Eq. (26) differs from the analogous Tolmachev[35] or Anderson and Morel[36] results by a factor $Z(0)$ in both denominator and numerator. Putting $\widetilde{N}(|\widetilde{\omega}_m|)$ as given by Eq. (16) into $P(\omega)$ Eq. (25) one can perform the summation over m, the result being easily expressed in terms of combinations of digamma functions $\Psi[1/2\pm i(x/2\pi T_c)]$,[20,34] where $x=s/Z(0)$ or ω. For $\pi T_c \ll \omega_{ph}$ we obtain

$$P(\omega)=\frac{1}{N(s)}\,\frac{1}{\omega^2}\left(\ln\frac{1.13\omega}{T_c}\ln\frac{2W}{\sqrt{|t^2-\omega^2|}} -\frac{1}{2}\ln^2\frac{1.13\omega}{T_c}+S_t(\omega,T_c)\right), \tag{27}$$

where in the last equation we have set $t=s/Z(0)$ and

$$S_t(\omega,T_c)=\frac{1}{2}\int\frac{dt^2}{t^2-\omega^2}\ln\frac{1.13t}{T_c}.$$

The integral $S_t(\omega,T_c)$ cannot be expressed in terms of elementary functions; but depending on the value of $s=tZ(0)$ it can be written as a sum of leading terms $S_i^o(\omega,T_c)$ plus correction terms $\delta_i(\omega)$; that is

$$S_t(\omega,T_c)=S_i^o(\omega,T_c)+\delta_i(\omega). \tag{28}$$

In the last equation the index i is 1 or 2 and labels the different domains in which the filling parameter s takes on values corresponding to $t<\omega$ $(i=1)$ and $t>\omega$ $(i=2)$, respectively (see Appendix A). It turns out that the explicit formula for $S_t(\omega,T_c)$ is a piecewise-continuous function of the filling parameter $s=tZ(0)$, and in passing through the point $s^\star=\omega Z(0)$ the functional dependence of $S_t(\omega,T_c)$ on s changes. In the meantime, it is noteworthy that the discontinuous character of $S_t(\omega,T_c)$ is an intrinsic property of the logarithmic VHS. Simple inspection convinces one that the situation does not change when other models such as the two-square well[27] or two-step[28] models are used for the gap function $\phi(\omega)$ in the eigenvalue equation (21) instead of $\phi^o(\omega)$ Eq. (8). It should also be noted that our representation (28) for $S_t(\omega,T_c)$ is not an expansion in the interaction parameter λ. We retain all terms $\delta_i(\omega)$ considering them as small corrections of $S_i^o(\omega,T_c)$. Taking into account the specific expressions for $S_t(\omega,T_c)$ in Eqs. (21)–(27) finally gives an eigenvalue equation from which T_c is ultimately determined. The final form of T_c depends on the relative positions of the filling parameter s with respect to the peaks in $S(\omega)$. One obtains the following distinct cases.

A. Case of small separations, $s<\omega Z(0)$

Here we assume that all phonon frequencies $\omega_{\rm ph}$ represented in the spectral density of interaction $S(\omega)$ satisfy the condition $t<\omega$. This is the most important case since the Fermi energy lies in a region where the EDOS exhibits a pronounced structure in energy. For simplicity we use the weak Coulomb pseudopotential (26) $\mu_\star=0$. Using the implicit expressions for $S_1(\omega,T_c)$ (Appendix A) in Eqs. (21)–(27), the eigenvalue equation (21) becomes quadratic in $\ln(1.13t/T_c)$ (Appendix B); from this the final result for T_c is just

$$T_c = 1.13\Omega \exp(-D), \tag{29}$$

$$\Omega = 2We^{-\eta^o/2}\left(\frac{\overline{\omega_{\rm ph}}}{t}\right), \tag{30}$$

$$D = \sqrt{\left[\ln\left(\frac{2W}{t}\right)+\frac{1}{g}\right]^2+\left[\ln\left(\frac{\overline{\omega_{\rm ph}}}{t}\right)-\frac{\eta^o}{2}\right]^2-\frac{1}{g^2}-\frac{2\ln Z(0)}{g}-2\overline{\delta}.} \tag{31}$$

In expressions (29)–(31) for T_c we set

$$\overline{\omega_{\rm ph}} = \exp\langle \ln\omega\rangle_{S(\omega)}, \quad \frac{1}{g}=\frac{1}{\nu}+\rho(s), \quad t=\frac{s}{Z(0)}, \tag{32}$$

$$\eta^o = \left\langle \left\langle \frac{\omega^2}{\omega^2-\omega_1^2}\ln\frac{\omega_1^2}{\omega^2}\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega)} \sim 1, \tag{33}$$

$$\overline{\delta} = \left\langle \left\langle \frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}\left(\frac{1}{\omega_1^2}\delta_1(\omega_1,t) - \frac{1}{\omega_2^2}\delta_1(\omega_2,t)\right)\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega_2)}, \tag{34}$$

where the factors $\rho(s)$ and $\delta_1(\omega,t)$ are determined correspondingly, by Eqs. (20) and (28); the symbol $\langle(\ldots)\rangle$ should be understood as

$$\langle(\ldots)\rangle = \frac{\int_0^\infty (d\omega^2/\omega^2)S(\omega)(\ldots)}{\int_0^\infty (d\omega^2/\omega^2)S(\omega)}. \tag{35}$$

This expression for T_c closely resembles that obtained in the van Hove scenario within the framework of BCS theory by Tsuei *et al.*[6] Physically, however, formulas (29)–(31) are much richer since they differ from the well-known result which is an exponential factor D premultipied by Ω. The Ω includes the full bandwidth $2W$ as well as the characteristic phonon energy $\overline{\omega_{\rm ph}}$. In addition Ω depends on the filling factor s as well as on the mass renormalization factor $Z(0)$. The exponential factor D also involves characteristic parameters which in turn depend upon electronic, phononic and electron-phonon properties. Formula (29) also has another very interesting feature. To illustrate this, in accordance with the ordinary Eliashberg theory [when $N(\epsilon)=\text{const}$] let us assume that $2W/\overline{\omega_{\rm ph}} \gg 1$. For not very small values of λ (approximately, for $\lambda > 0.5$) one can then neglect all terms in the exponential factor D Eq. (31) except the first. Thus, expression (29) for T_c takes the form

$$T_c = \frac{\overline{\omega_{\rm ph}}}{1.45}\exp\left\{-\frac{1+\lambda}{\lambda}(1+\rho)\right\}, \quad \rho \sim 10^{-2}$$

which is precisely the familiar McMillan formula[37] with zero Coulomb pseudopotential.

B. Case of large separations, $s > \omega Z(0)$

Let us consider values of the filling parameter s for which the condition $t > \omega$ is satisfied for all characteristic frequencies of $S(\omega)$. In analogy with the case of $s < \omega Z(0)$ we obtain (see Appendix C)

$$T_c(s) = 1.13\Omega(s)\exp\left(-\frac{A(s)}{g(s)-\mu_\star}\right), \tag{36}$$

where in Eq. (36) one introduces the notation

$$\Omega(s) = \overline{\omega_{\rm ph}}\exp(\gamma(s)+\delta(s)),$$

$$A(s) = \frac{N(s)}{N(t)} = 1-\frac{\ln Z(0)}{N(t)}, \tag{37}$$

$$g(s) = \frac{\nu}{1+\nu E(s)}, \quad E(s) = \rho(s)+\eta(s)+\delta'(s). \tag{38}$$

In expression (36) for T_c the factor $g(s)$ Eq. (38) plays the role of an effective interaction parameter, the Coulomb pseudopotential $\mu_\star$ being given by Eq. (26). Specific expressions for $\rho(s)$ Eq. (20), $\gamma(s)$, $\eta(s)$, and $\delta(s)$ in Eq. (36) are then determined by the shape of interaction spectrum $S(\omega)$, namely,

$$\gamma(s) = -\frac{1}{2N(t)}\left\langle \ln\frac{\omega}{t}\ln\left(1-\frac{\omega^2}{t^2}\right)\right\rangle_{S(\omega)},$$

$$\delta(s) = \frac{1}{N(t)}\langle \delta_2(\omega,t)\rangle_{S(\omega)}, \tag{39}$$

$$\eta(s) = \frac{1}{2N(s)}\left\{N(t)\eta^o + \frac{1}{2}\left\langle\left\langle \frac{\omega^2+\omega_1^2}{\omega^2-\omega_1^2}\left[\ln\frac{\omega_1}{t}\ln\left(1-\frac{\omega_1^2}{t^2}\right) - \ln\frac{\omega}{t}\ln\left(1-\frac{\omega^2}{t^2}\right)\right]\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega)}\right\}, \tag{40}$$

$$\delta'(s) = \frac{1}{2N(s)}\left\langle\left\langle \frac{\omega^2+\omega_1^2}{\omega^2-\omega_1^2}[\delta_2(\omega,t) - \delta_2(\omega_1,t)]\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega)}. \tag{41}$$

In the above formulas $N(t)=\ln(2W/t)$, $N(s)=\ln(2W/s)$; $\delta_2(\omega,t)$, $\overline{\omega_{\rm ph}}$, and η^o are given correspondingly, by Eqs. (28), (32), and (33).

According to Eq. (17) the values of parameters $A(s)$ Eq. (37), $\gamma(s)$ Eq. (39), and $\eta(s)$ Eq. (40) depend on the magnitude of $N(\epsilon_F)$. For a large shift s of ϵ_F from the saddle point the function (17) is slowly varying. Therefore, in agreement with the physical situation, far from the peak in EDOS we can suppress the logarithmic variation of the $N(\epsilon_F)$ that enters expressions (37)–(40) and take it as constant for large energy values s.[25] Fixing $N(\epsilon_F)$ at the interdomain boundary $s^\star$ we set

$$N(t)=\ln\frac{2W}{\overline{\omega_{\rm ph}}}.$$

If the interaction spectrum $S(\omega)$ is a δ function centered about some average frequency $\omega_{\rm ph}$, and in addition if $\mu^\star=0$, then a little algebra leads to the following expression for T_c:

$$T_c=\frac{\omega_{\rm ph}}{1.45}\exp\left(-\frac{1}{\nu}+R(s)\right), \tag{42}$$

where

$$R(s)=\frac{1}{\ln(2W/\omega_{\rm ph})}\left[\frac{\ln Z(0)}{\lambda}-\frac{1}{2}\ln\frac{\omega_{\rm ph}}{t}\ln\left(1-\frac{\omega_{\rm ph}^2}{t^2}\right)\right.$$
$$\left.-\frac{3}{2}\frac{\omega_{\rm ph}^2}{t^2-\omega_{\rm ph}^2}\ln\frac{\omega_{\rm ph}}{t}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{\omega_{\rm ph}^2}{t^2}\right)^k\right].$$

It is not difficult to see that formula Eq. (36) for T_c has the correct asymptotic ''behavior,'' namely, if $2W/\overline{\omega_{\rm ph}}\to\infty$ then Eq. (36) assumes the form

$$T_c^o=1.13\overline{\omega_{\rm ph}}\exp\left(-\frac{1}{g^o-\mu_\star}\right),\quad g^o=\frac{\nu}{1+(1/2)\,\eta^o\nu}. \tag{43}$$

Expression (43) is well known from the theory[38] with constant EDOS for intermediate coupling. In particular, this expression turns out to be extremely useful in understanding the differences in partial isotope shift factors $\alpha_i=(m_i/T_c)dT_c/dm_i$ in compounds arising from the dissimilar variation in the interaction spectrum $\alpha^2F(\omega)$ under substitutions of i type of atoms by different isotopes. On the basis of formula (43) it was shown,[39,40] though the critical temperature itself is a weak function of the shape of $\alpha^2F(\omega)$ (T_c is determined mainly by average phonon frequency $\overline{\omega}$, interaction parameter λ as well as weak Coulomb pseudopotential $\mu_\star$) the differential parameters, such as, e.g., the partial isotope shift factor and the pressure variation of T_c may be very sensitive to changes in the interaction spectrum.

It should be stressed that formulas (29)–(31) and (36)–(38) for T_c were obtained solely within the framework of Eliashberg theory. That is, we deal with the *real parameters* of the electron-phonon system, which has a specific *microscopic origin*. Again, the final expressions for T_c include parameters of electronic, phononic, and electron-phonon origin. Moreover, the results for T_c are obtained as a functional of the interaction spectrum $\alpha^2F(\omega)$. In arriving at final formulas (29)–(31) and (36)–(38) one need not specify the shape of $\alpha^2F(\omega)$, by which one understands *any spectrum*—including spectra associated with *monoatomic* or with *compound* superconductors (see Refs. 39 and 40).

IV. DISCUSSION OF RESULTS

Consider now the expression for the critical temperature T_c for various parameter values of the model adopted above. We are interested in cases when the Fermi level is either near or far from the saddle point in the EDOS. Physically, we expect higher values of T_c for small s. The shift of ϵ_F from the middle of the conduction band may be considered as comparatively small in the domains labeled by $i=1$ in Eq. (28), since in these cases the distance between ϵ_F and the EDOS peak is of the order of the phonon frequency $\overline{\omega_{\rm ph}}$ which is the characteristic energy scale important for superconductivity. By contrast, when ϵ_F is off the saddle point, we can expect a secondary T_c-enhancing effect due to the peak structure in the EDOS and we recover a result for T_c values consistent with those from ordinary Eliashberg theory [namely, $N(\epsilon)=$const, comparatively small values of T_c]. A significant separation of ϵ_F from the middle of the conduction band occurs when $s>\overline{\omega_{\rm ph}}Z(0)$ (in the domain labeled with $i=2$). It should be emphasized, that the width of the segment of the filling parameter values where we expect higher values of T_c implicitly depends on the interaction parameter λ. Actually, the energy dependence in $N(\epsilon)$ is incorporated into Eliashberg theory by introducing an additional factor $\widetilde{N}(|\widetilde{\omega}_m|)$ Eq. (4) into the sums of the ordinary equations. The factor $\widetilde{N}(|\widetilde{\omega}_m|)$ is determined not by ''bare'' Matsubara frequencies ω_m but rather by the renormalized ones $\widetilde{\omega}_m$ Eq. (2). The final expressions for T_c Eqs. (29)–(31) and (36)–(38) are obtained after a preliminary summation in eigenvalue equation (21) over discrete $\widetilde{\omega}_m$, which in turn depend on the value of the factor $Z(\omega_m)$. Because of the summation in Eq. (21) the superconducting critical temperature, T_c, depends on the filling parameter s by means of the ratio $t=s/Z(0)$. Separation into domains where one expects, respectively, large and moderate values of T_c occurs at $t^\star=\overline{\omega_{\rm ph}}$. Thus, the width of the segment of the filling parameter values, where a higher T_c is expected, depends on $Z(0)$.

Note that in the derivation of formulas for T_c Eqs. (29)–(31) we have used the digamma function representation[20]

$$\Psi(X)=\ln|X|-\frac{1}{2X}-\frac{1}{12X^2}+\cdots,$$

which is valid for $|X|>1$ (here $X=1/2-s/2\pi T_c$). Therefore, for very small separation s of the Fermi level from the saddle point, when the application of the above-mentioned formula is invalid, our result for T_c is not correct and the range of values of s, where the final expression for T_c Eqs. (29)–(31) is applicable, must be restricted from below, say at $\overline{\omega_{\rm ph}}/2$. Here, it was assumed that $\omega_{\rm ph}\gg 2\pi T_c$, which is the case for La-Sr-Cu-O (LSCO) superconductors. Experimentally, LSCO electrons interact mainly with the O and Cu atoms in the conducting planes for which $\omega_{\rm ph}\sim 300$ K and $T_c\sim 40$ K. Thus, if $s>\omega_{\rm ph}/2$ then the condition $s>2\pi T_c$ holds.

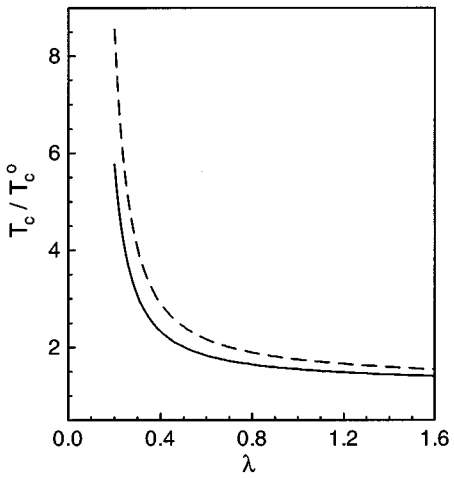

FIG. 1. T_c/T_c^o vs λ, for $(2W/\omega_{\rm ph})=200$ (solid line) and 40 (dashed line), the so-called ''high-temperature region.''

Next we analyze the T_c formulas. Direct inspection of T_c Eqs. (29)–(31) allows one to conclude that for values of the filling parameter s such that $\overline{\omega_{\rm ph}}/2<s<\overline{\omega_{\rm ph}}Z(0)$, the transition temperature T_c exceeds the value T_c^o corresponding to constant $N(\epsilon)$. However, enhancement of T_c will depend on the value of the interaction parameter. The relative enhancement of T_c is more noticeable for small λ, viz., as seen from Fig. 1 where T_c/T_c^o versus λ appears. For $\lambda=0.3$ the critical temperature increases nearly four times, while for $\lambda=1.5$ the increase in T_c is only about 50%. Actually, in the modified Eliashberg theory for T_c the factor $\widetilde{N}(\widetilde{\omega}_n)$ Eq. (4) describes the broadening in the EDOS peak structure. Washing out of the EDOS peaks due to this broadening is more pronounced for large λ. Therefore, for strongly coupled superconductors a sharp structure in $N(\epsilon)$ is less effective for T_c enhancement than for weakly coupled ones.

Figure 2 shows plots of T_c as a function of s $(s<s^\star)$, for different values of $2W/\overline{\omega_{\rm ph}}$. The magnitude of T_c^o is indicated by the horizontal line at the bottom. This figure clearly illustrates the considerably higher T_c compared with T_c^o throughout the entire range of variation of s from $\overline{\omega_{\rm ph}}/2$ to $\overline{\omega_{\rm ph}}Z(0)$. Henceforth, this domain will be referred to as the ''high-temperature region.'' Significantly, however, in this region T_c is nonmonotonic in the filling parameter s. As s decreases from $\overline{\omega_{\rm ph}}Z(0)$, T_c increases initially and then reaches a maximum value at some s'. Further decrease of s, namely, further increasing $N(\epsilon_F)$ through the saddle point, produces a gradual decrease of T_c, in agreement with the Pickett result[18] for $A15$ compounds. The reason for the reduction of T_c for small separation s of ϵ_F from the middle of the conducting band can be understood if we recall the substantial decrease of the factor $\widetilde{N}(\widetilde{\omega}_n)$ Eq. (16) produced by decreasing s. Because $\widetilde{N}(\widetilde{\omega}_n)$ is proportional to $1/N(\epsilon_F)$ which vanishes rapidly as ϵ_F varies up to the saddle point, we have a reduction of T_c for small s instead of its expected increase. Another remarkable feature of T_c is that the strongest variation in T_c does not occur in our so-called high-temperature region. Namely, $\Delta T_c/T_c$ is of order 0.1, where ΔT_c is the total change in T_c as s varies from $\overline{\omega_{\rm ph}}/2$ to $\overline{\omega_{\rm ph}}Z(0)$. We again stress that the width of the ''high-temperature region'' thus defined depends on the value of the interaction parameter: large values of λ correspond to a broader such region.

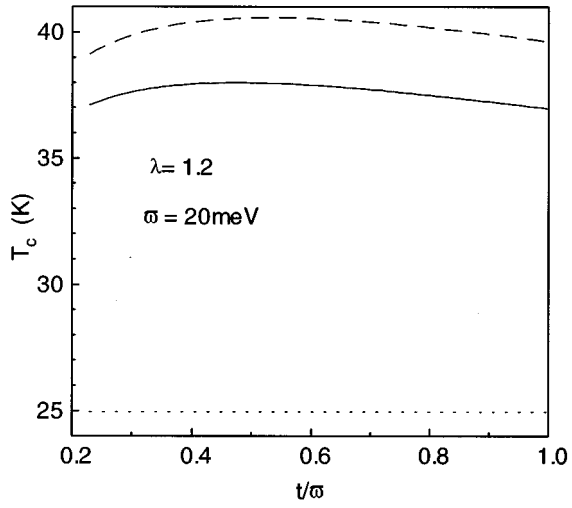

FIG. 2. T_c as a function of $t/\overline{\omega_{\rm ph}}$ (high-temperature region). Curves correspond to $2W/\omega_{\rm ph}=200$ and 40. Here $\lambda=1.2$, $\omega_{\rm ph}=20$ meV. The horizontal dotted line is T_c^o.

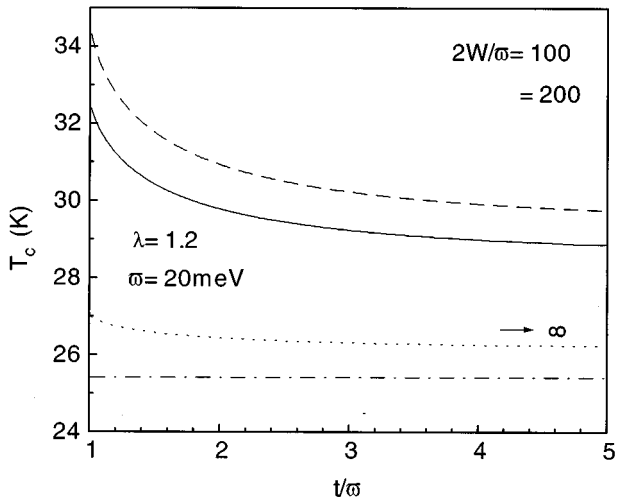

FIG. 3. T_c as function of $t/\omega_{\rm ph}$ for large separation of ϵ_F from the saddle point. $2W/\omega_{\rm ph}=200$ (solid line) and 100 (dashed line). Dotted and dash-dotted lines at bottom correspond, respectively, to cases $2W/\omega_{\rm ph}\to\infty$ and constant EDOS. Here $\lambda=1.2$, $\omega_{\rm ph}=20$ meV.

Figure 3 displays plots of T_c vs s for large separations s $(s>s^\star)$. As seen from Figs. 2 and 3, both drawn for the δ-like interaction spectrum and for the same values of λ and $2W/\omega_{\rm ph}$, for a value of $\overline{\omega_{\rm ph}}Z(0)$ for the filling parameter s, T_c drops discontinuously toward the T_c^o lying a little higher than T_c^o. The difference $T_c-T_c^o$ tends to zero for significant values of the ratio $2W/\overline{\omega_{\rm ph}}$, in accordance with the constant EDOS case. In fact, variation of the functional form of $T_c(t)$ by increasing the filling parameter [that is, transfer of expressions (29)–(31) for T_c to formulas of the form (36)–(41)] takes place over the full ''width'' of the spectral density $S(\omega)$. The Eliashberg function $S(\omega)$ consists of sev-

eral narrow peaks at certain $\{\omega_i\}$ $(i=1,2,\ldots)$ arising from the interaction of electrons with individual groups of phonons with frequencies ω_i.[32] The δ-like structure of $S(\omega)$ leads to a discontinuity of T_c as a function of $s=tZ(0)$. Note that the magnitude of $T_c-T_c^o$ at large s (Fig. 3) is associated with the factor $\ln Z(0)$ in the expression for T_c [Eq. (42)] arising from using renormalized frequencies $\tilde{\omega}_n=\omega_n\tilde{Z}(0)$ in the broadening factor $\tilde{N}(\tilde{\omega}_n)$ Eq. (4). Using in our T_c calculations the ''bare'' Matsubara frequencies ω_n, instead of $\tilde{\omega}_n$, then beginning approximately from $(t/\overline{\omega_{\rm ph}})=2$ the critical temperature T_c approaches T_c^o. The qualitative picture shown in Fig. 2 does not change on substitution of ω_n for the renormalized $\tilde{\omega}_n$, but the λ dependence of T_c (Fig. 1) steepens, and for $\lambda=1.5$ we have only $\sim 25\%$ of the T_c enhancement. It is easy to understand that representing the spectral function $S(\omega)$ as a sum of several δ-like peaks, that is, allowing for simultaneous contribution of frequencies from two different domains with $t<\omega$ and with $t>\omega$ to the integrals (23) and (24), does not alter the discontinuous character of T_c nor its asymptotic behavior at large values of s. Therefore, *discontinuity of T_c as a function of the filling parameter is an intrinsic feature of quasi-two-dimensional superconductivity.* Taking into account the finite widths of the peaks in $S(\omega)$, as well as the transition matrix elements for electrons between adjacent conducting layers will ''spoil'' this ideal discontinuous picture in the T_c behavior and smooth out the transition from the high-temperature region to a region where T_c is moderate. We believe that, physically, the discontinuous nature of T_c as a function of the distance between ϵ_F and saddle point in $N(\epsilon)$ is a result of the *short-range character in the energy space* of the indirect electron-electron interaction via phonons. The mathematical origin of this discontinuity is hidden in the nonanalytic behavior of the integral (28). In energy space this interaction is spread over a region ω_0 around the Fermi surface. Let us assume that the singularity in EDOS is located at exactly a distance ω_0 from ϵ_F. Then any small increase of s will remove the singularity from the energy shell $\epsilon_F-\omega_0<\epsilon<\epsilon_F+\omega_0$ and lead to exclusion of the contribution to T_c due to the singularity in $N(\epsilon)$. Namely, the latter contribution is responsible for the discontinuous nature of T_c.

It is not difficult to see that $(\delta T_c/\delta W)<0$ everywhere. Thus, decreasing $2W/\overline{\omega_{\rm ph}}$ always results in a slight increase in T_c, the influence of bandwidth on T_c becoming more significant for relatively smaller ratios of $2W/\overline{\omega_{\rm ph}}$.

In Fig. 4 the influence of the correction terms $\delta_i(\omega,s)$ on T_c is exhibited, the dashed curve corresponding to a plot of T_c including correction terms in expressions (29)–(31) and (36)–(38). The full curve is the same plot but without the $\delta_i(\omega,s)$. Evidently, over the entire range of variation of the filling parameter, taking these correction terms $\delta_i(\omega,s)$ into account leads to small changes in the magnitude of T_c.

Our results may be useful in understanding the experimental data of Ref. 41 where it was established that the phase diagram of well-annealed $La_{2-x}Sr_xCuO_4$ oxides differs sharply from that held by conventional wisdom. The concentration dependence of T_c in $La_{2-x}Sr_xCuO_4$ does not follow the commonly assumed ''inverted-parabola'' behavior; rather it remains in the 35–40 K range for

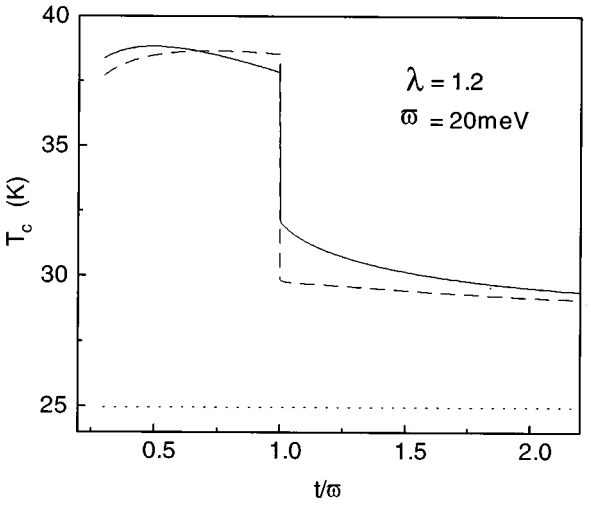

FIG. 4. T_c as function of $t/\omega_{\rm ph}$. Dashed curve is plot of T_c when correction terms discussed in text are included. Full curve is same plot but without $\delta_i(\omega,s)$. $2W/\omega_{\rm ph}=200$; λ and ω are as in Figs. 2 and 3.

$x=0.14$–0.21. At $x_{\rm cr}\sim 0.21-0.22$, T_c drops discontinuously to zero. If one assumes that critical doping $x_{\rm cr}$ corresponds to filling parameter $s^\star=\omega_{\rm ph}Z(0)$ in our consideration, then one can explain T_c as nearly constant over the concentration range $x=0.14$–0.21, and one can understand its discontinuity as well. It has also been found[41] that at a concentration where T_c experiences an abrupt decrease, a second-order structural transition takes place from a low-temperature orthorhombic phase to a high-temperature tetragonal phase. Such drastic changes in the material at nearly the same concentration $x_{\rm cr}$ may, in principle, have an identical origin. Thus, it would be extremely interesting to investigate this phase transition in layered systems by assuming a van Hove scenario.

To conclude, an analytic expression is obtained for the superconducting critical temperature T_c in the logarithmic van Hove scenario (VHS) within the framework of the modified Eliashberg equations allowing for sharp variations in the electronic density of states. The behavior of T_c as a function of the Fermi energy shift from the van Hove saddle point is analyzed, and we identify the region of the filling parameter values where the effectiveness of VHS in enhancing T_c is most appreciable. The dependence of T_c enhancing on the value of interaction parameter λ in VHS is thus established. It is shown that though an increasing T_c is more effective for weak coupling, nevertheless, for intermediate coupling the critical temperature is still considerably higher than T_c^o, corresponding to a constant EDOS. Based on the existence of low-dimensional structural elements such as planes, a possible explanation is suggested for a curious discontinuous change in transition temperature observed experimentally in $La_{2-x}Sr_xCuO_4$ oxides.

ACKNOWLEDGMENTS

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APPENDIX A

To obtain concrete expressions for $S_t(\omega,T_c)$ Eq. (28) integral tables[34] give

$$S_1(\omega,T_c)=\ln\frac{1.13\omega/T_c}{\ln\dfrac{1.13\sqrt{\omega^2-t^2}}{T_c}}+\delta_1(\omega,t),\quad t<\omega; \tag{A1}$$

$$S_2(\omega,T_c)=\ln\frac{1.13t}{T_c}\ln\frac{1.13\sqrt{t^2-\omega^2}}{T_c}-\frac{1}{2}\ln^2\frac{1.13t}{T_c}+\delta_2(\omega,t),\quad t>\omega, \tag{A2}$$

where

$$\delta_1(\omega)=-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{\omega^2-t^2}{\omega^2}\right)^k,\quad t<\omega,$$

$$\delta_2(\omega)=-\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{\omega^2}{t^2}\right)^k,\quad t>\omega.$$

If the Eliashberg function $S(\omega)$ is located in the interval $(\omega_{\min},\omega_{\max})$ then, depending on the relation $t<\omega_{\min}$ or $t>\omega_{\max}$, the integrals (23) and (24) are determined correspondingly by Eq. (A1) or (A2). The use of $S_1(\omega,T_c)$ or $S_2(\omega,T_c)$, for Eqs. (21)–(27), results in expressions for T_c which have different functional forms (see below).

APPENDIX B

Substitution of $S_1(\omega,T_c)$ (Appendix A) into Eq. (27) after rearrangement gives

$$P(\omega)=\frac{1}{N(s)}\frac{1}{\omega^2}\left(\frac{1}{2}L^2+L\ln\frac{2W\omega}{t^2}+\ln\frac{\omega}{t}\ln\frac{2W}{t}+\delta_1(\omega,t)\right),$$

where $L=\ln(1.13t/T_c)$; $\delta_i(\omega,t)$ for $i=1$ is given by Eq. (A1). From definitions (23) and (35) we have

$$I_1=\left\langle\left\langle\frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}[P(\omega_1)-P(\omega_2)]\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega_2)}.$$

Using in the last equality the relations

$$\left\langle\left\langle\frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}\left(\frac{1}{\omega_1^2}\ln\frac{2W\omega_1}{t^2}-\frac{1}{\omega_2^2}\ln\frac{2W\omega_2}{t^2}\right)\right\rangle_{S(\omega)}\right\rangle_{S(\omega)}$$

$$=\ln\frac{2W\overline{\omega}}{t^2}-\frac{1}{2}\eta^o,$$

$$\left\langle\left\langle\frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}\left(\frac{1}{\omega_1^2}\ln\frac{\omega_1}{t}-\frac{1}{\omega_2^2}\ln\frac{\omega_2}{t}\right)\right\rangle\right\rangle=\ln\frac{\overline{\omega}}{t}-\frac{1}{2}\eta^o,$$

where the factors $\overline{\omega}$ and η^o in the above expressions are given by Eqs. (31) and (33), and inserting a zero Coulomb pseudopotential $\mu^\star=0$ into the eigenvalue equation (21) we arrive at

$$L^2+2L\left(\ln\frac{2W\overline{\omega}}{t^2}-\frac{\eta^o}{2}\right)+2\ln\frac{2W}{t}\left(\ln\frac{\overline{\omega}}{t}-\frac{\eta^o}{2}\right)+2\overline{\delta}=\frac{2N(s)}{g}. \tag{B1}$$

The solution for L of the last equation gives

$$L_{\pm}=-\left(\ln\frac{2W\overline{\omega}}{t^2}-\frac{\eta^o}{2}\right)\pm\sqrt{\left(\ln\frac{2W\overline{\omega}}{t^2}-\frac{\eta^o}{2}\right)^2-2\ln\frac{2W}{t}\left(\ln\frac{\overline{\omega}}{t}-\frac{\eta^o}{2}\right)+\frac{2N(s)}{g}-2\overline{\delta}.}$$

Taking into account in $L_{\pm}$ the identity

$$N(s)=\ln\frac{2W}{t}+\ln\frac{t}{s},$$

we get, after simple algebra,

$$L=-\left(\ln\frac{2W\overline{\omega}}{t^2}-\frac{\eta^o}{2}\right)+\sqrt{\left(\ln\frac{2W}{t}+\frac{1}{g}\right)^2+\left(\ln\frac{\overline{\omega}}{t}-\frac{\eta o}{2}\right)^2-\frac{1}{g^2}-\frac{2\ln Z(0)}{g}-2\overline{\delta},}$$

which is the physically possible solution L_+ of equation (B1). One finally obtains the expression for T_c in Eqs.(29)–(31).

APPENDIX C

Let us rewrite the eigenvalue equation (21) for T_c in the form

$$\frac{\widetilde{Z}(0)}{\lambda}-(I_1-I_2)=\frac{I_2}{1+\mu_\star I_2}. \tag{C1}$$

In the denominator Eq. (26) was used, and $\widetilde{Z}(0)$ is given by Eq. (20). Recalling Eqs. (23) and (24) the concrete expression for $S_2(\omega,T_c)$ Eq. (A2) gives

$$I_1=\frac{1}{N(s)}(L+\eta_1+\overline{\delta_2}), \tag{C2}$$

$$I_2=\frac{1}{N(s)}(L+\eta_2+\overline{\delta_2'}), \tag{C3}$$

where the following notation was introduced:

$$L=\ln\frac{1.13t}{T_c}\ln\frac{2W}{t},$$

$$\eta_1=\left\langle\left\langle\frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}\left(\frac{1}{\omega_1^2}\ln\frac{\omega_1}{t}\ln\frac{2W}{\sqrt{t^2-\omega_1^2}}-\frac{1}{\omega_2^2}\ln\frac{\omega_2}{t}\ln\frac{2W}{\sqrt{t^2-\omega_2^2}}\right)\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega_2)},$$

$$\eta_2=\left\langle\ln\frac{\omega}{t}\ln\frac{2W}{\sqrt{t^2-\omega^2}}\right\rangle_{S(\omega)},$$

$$\overline{\delta_2}=\left\langle\left\langle\frac{\omega_1^2\omega_2^2}{\omega_2^2-\omega_1^2}\left(\frac{1}{\omega_1^2}\delta_2(\omega_1)-\frac{1}{\omega_2^2}\delta_2(\omega_2)\right)\right\rangle_{S(\omega_1)}\right\rangle_{S(\omega_2)},$$

$$\overline{\delta_2'}=\langle\delta_2(\omega,t)\rangle_{S(\omega)}.$$

The left-hand side of Eq. (C1) does not depend on T_c and is denoted by $1/g(s)$. Then from Eq. (C1) we have $I_2=1/g-\mu_\star$ which in conjunction with Eq. (C3) leads to the final expressions (36)–(38) for T_c.

APPENDIX D

In this appendix we derive formulas for parameters appearing in the final expressions (29)–(31) and (36)–(38) for T_c, assuming that the interaction spectrum $S(\omega)$ has a δ-like shape:

$$\overline{\omega_{\rm ph}}=\omega_{\rm ph},\quad \eta^o=1,$$

$$\overline{\delta(s)}=\frac{1}{8}\left[\frac{t^2}{\omega_{\rm ph}^2-t^2}\ln\frac{\omega_{\rm ph}^2}{t^2}-\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{\omega_{\rm ph}^2-t^2}{\omega_{\rm ph}^2}\right)^k\right],$$

$$\rho(s)=\frac{1}{N(s)}\left(\ln Z(0)+\frac{\omega_{\rm ph}^2}{t^2-\omega_{\rm ph}^2}\ln\frac{\omega_{\rm ph}}{t}\right),$$

$$\gamma(s)=-\frac{1}{2N(t)}\ln\frac{\omega_{\rm ph}}{t}\ln\left(1-\frac{\omega_{\rm ph}^2}{t^2}\right),$$

$$\eta(s)=\frac{1}{2}\frac{N(t)}{N(s)}-\frac{1}{4N(s)}\left\{\ln\left(1-\frac{\omega_{\rm ph}^2}{t^2}\right)-\frac{2\omega_{\rm ph}^2}{t^2-\omega_{\rm ph}^2}\ln\frac{\omega_{\rm ph}}{t}\right\},$$

$$\delta(s)=-\frac{1}{4N(t)}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{\omega_{\rm ph}^2}{t^2}\right)^k,$$

$$\delta'(s)=-\frac{1}{4N(s)}\ln\left(1-\frac{\omega_{\rm ph}^2}{t^2}\right),$$

where $N(t)=\ln(2W/t)$ and $N(s)=\ln(2W/s)$.

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