Path renormalization of quasiperiodic generalized Fibonacci chains

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A renormalization group is introduced which is based upon a real-space rescaling procedure of the Dyson equation for one-band tight-binding models of generalized Fibonacci quasicrystals. We present an approach for finding possible successions of given elementary rescaling transformations to calculate the *diagonal* elements of the Green function for all sites. [S0163-1829(97)05613-0]

I. INTRODUCTION

Inspired by the discovery of quasicrystals by Shechtman *et al.*,[1] quasiperiodic structures have attracted much interest.[2] In particular one-dimensional (1D) quasiperiodic systems such as the Fibonacci chain have been theoretically investigated in great detail, e.g., Refs. 3–6. Moreover, in the past few years an understanding of a close connection between the structural description of the different classes of quasicrystals and the theory of ''*the isomorphic group of the free group*'' developed by Nielsen[7] in 1918/24 came forward.[8]

In this context, generalized Fibonacci chains (GFC) play an important role in the development of an adequate mathematical apparatus for the computation of physical quantities such as the density of states or conductivity. Analytical methods, based upon *trace maps*[4] in the framework of the *transfer-matrix approach* and *real-space decimation techniques*,[5,6] are central to the understanding of the physical properties of these systems. Unfortunately, a drawback of the renormalization schemes, as presented in Refs. 5 and 6, is the absence of an algorithm which determines the correlation between the diagonal elements of the Green function and a possible choice of the successional rescaling transformations decimating the Dyson equation. It will be apparent that our approach yields just this missing algorithm. In doing so, the objective of this and a further article[9] is to exhibit the relationship between the geometric structure of a class of GFC (represented by the so-called positive primitive elements of the free group $\boldsymbol{F}_2$), and the abstract group of automorphisms $\boldsymbol{\Phi}_2=\boldsymbol{Aut}(\boldsymbol{F}_2)$ on the one hand, and the renormalization *group* $\boldsymbol{RG}(\boldsymbol{F}_2)$ associated with a tight-binding Hamiltonian on the free group $\boldsymbol{F}_2$ of rank 2 on the other. The latter provides the tool for solving the Dyson equation for the Green function. Our main issue is to construct a renormalization scheme for a class of GFC via a real-space decimation technique on graphs (*path renormalization*). In this article we will present that part of our approch which yields the *diagonal* elements of the Green function.

We will go into the presentation of the *path renormalization* scheme as follows. In Sec. II we consider the geometrical structure of the GFC and the physical model. In the first part of Sec. III we address several *formal* aspects of the structural description of the quasiperiodic chains and its relation to the group of automorphisms $\boldsymbol{\Phi}_2$. In the second part we introduce a number system which is related to the GFC. These concepts form the foundation for the following sections. Section IV is devoted to the construction of the real-space rescaling procedure. We will show that the set of the so generated renormalization transformations (RT) defines a *group* that we denote the renormalization group $\boldsymbol{RG}_{\downarrow x,y}<\boldsymbol{RG}(\boldsymbol{F}_2)$. The computation of the diagonal elements of the Green function and the local density of states (LDOS) is presented in Sec. V. In Sec. VI we introduce the *path renormalization* scheme and examine its properties. Finally, in Secs. VII and VIII numerical results and a summary are given.

II. FIBONACCI CHAINS

We consider 1D quasiperiodic systems with two different types of nearest neighbor interactions specified by t_L, and t_S, respectively. One may describe the geometric structure of such lattices by *words* $w(L,S)$, i.e., strings in the symbols L,S representing the corresponding linear arrangement of the long and short bonds L and S. These words obey the following recursion law:

$$w_n=(w_{n-1})^{M_n-s_n} * w_{n-2} * (w_{n-1})^{s_n}, \tag{1}$$

where $w_{-1}=S$, $w_0=L$, $M_n\in\mathbb{N}^+$, and $s_n\in\{0,1,\ldots,M_n\}$. The *Nielsen transformation* is given by

$$\hat{\boldsymbol{X}}_{M_n-s_n,s_n}(w_{n-1},w_{n-2})=(w_n,w_{n-1}). \tag{2}$$

The *length* $|w_n|=F_n$, i.e., the power sum of symbols L and S in $w_n(L,S)$,[10] satisfies the recursion relation for the generalized Fibonacci numbers

$$F_n=M_nF_{n-1}+F_{n-2},\quad F_{-1}=F_0=1. \tag{3}$$

The group product $*$ is defined as the concatenation of two strings. The corresponding generalization is $(w_n)^{M_n}=w_n*w_n*\cdots*w_n$. To lighten the notation, the asterisk symbol will be left out in the following. The special case $M_n=1,s_n=0,\ \forall n$, yields the standard Fibonacci chain.

Alternatively to the Nielsen transformations (2), the GFC can be generated by *substitutions* (*morphisms*), which operate on the symbols L,S rather than on words $w_n(L,S)$. Starting with $w_0(L,S)=L$, the sequential operation $\hat{\boldsymbol{\mathcal{X}}}_{N_n-r_n,r_n}$ on L and S

$$\hat{\boldsymbol{\mathcal{X}}}_{N_n-r_n,r_n}:\ L\to L^{N_n-r_n}SL^{r_n},\ S\to L,$$

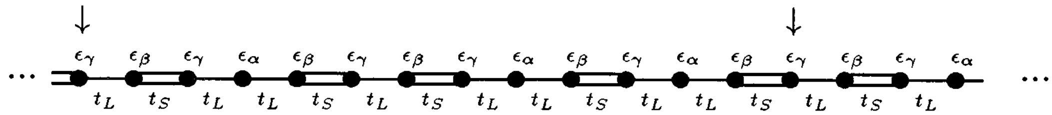


FIG. 1. Fifth generation standard Fibonacci lattice $w_5 = LSLLSLSLLSLLS$ ($F_5 = 13$). The arrows over the parameters ϵ_γ serve to emphasize the border of the unit cell with the quasiperiodic structure [compare with Fig. 3(a)].

$$\hat{\mathcal{X}}_{N_n-r_n,r_n}(w_{n-1}(L,S)) = w_{n-1}(L^{N_n-r_n}SL^{r_n},L) = w_n(L,S), \tag{4}$$

with $N_n \in \mathbb{N}^+$ and $r_n \in \{0,1,\ldots,N_n\}$ generates the same set of words $w_n(L,S)$ as Eq. (2). Note the *reversed order* of the substitutions when compared with the Nielsen transformations, i.e.,

$$\mathcal{N}_n \equiv (N_1,N_2,\ldots,N_n) = (M_n,M_{n-1},\ldots,M_1),$$

$$\mathcal{R}_n \equiv (r_1,r_2,\ldots,r_n) = (s_n,s_{n-1},\ldots,s_1). \tag{5}$$

In the following, we will specifically use the letters M_k, s_k for the Nielsen transformations (as well as for the *inverse* substitutions which will be introduced in the following section), and the letters N_k, r_k for the morphisms.

In our study, we employ the following one-particle tight-binding Hamiltonian

$$\mathcal{H} = \sum_\mu |\mu\rangle \epsilon_\mu \langle \mu| + \sum_{\langle \mu,\nu\rangle} |\mu\rangle t_{\mu,\nu}\langle \nu|, \tag{6}$$

where $|\mu\rangle$ are Wannier states (atomiclike orbitals) centered at sites μ. $\epsilon_\mu \in \{\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma, \epsilon_\delta\}$, $t_{\mu,\mu+1} \in \{t_L, t_S\}$ denote the site energy and the nearest neighbor transfer integrals, respectively, with

$$\epsilon_\mu = \epsilon_\alpha \quad \text{if} \quad t_{\mu-1,\mu} = t_{\mu,\mu+1} = t_L,$$

$$\epsilon_\mu = \epsilon_\beta \quad \text{if} \quad t_{\mu-1,\mu} = t_L \quad \text{and} \quad t_{\mu,\mu+1} = t_S,$$

$$\epsilon_\mu = \epsilon_\gamma \quad \text{if} \quad t_{\mu-1,\mu} = t_S \quad \text{and} \quad t_{\mu,\mu+1} = t_L,$$

$$\epsilon_\mu = \epsilon_\delta \quad \text{if} \quad t_{\mu-1,\mu} = t_{\mu,\mu+1} = t_S. \tag{7}$$

The one-particle Green function is defined through $(z - \mathcal{H})\boldsymbol{G}(z) = \mathbf{1}$ and obeys the Dyson equation

$$(z-\epsilon_\mu)G_{\mu\nu}(z) = \delta_{\mu\nu} + t_{\mu\mu-1}G_{\mu-1\nu}(z) + t_{\mu\mu+1}G_{\mu+1\nu}(z), \tag{8}$$

with $z = E + i\eta$. In particular, the local density of states (LDOS) at site μ is given by

$$\rho_\mu(E) = -\frac{1}{\pi}\mathrm{Im}G_{\mu\mu}(E+i0^+). \tag{9}$$

III. COMBINATORIAL STRUCTURAL DESCRIPTION

A. Group theoretical description

Mathematical tools for analyzing aperiodic structures are found in the mathematical discipline of combinatorial group theory, as laid down, e.g., in Ref. 11. The GFC are special cases, which may be described by words $w = \{y_\mu\}_{\mu=1}^k \equiv y_1 \ldots y_k$ in the symbols $y_\mu \in \mathcal{A}_2$ over a two-element set $\mathcal{A}_2$ (''alphabet''). The set of all words over $\mathcal{A}_2$ together with the product $*$ defined in the last section creates the free semigroup $\boldsymbol{F}_2^+$. Appending the empty word $\mathbf{1}$: $\mathbf{1}w = w\mathbf{1} = w$ as well as the inverse elements y_μ^{-1}: $y_\mu^{-1}y_\mu = y_\mu y_\mu^{-1} = \mathbf{1}$, $y_\mu \in \mathcal{A}_2$, we extend $\boldsymbol{F}_2^+$ to the group $\boldsymbol{F}_2 = \langle L,S\rangle$ with the generating elements L,S. Except for the trivial relations $LL^{-1} = L^{-1}L = \mathbf{1}$ and $SS^{-1} = S^{-1}S = \mathbf{1}$ there are no defining relations between the generating elements, i.e., $\boldsymbol{F}_2$ is a *free group*.

We assign (cf. Fig. 1) to a word $w \in \boldsymbol{F}_2$, a *dual word* $\Sigma = \{\sigma_\mu\}_{\mu=0}^N$, where a symbol $\sigma \in \{\alpha_i^j, \beta_i^j, \gamma_i^j, \delta_i^j\}$ will be related to each pair $(y_\mu, y_{\mu+1})$ by the map $(L^iL^j, L^iS^j, S^iL^j, S^iS^j) \leftrightarrow (\alpha_i^j, \beta_i^j, \gamma_i^j, \delta_i^j)$, $i,j \in \{-1,+1\}$, e.g., the dual word pertaining to w_5 in Fig. 1 is $\Sigma_5 = \gamma\beta\gamma\alpha\beta\gamma\beta\gamma\alpha\beta\gamma\alpha\beta$. As a rule, we will use the abbreviations $\alpha = \alpha_+^+, \beta = \beta_+^+, \gamma = \gamma_+^+, \delta = \delta_+^+$. As we limit ourselves to the case of *periodic boundary condition* this map is unique. Let Σ_u, Σ_v, and Σ_w be the dual words of u,v, and $w = uv$, respectively. We define the product of two dual words by $\Sigma_u\Sigma_v = \Sigma_w$.

In the following, we will employ exclusively *presentations of groups* in the form of generators and a complete set of defining relations. Consider (cf. Ref. 12) the following *presentation of the automorphism group* $\boldsymbol{\Phi}_2$ of $\boldsymbol{F}_2$ found by Nielsen in 1924.[7,11] It will be apparent that $\boldsymbol{\Phi}_2$ is isomorphic to the *renormalization group* $\boldsymbol{RG}_{\downarrow x,y}$ which will be introduced in the following section:

$$\boldsymbol{\Phi}_2 = \langle \hat{\mathcal{P}}, \hat{\mathcal{O}}, \hat{\mathcal{U}} | \hat{\mathcal{P}}^2 = \hat{\mathcal{O}}^2 = (\hat{\mathcal{O}}\hat{\mathcal{P}})^4 = (\hat{\mathcal{P}}\hat{\mathcal{O}}\hat{\mathcal{P}}\hat{\mathcal{U}})^2$$

$$= (\hat{\mathcal{O}}\hat{\mathcal{P}}\hat{\mathcal{U}})^3 = [\hat{\mathcal{U}}, \hat{\mathcal{O}}\hat{\mathcal{U}}\hat{\mathcal{O}}] = \mathbf{1}\rangle. \tag{10}$$

$\hat{\mathcal{P}}, \hat{\mathcal{O}}$, and $\hat{\mathcal{U}}$ are the generating elements of $\boldsymbol{\Phi}_2$ represented by

$$\hat{\mathcal{P}}\colon\ L \to S,\ S \to L \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\mathcal{O}}\colon\ L \to L^{-1},\ S \to S \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{\mathcal{U}}^{\pm 1}\colon\ L \to LS^{\pm 1},\ S \to S \quad \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

The corresponding substitution matrices are elements of the *unimodular group* $\boldsymbol{GL}(2,\mathbb{Z})$ (cf. Refs. 11 and 13). $[x,y] \equiv xyx^{-1}y^{-1}$ is the group commutator of x and y. Note, the group of Nielsen transformations of rank 2, $\boldsymbol{\Gamma}_2$, is *anti*isomorphic to the automorphism group $\mathbf{\Phi}_2 = \boldsymbol{Aut}(\boldsymbol{F}_2)$ [Ref. 11, cf. Eq. (5)]. Later, we will frequently encounter a second ''elementary automorphism''

$$\hat{\mathcal{A}}^{\pm 1} = \hat{\mathcal{O}}\hat{\mathcal{U}}^{\mp 1}\hat{\mathcal{O}}:\ L \to S^{\pm 1}L,\ S \to S. \tag{12}$$

$\mathbf{\Phi}_2$ alternatively can be generated by the automorphisms $\hat{\mathcal{P}}, \hat{\mathcal{A}}$, and $\hat{\mathcal{U}}$ or by the automorphisms $\hat{\mathcal{P}}, \hat{\mathcal{L}} = \hat{\mathcal{P}}\hat{\mathcal{A}}$ and $\hat{\mathcal{R}} = \hat{\mathcal{P}}\hat{\mathcal{U}}$, where $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ are the generators of all cyclic permutations of the standard Fibonacci chain. Note that $\hat{\mathcal{A}}$ and $\hat{\mathcal{U}}$ correspond to the same element of $\boldsymbol{GL}(2,\mathbb{Z})$. Adding the relation $\hat{\mathcal{U}} = \hat{\mathcal{A}} = \hat{\mathcal{O}}\hat{\mathcal{U}}^{-1}\hat{\mathcal{O}}$ or

$$(\hat{\mathcal{O}}\hat{\mathcal{U}})^2 = \mathbf{1} \tag{13}$$

to the set of relations (10) yields a presentation of $\boldsymbol{GL}(2,\mathbb{Z})$, where its generators $\hat{\mathcal{P}}, \hat{\mathcal{O}}$, and $\hat{\mathcal{U}}$ may be represented by the substitution matrices (11). It is also well known that the *projective unimodular group* $\boldsymbol{PGL}(2,\mathbb{Z})$ is defined by the system of relations (10) and (13) along with a further relation

$$(\hat{\mathcal{O}}\hat{\mathcal{P}})^2 = \mathbf{1}. \tag{14}$$

Thus, if the matrix

$$\hat{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

represents an element of $\boldsymbol{GL}(2,\mathbb{Z})$, then $\pm\hat{\mathcal{B}}$ determine an element $\hat{\mathcal{B}}(z) = (az+b)/(cz+d)$ in $\boldsymbol{PGL}(2,\mathbb{Z})$,[7,11] The usage of the same letters for elements of different groups should not cause any confusion.

We now turn to the geometric structures considered in Sec. II. All corresponding lattices may be described by words of the form $w(L,S) = \hat{\mathcal{B}}(L) \in \boldsymbol{F}_2^+$ with $\hat{\mathcal{B}} \in \mathbf{\Phi}_2^+$, where the semigroup $\mathbf{\Phi}_2^+(\mathbf{\Phi}_2^-) \subset \mathbf{\Phi}_2$ is the set of all positive (negative) words in $\hat{\mathcal{P}}, \hat{\mathcal{A}}$, and $\hat{\mathcal{U}}$. One defines $w(\hat{\mathcal{P}}, \hat{\mathcal{A}}, \hat{\mathcal{U}})$ as a *positive* (*negative*) word if *no* $\hat{\mathcal{P}}^k, \hat{\mathcal{A}}^k$, or $\hat{\mathcal{U}}^k$ with $k<0$ $(k>0)$ occur in w. We found the following presentation:

$$\mathbf{\Phi}_2^+ = \langle \hat{\mathcal{P}}, \hat{\mathcal{A}}, \hat{\mathcal{U}}; \hat{\mathcal{P}}^2 = \mathbf{1}, \hat{\mathcal{U}}\hat{\mathcal{A}} = \hat{\mathcal{A}}\hat{\mathcal{U}},$$

$$\hat{\mathcal{U}}\hat{\mathcal{P}}\hat{\mathcal{U}}^k\hat{\mathcal{P}}\hat{\mathcal{A}} = \hat{\mathcal{A}}\hat{\mathcal{P}}\hat{\mathcal{A}}^k\hat{\mathcal{P}}\hat{\mathcal{U}},\ \ k \in \mathbb{N}^+\rangle. \tag{15}$$

$\mathbf{\Gamma}_2^+$ is the pendant of $\mathbf{\Phi}_2^+$ in $\mathbf{\Gamma}_2$.

Since $\hat{\mathcal{P}}^2 = \mathbf{1}$, and because $\hat{\mathcal{A}}$ commutes with $\hat{\mathcal{U}}$, each automorphism $\hat{\mathcal{B}} \in \mathbf{\Phi}_2^+$ has the form $\hat{\mathcal{B}} = \hat{\mathcal{P}}^{p_1}\hat{\mathcal{X}}_{(r_n \cdots r_1)}\hat{\mathcal{P}}^{p_2}, p_i \in \{1,2\}$, where[12]

$$\hat{\mathcal{X}}_{(r_n \cdots r_1)} = \prod_{k=1}^{n} \hat{\mathcal{X}}_{N_k - r_k, r_k} \in \mathbf{\Phi}_2^+ \tag{16}$$

denotes the product of the automorphisms

$$\hat{\mathcal{X}}_{N-r,r} = \hat{\mathcal{P}}\hat{\mathcal{A}}^{N-r}\hat{\mathcal{U}}^r:\ \ \mathbf{S}_N = \begin{pmatrix} N & 1 \\ 1 & 0 \end{pmatrix}, \forall r \tag{17}$$

defined in Eq. (4). The substitution matrix corresponding to Eq. (16) is given by

$$\boldsymbol{S}_{(\mathcal{N}_n)} = \prod_{k=1}^{n} \boldsymbol{S}_{M_k} = \begin{pmatrix} P_n & Q_n \\ P_{n-1} & Q_{n-1} \end{pmatrix} \in \boldsymbol{GL}(2,\mathbb{Z}). \tag{18}$$

P_n (Q_n) is the number of the symbols L (S) in $w_n(L,S)$. They obey the recursion relation $(P_n, Q_n) = M_n(P_{n-1}, Q_{n-1}) + (P_{n-2}, Q_{n-2})$ [cf. Eq. (5)] with $(P_{-1}, Q_{-1}) = (0,1)$ and $(P_0, Q_0) = (1,0)$.

Thus, in the following we may restrict ourselves to the investigation of those lattices which may be described by the words

$$w_n^{r_n \cdots r_1}(L,S) = \hat{\mathcal{X}}_{(r_n \cdots r_1)}(L), \tag{19}$$

where $w_0 = L, w_{-1} = S$. Let us consider an important property of these words. By induction, we found for given $\mathcal{N}_n$ the following correspondence between the words belonging to different $\mathcal{R}_n$:

$$w_n^{r_n \cdots r_1} = v_n^{-1} w_n^{0 \cdots 0} v_n \in \boldsymbol{F}_2^+, \tag{20}$$

with

$$v_n = \left[\prod_{i=1}^{n-1} (w_i^{0 \cdots 0})^{r_{n-i}}\right](w_0)^{r_n} \in \boldsymbol{F}_2^+. \tag{21}$$

As a result, the morphisms $\hat{\mathcal{X}}_{(r_n \cdots r_1)}$ generate all cyclical permutations of the standard GFC obeying the standard recursion relation $w_n = (w_{n-1})^{M_n} w_{n-2}$ with $s_n = 0$ [cf. Eq. (1)].

All relations between the different automorphisms $\hat{\mathcal{B}}, \hat{\mathcal{B}}' \in \mathbf{\Phi}_2$ are reducible to the defining relations of $\mathbf{\Phi}_2$. But $\hat{\mathcal{B}} \neq \hat{\mathcal{B}}' \in \mathbf{\Phi}_2$ does *not* necessarily imply $\hat{\mathcal{B}}(L) \neq \hat{\mathcal{B}}'(L) \in \boldsymbol{F}_2$. The reason is found in a further relation

$$\hat{\mathcal{B}}\hat{\mathcal{P}}\hat{\mathcal{U}}(L) = \hat{\mathcal{B}}\hat{\mathcal{A}}(L), \quad \forall \hat{\mathcal{B}} \in \mathbf{\Phi}_2, \tag{22}$$

which is not contained in the system of relations (10). Note that Eq. (22) is a relation between *words*. Together with the system of defining relations (15) we have thus found all independent relations between the *positive words* $\hat{\mathcal{B}}(L) \in \boldsymbol{F}_2^+$ with $\hat{\mathcal{B}} \in \mathbf{\Phi}_2^+$.

B. A number system—representation of Eq. (16) by integers $\xi^{(n)}$

It will be apparent that a link between the real-space rescaling, which will be introduced in Sec. IV, on the one hand, and the following ideas on the other hand, will yield for given GFC a correspondence between the choice of the index μ of the diagonal elements of the Green function and a possible succession of rescaling transformations determining $G_{\mu\mu}(E)$ (cf. Sec. V).

We define $\hat{\mathcal{T}}_{F_n}(y_1 y_2 \ldots y_{F_n}) = y_2 \ldots y_{F_n} y_1, y_\mu \in \{L,S\}$ with *length* $|w_n| = F_n = P_n + Q_n \in \mathbb{N}^+$, defined in Sec. II. From Eq. (20) it follows

$$w_n^{r_n \cdots r_1}(L,S) = \hat{\mathcal{T}}_{F_n}^{\xi^{(n)}}(w_n^{0 \cdots 0}), \tag{23}$$

where

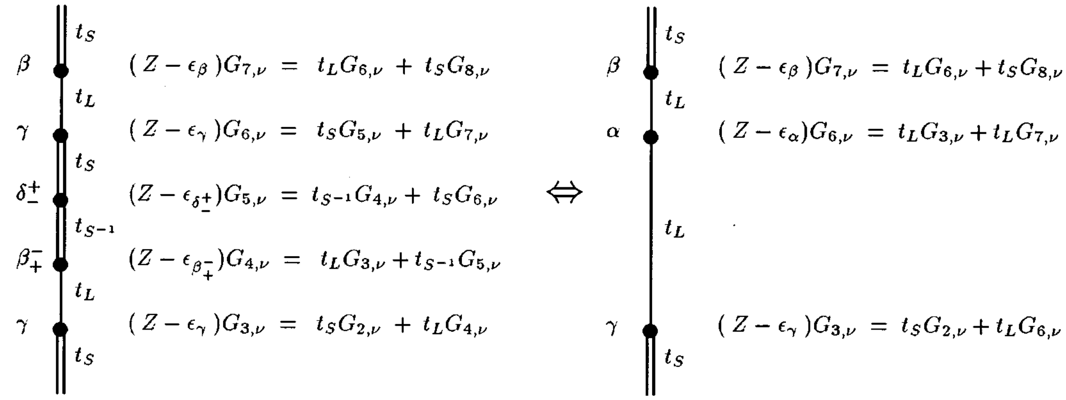


FIG. 2. *For given ν to each symbol σ_μ in the dual word of w corresponds one equation $(z-\epsilon_\mu)G_{\mu\nu}=\dots$* . Obviously, $w=\dots SLS^{-1}SLS\dots=\dots SLLS\dots=w'$. Thus the homogeneous part of the Dyson equation corresponding to a word w must be equivalent to the homogeneous part of the Dyson equation corresponding to the *freely reduced* word pertaining to w. From Eq. (29) it follows that $G_{4,\nu}\equiv G_{6,\nu}$, and consequently $\epsilon_{\beta^-_+}=\epsilon_\alpha-\epsilon_\gamma+z$.

$$\xi^{(n)}=\sum_{k=1}^{n} r_k F_{n-k} \tag{24}$$

takes *all* numbers in the interval $I_n:\quad 0\leq\xi^{(n)}\leq F_n+F_{n-1}-2$. Relations (23) and (24) have been found by Wai-fong Chuan[14] for the special case $M_k=1,\forall k$, by using another approach.

Using the fact that $gcd(F_n,F_{n-1})=1,\ n\geq 3$, one gets

$$\mathcal{N}_n^{(1)}=\mathcal{N}_m^{(2)}\Leftrightarrow \boldsymbol{S}_{(\mathcal{N}_n^{(1)})}=\boldsymbol{S}_{(\mathcal{N}_m^{(2)})}. \tag{25}$$

From Eqs. (15), (23), and (25), *for given $\mathcal{N}_n$* we discover the following important relation:

$$\xi_1^{(n)}=\xi_2^{(n)}\Leftrightarrow \hat{\boldsymbol{\mathcal{X}}}_{(r_n^{(1)}\cdots r_1^{(1)})}=\hat{\boldsymbol{\mathcal{X}}}_{(r_n^{(2)}\cdots r_1^{(2)})}. \tag{26}$$

It guarantees the existence and uniqueness of the representation of the morphisms $\hat{\boldsymbol{\mathcal{X}}}_{(r_n\cdots r_1)}$ by the integers $\xi^{(n)}$. As a result, it will be possible to define $\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}$ as a representative of the set of *equivalent* substitutions

$$\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}\in\left\{\hat{\boldsymbol{\mathcal{X}}}_{(r_n\cdots r_1)}\,\middle|\,\xi^{(n)}=\sum_{k=1}^{n} r_k F_{n-k}\right\}. \tag{27}$$

As stated above the substitutions $\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}},0\leq\xi^{(n)}\leq F_n+F_{n-1}-2$, generate all cyclic permutations of $w_n^{0\cdots 0}$ One can show that $\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}(L)=\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}+F_n}(L)$, which follows from Eqs. (15) and (22), provided $\xi^{(n)}+F_n\in I_n$.

Now let $\mathrm{int}(x)$ be the integer part of x, and $\xi^{(n)}=\mu\mathrm{mod}F_n$ (cf. Sec. V), respectively. It can be shown that

$$s_k=\mathrm{int}\left[\frac{(\xi^{(k)}\mathrm{mod}F_k)}{F_{k-1}}\right],\quad \xi^{(k-1)}=\xi^{(k)}-s_kF_{k-1}, \tag{28}$$

for $k=n,n-1,\dots,1$, successively yields, for given $\xi^{(n)}\in I_n$ a possible solution $\mathcal{R}_n$ of Eq. (24) determining the substitution $\hat{\boldsymbol{\mathcal{X}}}_{(r_n\cdots r_1)}=\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}$. In this manner one can determine the representatives $\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}$ *for all $\xi^{(n)}\in I_n$*.

IV. REAL–SPACE RESCALING — THE RENORMALIZATION GROUP $\boldsymbol{RG}_{\downarrow x,y}$

In this section we introduce, in a generalized form, a well-known real-space rescaling method developed by de Silva and Koiller.[15] The fundamental idea of the real-space rescaling is to achieve a reduction of degrees of freedom through appropriate elimination of equations from the set of equations (8). In the scope of aperiodic 1D systems, applications of this approach were found in a number of works (e.g., in Refs. 5 and 6), considering explicitly the symmetry of the lattices by the corresponding elimination steps. The advantage is clear; the number of parameters $\epsilon_\alpha,\dots,t_S$ remains constant. We generalize that real-space rescaling procedure as follows. In a first step, by means of $t_L\leftrightarrow L,\dots,t_{S^{-1}}\leftrightarrow S^{-1}$ and $z-\epsilon_\sigma\leftrightarrow\sigma\in\{\alpha,\dots,\delta_-^-\}$ we represent the set of equations $(z-\epsilon_\mu)G_{\mu\nu}=\delta_{\mu\nu}+\cdots$, where ν is fixed, by the words $w(L,\dots,S^{-1})$, and $\Sigma(\alpha,\dots,\delta_-^-)$, respectively (cf. Fig. 2). Thus, for given ν to each symbol σ_μ in the dual word of w corresponds *one* equation $(z-\epsilon_\mu)G_{\mu\nu}=\delta_{\mu\nu}+\cdots$. Note, only the correspondence between the *homogeneous* part of the Dyson equation and the word w is unique. As a result, we extend the original set of binary chains described by $w\in\boldsymbol{F}_2^+$ to a set of *fictitious* chains with *four* different hopping elements $t_L,t_{L^{-1}},t_S$, and $t_{S^{-1}}$ described by words w of the *free group* $\boldsymbol{F}_2$, rather than of the *semigroup* $\boldsymbol{F}_2^+$. Next, we will consider only such rescaling transformations, which do *not* eliminate the equation $(z-\epsilon_\mu)G_{\mu\mu}=1+\cdots$ containing the inhomogeneous part of the Dyson equation.[16] A succession of these transformations will just yield the *diagonal elements* of the Green function. A generalization of this method to obtain the nondiagonal elements $G_{\mu\nu}$ [i.e., the extention of $\boldsymbol{RG}_{\downarrow x,y}$ to the total renormalization group $\boldsymbol{RG}(\boldsymbol{F}_2)$ and its generalization to $\boldsymbol{RG}(\boldsymbol{F}_n)$] will be presented elsewhere.[9]

In the following we require that the *homogeneous* part of the set of equations which corresponds to a given word $w\in\boldsymbol{F}_2$ must be equivalent to the *homogeneous* part of another set of equations, which belongs to the freely reduced word w' pertaining to w (compare Fig. 2). A word w is called *freely reduced* if no y_μ^k, y_μ^{-k} with $k=\pm 1, y_\mu \in \{L,S\}$ follows one another in w. This precondition has far-reaching consequences for the concrete choice of the parameters $\epsilon_{\alpha_\pm^\mp}, \ldots, t_{S^{-1}}$. One can show that these parameters must satisfy the following conditions:

$$\epsilon_{\delta_\pm^\mp}=z, \quad \epsilon_{\alpha_\pm^\mp}=z, \quad t_{L^{-1}}=-t_L, \quad t_{S^{-1}}=-t_S, \tag{29}$$

$$z-\epsilon_{\alpha_-^-}=\epsilon_\alpha-z, \quad z-\epsilon_{\delta_-^-}=\epsilon_\delta-z, \quad z-\epsilon_{\beta_\mp^\pm}=\epsilon_{\gamma_\mp^\pm}-z,$$

$$z-\epsilon_{\beta_\pm^\pm}=\epsilon_{\gamma_\mp^\mp}-z, \quad \epsilon_{\delta_\pm^\pm}=\epsilon_{\beta_\pm^\pm}+\epsilon_{\gamma_\pm^\mp}-z, \tag{30}$$

as well as

$$\epsilon_\alpha=\epsilon_\beta+\epsilon_\gamma-\epsilon_\delta. \tag{31}$$

From Eqs. (29)–(31) one recognizes that only the parameters $t_L, t_S, \epsilon_\beta, \epsilon_\gamma$, and ϵ_α (or ϵ_δ) can be chosen independently. If these parameters are defined all other parameters in the Dyson equation (8) are fixed.

Let us consider the tight-binding model for a linear chain described by an *arbitrary* word $w\in \boldsymbol{F}_2$. We introduce the rescaling transformations ${}^R\hat{\mathcal{P}}, {}^R\hat{\mathcal{O}}$, and ${}^R\hat{\mathcal{U}}^{-1}$ of the parameters $\epsilon_\alpha, \ldots, t_{L^{-1}}$, which are induced by the elementary automorphisms $\hat{\mathcal{P}}, \hat{\mathcal{O}}$, and $\hat{\mathcal{U}}^{-1}$. In observing the relations (29)–(31), we can limit ourselves to the examination of the transformation behavior of the parameters $t_L, t_S, \epsilon_\beta, \epsilon_\gamma$, and ϵ_α, where, for reasons of convenience, we include also ϵ_δ. A representation of ${}^R\hat{\mathcal{P}}={}^R\hat{\mathcal{P}}^{-1}$ is given by

$$\epsilon_{\alpha|\hat{\mathcal{P}}}=\epsilon_\delta, \quad \epsilon_{\beta|\hat{\mathcal{P}}}=\epsilon_\gamma, \quad \epsilon_{\gamma|\hat{\mathcal{P}}}=\epsilon_\beta, \quad \epsilon_{\delta|\hat{\mathcal{P}}}=\epsilon_\alpha,$$

$$t_{L|\hat{\mathcal{P}}}=t_S, \quad t_{S|\hat{\mathcal{P}}}=t_L, \tag{32}$$

and for ${}^R\hat{\mathcal{O}}={}^R\hat{\mathcal{O}}^{-1}$ one obtains [cf. Eqs. (29) and (30)]

$$\epsilon_{\alpha|\hat{\mathcal{O}}}=2z-\epsilon_\alpha, \quad \epsilon_{\beta|\hat{\mathcal{O}}}=z+\epsilon_\delta-\epsilon_\gamma, \quad \epsilon_{\gamma|\hat{\mathcal{O}}}=z+\epsilon_\delta-\epsilon_\beta,$$

$$\epsilon_{\delta|\hat{\mathcal{O}}}=\epsilon_\delta, \quad t_{L|\hat{\mathcal{O}}}=-t_L, \quad t_{S|\hat{\mathcal{O}}}=t_S. \tag{33}$$

Let $\varrho_l^{-1}: L\mapsto L\underline{S}S^{-1}, S\mapsto S$ be an ''*inverse reduction process*,'' and let $\overline{\mathcal{U}}^{-1}: LS\mapsto L, S\mapsto S$ be a *deflation operation*. ϱ_l^{-1} induces a ''lifting'' of the Dyson equation corresponding to the freely reduced word w' to the Dyson equation corresponding to w (cf. Fig. 2), where the parameters $\epsilon_\alpha, \ldots, t_{S^{-1}}$ are determined by Eqs. (29)–(31).[16] Thus, $\hat{\mathcal{U}}^{-1}=\overline{\mathcal{U}}^{-1}\varrho_l^{-1}$ induces ${}^R\hat{\mathcal{U}}^{-1}$, which possesses the representation

$$\epsilon_{\alpha|\hat{\mathcal{U}}^{-1}}=\epsilon_\gamma+\frac{t_L^2+t_S^2}{z-\epsilon_\beta}, \quad \epsilon_{\beta|\hat{\mathcal{U}}^{-1}}=\epsilon_\delta+\frac{t_S^2}{z-\epsilon_\beta},$$

$$\epsilon_{\gamma|\hat{\mathcal{U}}^{-1}}=\epsilon_\gamma+\frac{t_L^2}{z-\epsilon_\beta}, \quad \epsilon_{\delta|\hat{\mathcal{U}}^{-1}}=\epsilon_\delta,$$

$$t_{L|\hat{\mathcal{U}}^{-1}}=\frac{t_L t_S}{z-\epsilon_\beta}, \quad t_{S|\hat{\mathcal{U}}^{-1}}=t_S. \tag{34}$$

One obtains ${}^R\hat{\mathcal{A}}^{-1}$ from Eq. (34) by interchanging the symbols β and γ. Let $\underline{\varrho_r}^{-1}: L\mapsto LS^{-1}S, S\mapsto S$ and $\overline{\mathcal{U}}: LS^{-1}\mapsto L, S\mapsto S$. Thus $\hat{\mathcal{U}}=\mathcal{U}\varrho_r^{-1}$. As a result, one obtains as a representation for ${}^R\hat{\mathcal{U}}$

$$\epsilon_{\alpha|\hat{\mathcal{U}}}=\epsilon_\gamma-\epsilon_\delta+z-\frac{t_L^2+t_S^2}{\epsilon_\beta-\epsilon_\delta}, \quad \epsilon_{\beta|\hat{\mathcal{U}}}=-\frac{t_S^2}{\epsilon_\beta-\epsilon_\delta}+z,$$

$$\epsilon_{\gamma|\hat{\mathcal{U}}}=\epsilon_\gamma-\frac{t_L^2}{\epsilon_\beta-\epsilon_\delta}, \quad \epsilon_{\delta|\hat{\mathcal{U}}}=\epsilon_\delta,$$

$$t_{L|\hat{\mathcal{U}}}=\frac{t_L t_S}{\epsilon_\beta-\epsilon_\delta}, \quad t_{S|\hat{\mathcal{U}}}=t_S. \tag{35}$$

Obviously, ${}^R\hat{\mathcal{U}}^{-1}\circ{}^R\hat{\mathcal{U}}={}^R\hat{\mathcal{U}}\circ{}^R\hat{\mathcal{U}}^{-1}=\mathbf{1}$ holds.

Note that ${}^R\hat{\mathcal{X}}_{j,i}^{-1}, i,j\in\mathbb{Z}$ (!), are determined by means of these three generating RT,

$$\epsilon_{\alpha|\hat{\mathcal{X}}_{j,i}^{-1}}=\epsilon_\alpha+t_L\frac{\Xi_{i-1,j}+\Xi_{i,j-1}}{\Xi_{i,j}}, \quad \epsilon_{\delta|\hat{\mathcal{X}}_{j,i}^{-1}}=\epsilon_\alpha,$$

$$\epsilon_{\beta|\hat{\mathcal{X}}_{j,i}^{-1}}=\epsilon_\alpha+t_L\frac{\Xi_{i-1,j}}{\Xi_{i,j}}, \quad \epsilon_{\gamma|\hat{\mathcal{X}}_{j,i}^{-1}}=\epsilon_\alpha+t_L\frac{\Xi_{i,j-1}}{\Xi_{i,j}},$$

$$t_{L|\hat{\mathcal{X}}_{j,i}^{-1}}=\frac{t_S}{\Xi_{i,j}}, \quad t_{S|\hat{\mathcal{X}}_{j,i}^{-1}}=t_L. \tag{36}$$

Thereby, the polynomials

$$\Xi_{i,j}=\left[\frac{z-\epsilon_\gamma}{t_L}\mathcal{U}_i-\mathcal{U}_{i-1}\right]\left[\frac{z-\epsilon_\beta}{t_L}\mathcal{U}_j-\mathcal{U}_{j-1}\right]-\frac{t_S^2}{t_L^2}\mathcal{U}_i\mathcal{U}_j \tag{37}$$

were introduced with the *Chebyshev polynomials* of the second kind $\mathcal{U}_n=\mathcal{U}_n(x_\alpha/2)$.[17] $\Xi_{i,j}=\Xi_{i,j}(x_\alpha/2)$ fulfill the recursion formulas

$$\Xi_{i,j+1}=x_\alpha\Xi_{i,j}-\Xi_{i,j-1}, \quad \Xi_{i+1,j}=x_\alpha\Xi_{i,j}-\Xi_{i+1,j}, \tag{38}$$

yet with the initial values

$$\Xi_{00}=1, \quad \Xi_{10}=\frac{z-\epsilon_\gamma}{t_L}, \quad \Xi_{01}=\frac{z-\epsilon_\beta}{t_L},$$

$$\Xi_{11}=\frac{(z-\epsilon_\beta)(z-\epsilon_\gamma)-t_S^2}{t_L^2}, \tag{39}$$

and with the abbreviation $x_\alpha=(z-\epsilon_\alpha)/t_L$.

We denote the parameter space by $\boldsymbol{\Pi}_\epsilon=\{\boldsymbol{\epsilon}=(\epsilon_\alpha,\epsilon_\beta,\epsilon_\gamma,t_L,t_S)|\epsilon_\alpha,\epsilon_\beta,\epsilon_\gamma,t_L,t_S\in\mathbb{C}\}$. Each *automorphism* $\hat{\mathcal{B}}\in\boldsymbol{\Phi}_2$ corresponds uniquely to one *renormalization transformation (RT)* ${}^R\hat{\mathcal{B}}\in\boldsymbol{RG}_{\downarrow x,y}$ (i.e., a homomorphism of $\boldsymbol{\Pi}_\epsilon$ into itself), which converts the set of equations with the parameters $\boldsymbol{\epsilon}\in\boldsymbol{\Pi}_\epsilon$ into a new set of equations with new parameters $\boldsymbol{\epsilon}'\in\boldsymbol{\Pi}_\epsilon$

$${}^R\hat{\mathcal{B}}: \quad \boldsymbol{\epsilon}\mapsto\boldsymbol{\epsilon}'=\boldsymbol{\epsilon}_{|\hat{\mathcal{B}}}\equiv{}^R\hat{\mathcal{B}}(\boldsymbol{\epsilon}) \tag{40}$$

(generalized real-space rescaling method). Now, it will be possible to define the renormalization *group*. The product of two RT is given through

$$^{R}(\hat{\mathcal{B}}_2\hat{\mathcal{B}}_1)={}^{R}\hat{\mathcal{B}}_2\circ{}^{R}\hat{\mathcal{B}}_1:\ \boldsymbol{\epsilon}_{|\hat{\mathcal{B}}_2\hat{\mathcal{B}}_1}=(\boldsymbol{\epsilon}_{|\hat{\mathcal{B}}_1})_{|\hat{\mathcal{B}}_2}. \tag{41}$$

The RT ${}^{R}\hat{\mathcal{B}}\in\boldsymbol{RG}_{\downarrow x,y}$ satisfy the same system of relations as the accompanying automorphisms $\hat{\mathcal{B}}\in\boldsymbol{\Phi}_2$. Thus $\boldsymbol{RG}_{\downarrow x,y}$ is isomorphic to $\boldsymbol{\Phi}_2$ ($\boldsymbol{RG}_{\downarrow x,y}\cong\boldsymbol{\Phi}_2$), generated by the three RT ${}^{R}\hat{\mathcal{P}},{}^{R}\hat{\mathcal{O}}$ (linear) and ${}^{R}\hat{\mathcal{U}}$ (nonlinear).

Note, employing the simple map $\epsilon'_\alpha=z-\epsilon_\alpha,\epsilon'_\beta=z-\epsilon_\beta,\epsilon'_\gamma=z-\epsilon_\gamma,t'_L=t_L,t'_S=t_S$ to Eqs. (32)–(35) yields equivalent parameter transformations where the energy z does not occur explicitly. Thus the *RT considered as homomorphisms* ${}^{R}\hat{\mathcal{B}}:\boldsymbol{\Pi}_\epsilon\mapsto\boldsymbol{\Pi}_{\epsilon'}$ are independent upon the energy.

Remarkably, the parameters ϵ_α and t_L satisfy yet another relation, which is the pendant to Eq. (22), but does not follow from the relations defining $\boldsymbol{RG}_{\downarrow x,y}$ [cf. Eq. (10)]:

$$\epsilon_{\alpha|\hat{\mathcal{A}}^{-1}\hat{\mathcal{P}}\hat{\mathcal{B}}^{-1}}=\epsilon_{\alpha|\hat{\mathcal{U}}^{-1}\hat{\mathcal{B}}^{-1}},\quad t_{L|\hat{\mathcal{A}}^{-1}\hat{\mathcal{P}}\hat{\mathcal{B}}^{-1}}=t_{L|\hat{\mathcal{U}}^{-1}\hat{\mathcal{B}}^{-1}}. \tag{42}$$

for all $\hat{\mathcal{B}}^{-1}\in\boldsymbol{\Phi}_2$. The remaining parameters $\epsilon_\beta,\epsilon_\gamma,\epsilon_\delta$, and t_S, however, do *not* fulfill such a relation.

In the remainder of this section we employ the following map of the parameter space $\boldsymbol{\Pi}_\epsilon\mapsto\boldsymbol{\Pi}_{x,y}\subseteq\boldsymbol{\Pi}_\epsilon$ onto itself, which reveals some new properties of the RT. We have introduced the Cartesian product $\boldsymbol{\Pi}_{x,y}=\boldsymbol{\Pi}_x\times\boldsymbol{\Pi}_y$ of $\boldsymbol{\Pi}_x=\{\mathbf{X}=(x_\delta,x_\alpha,x'_\alpha)|x_\delta,x_\alpha,x'_\alpha\in\mathbb{C}\}$ with

$$x_\delta=\frac{z-\epsilon_\delta}{t_S},\quad x_\alpha=\frac{z-\epsilon_\alpha}{t_L},$$

$$x'_\alpha=\frac{(z-\epsilon_\beta)(z-\epsilon_\gamma)-t_L^2-t_S^2}{t_Lt_S}=x_{\alpha|\hat{\mathcal{A}}^{-1}}=x_{\alpha|\hat{\mathcal{U}}^{-1}}, \tag{43}$$

and $\boldsymbol{\Pi}_y=\{\mathbf{Y}=(y_S,y_L,y_\beta,y_\gamma)|y_S,y_L,y_\beta,y_\gamma\in\mathbb{C}\}$ with

$$y_S=\frac{1}{t_S},\quad y_L=\frac{1}{t_L},\quad y_\beta=y_{L|\hat{\mathcal{U}}^{-1}}=\frac{z-\epsilon_\beta}{t_Lt_S}=\frac{1}{2}(\chi+\Delta),$$

$$y_\gamma=y_{L|\hat{\mathcal{A}}^{-1}}=\frac{z-\epsilon_\gamma}{t_Lt_S}=\frac{1}{2}(\chi-\Delta), \tag{44}$$

where $\Delta^2\equiv\chi^2-4\eta=(y_\beta-y_\gamma)^2$, $\chi=x_\alpha y_S+x_\delta y_L$, and $\eta=y_Sy_Lx'_\alpha+y_L^2+y_S^2$.

It will be instructive to examine the transformation behavior of the new parameters under ${}^{R}\hat{\mathcal{P}},{}^{R}\hat{\mathcal{A}}^{-1}$, and ${}^{R}\hat{\mathcal{U}}^{-1}$. One obtains for the parameter tuple $\boldsymbol{X}$

$$\boldsymbol{X}_{|\hat{\mathcal{P}}}=(x_\alpha,x_\delta,x'_\alpha),\quad \boldsymbol{X}_{|\hat{\mathcal{A}}^{-1}}=\boldsymbol{X}_{|\hat{\mathcal{U}}^{-1}}=(x_\delta,x'_\alpha,x_\delta x'_\alpha-x_\alpha). \tag{45}$$

This dynamical system possesses an *invariant*,[18] i.e., the quantity

$$\Lambda=x_\delta^2+x_\alpha^2+{x'_\alpha}^2-x_\delta x_\alpha x'_\alpha \tag{46}$$

obeys $\Lambda=\Lambda_{|\hat{\mathcal{B}}^{-1}}$ for all $\hat{\mathcal{B}}^{-1}\in\boldsymbol{\Phi}_2$.

We define the *restriction* ${}^{R}\hat{\mathcal{B}}_{\downarrow_x}$ of the RT ${}^{R}\hat{\mathcal{B}}$ to $\boldsymbol{\Pi}_x$ as the RT ${}^{R}\hat{\mathcal{B}}$ with the domain $\boldsymbol{\Pi}_x\subset\boldsymbol{\Pi}_{x,y}$. For all $\boldsymbol{X}\in\boldsymbol{\Pi}_x$, ${}^{R}\hat{\mathcal{B}}_{\downarrow_x}(\boldsymbol{X})={}^{R}\hat{\mathcal{B}}(\boldsymbol{X})\equiv\boldsymbol{X}_{|\hat{\mathcal{B}}}$. From Eq. (45) it is apparent that the restriction of ${}^{R}\hat{\mathcal{A}}$ and ${}^{R}\hat{\mathcal{U}}$ to $\boldsymbol{\Pi}_x$ yields ${}^{R}\hat{\mathcal{A}}_{\downarrow_x}={}^{R}\hat{\mathcal{U}}_{\downarrow_x}$, i.e., it induces a homomorphism $\boldsymbol{RG}_{\downarrow x,y}\to\boldsymbol{GL}(2,\mathbb{Z})$ [cf. Eq. (13)]. There is one *additional* relation $({}^{R}\hat{\mathcal{P}}_{\downarrow_x}\circ{}^{R}\hat{\mathcal{O}}_{\downarrow_x})^2=\mathbf{1}$ [but $(\hat{\mathcal{P}}\hat{\mathcal{O}})^2=-\mathbf{1}$] not contained in the presentation of $\boldsymbol{RG}_{\downarrow x,y}$. As a result, the restriction $\boldsymbol{RG}_{\downarrow x}$ of $\boldsymbol{RG}_{\downarrow x,y}$ to $\boldsymbol{\Pi}_x$ is isomorphic to $\boldsymbol{PGL}(2,\mathbb{Z})$ rather than to $\boldsymbol{GL}(2,\mathbb{Z})$ [cf. Eq. (14)].

Note the formal correspondence between Eqs. (45) and (46) to the so-called trace map as well as to the accompanying invariant (see, e.g., Refs. 4, 8, and 19). In contrast to the transfer matrix method, the invariant (46) depends upon the energy. An exception is the so-called *off diagonal* model ($\epsilon_\alpha=\cdots=\epsilon_\delta=\epsilon$), for which $\Lambda=(t_L/t_S+t_S/t_L)^2\geqslant4$ is valid.

The transformations of $\boldsymbol{Y}$ lead to

$$\boldsymbol{Y}_{|\hat{\mathcal{P}}}=(y_L,y_S,y_\gamma,y_\beta),$$

$$\boldsymbol{Y}_{|\hat{\mathcal{A}}^{-1}}=(y_S,y_\gamma,x'_\alpha y_S+y_L,x_\delta y_\gamma-y_L),$$

$$\boldsymbol{Y}_{|\hat{\mathcal{U}}^{-1}}=(y_S,y_\beta,x_\delta y_\beta-y_L,x'_\alpha y_S+y_L). \tag{47}$$

From Eq. (44) it is obvious that y_β and y_γ are not independent parameters.

Contrary to the rescaling transformations considered above, the parameters $(\boldsymbol{X},\boldsymbol{Y})$ are polynomials in the energy. These polynomials have an important property. They can be represented by *orthogonal polynomials*. Therefore, in the remainder of this section, we will represent our renormalization theory from a slightly different point of view.

Consider the tridiagonal *Jacobi matrix* [$t_\mu=t_{\mu\mu+1}$; cf. Eq. (7)]

$$\boldsymbol{J}(\Sigma)=\begin{pmatrix}\epsilon_0 & -t_0 & 0 & & 0 & 0\\ -t_0 & \epsilon_1 & -t_1 & \cdots & 0 & 0\\ 0 & -t_1 & \epsilon_2 & & 0 & 0\\ \vdots & & & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \epsilon_{N-2} & -t_{N-2}\\ 0 & 0 & 0 & \cdots & -t_{N-2} & \epsilon_{N-1}\end{pmatrix} \tag{48}$$

associated with the dual word $\Sigma=\sigma_0\sigma_1\cdots\sigma_{N-1}$ of the freely reduced word $u=\hat{\mathcal{B}}(L)\in\boldsymbol{F}_2,\hat{\mathcal{B}}\in\boldsymbol{\Phi}_2$. Let Σ_u,Σ_v, and Σ_w be the dual words of u,v, and $w=uv$, respectively. We define $\boldsymbol{J}(\Sigma_u\Sigma_v)=\boldsymbol{J}(\Sigma_w)$. The *associated monic orthogonal polynomials* may be defined by (cf., e.g., Ref. 20)

$$\pi^{(k)}_{\Sigma,N}=\det(z\mathbf{1}-\boldsymbol{J}[\hat{\mathcal{T}}^k_N(\Sigma)]). \tag{49}$$

satisfying the recurrence relation

$$\pi^{(k)}_{\Sigma,i+1}=(z-\epsilon_{i+k})\pi^{(k)}_{\Sigma,i}-t_{i+k}^2\pi^{(k)}_{\Sigma,i-1},\quad \pi^{(k)}_{\Sigma,-1}=0,\pi^{(k)}_{\Sigma,0}=1 \tag{50}$$

(remember, $\sigma_k=\sigma_{k+N}$). For convenience we introduce the polynomials

$$\Pi^{(k)}_{\Sigma,i}=\frac{\pi^{(k)}_{\Sigma,i}}{\prod_{p=0}^{i}t_{p+k-1}}. \tag{51}$$

Let Σ' be the dual word of the freely reduced word $v=\hat{\mathcal{B}}(S)$ with norm $M=\|v\|$ (cf. Ref. 10). Then, the relations

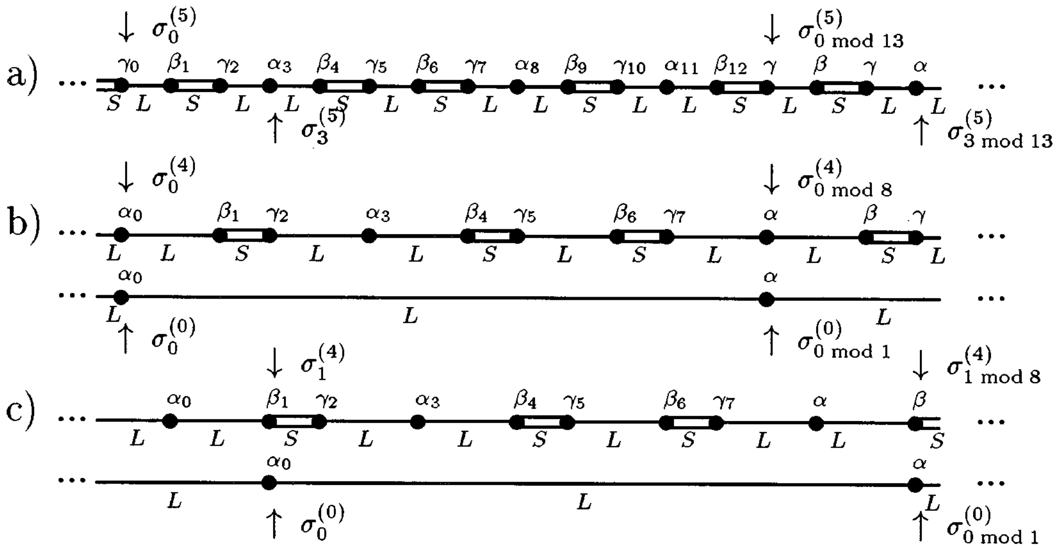


FIG. 3. (a) Standard Fibonacci chain [cf. Eq. (19)] $w_5(L,S)=w_5^{00000}(L,S)$. The arrows over the symbols $\sigma_0^{(5)}=\gamma$ and $\sigma_{13}^{(5)}=\gamma$ serve to emphasize the border of the unit cell containing the quasiperiodic structure. The arrows below the symbols α serve to emphasize the border of the unit cell of the lattice $\hat{\mathcal{T}}_{13}^{-3}(w_5(L,S))=w_5^{11000}(L,S)=w_5^{00100}(L,S)$. The *first* symbol of the dual word pertaining to the latter word corresponds to $\sigma_3^{(5)}$ of the dual lattice of w_5^{00000}. (b) The lattices $\hat{\boldsymbol{\mathcal{X}}}_{1,0}^{-1}(w_5^{00000}(L,S))=w_5^{00000}(S,S^{-1}L)=w_4^{0000}(L,S)$, and $\hat{\boldsymbol{\mathcal{X}}}_{\xi=0}^{-1}(w_5^{00000})=L$, respectively. (c) The lattices $\hat{\boldsymbol{\mathcal{X}}}_{0,1}^{-1}(w_5^{11000}(L,S))=w_5^{11000}(S,LS^{-1})=w_4^{1000}(L,S)$ and $\hat{\boldsymbol{\mathcal{X}}}_{\xi=3}^{-1}(w_5^{11000})=L$, respectively.

$$y_{L|\hat{\mathcal{B}}^{-1}}=\Pi_{\Sigma,N-1}^{(1)},\quad x_{\alpha|\hat{\mathcal{B}}^{-1}}=t_{-1}(\Pi_{\Sigma,N}^{(0)}-\Pi_{\Sigma,N-2}^{(1)}) \tag{52}$$

and $x'_{\alpha|\hat{\mathcal{B}}^{-1}}=t'_{-1}(\Pi_{\Sigma\Sigma',M+N}^{(0)}-\Pi_{\Sigma\Sigma',M+N-2}^{(1)})$, $x_{\delta|\hat{\mathcal{B}}^{-1}}=t'_{-1}(\Pi_{\Sigma',M}^{(0)}-\Pi_{\Sigma',M-2}^{(1)})$ as well as $y_{S|\hat{\mathcal{B}}^{-1}}=\Pi_{\Sigma',M-1}^{(1)}$, $y_{\beta|\hat{\mathcal{B}}^{-1}}=\Pi_{\Sigma\Sigma',M+N-1}^{(1)}$, $y_{\gamma|\hat{\mathcal{B}}^{-1}}=\Pi_{\Sigma\Sigma',M+N-1}^{(M+1)}$ are valid. From this point of view, *real-space rescaling* yields

$$\Pi_{\Sigma,N|\hat{\mathcal{B}}^{-1}}^{(0)}=\Pi_{\hat{\mathcal{B}}(\Sigma),N'}^{(0)}, \tag{53}$$

where $N'=\|\hat{\boldsymbol{\mathcal{B}}}(\Sigma)\|$. Taking into account Eq. (52), the *renormalization transformations* are given by Eq. (45), and Eq. (47), respectively. Finally, we note a relation which corresponds to Eq. (68)

$$t_{-1}(\Pi_{\Sigma,N}^{(0)}-\Pi_{\Sigma,N-2}^{(1)})=t_{k-1}(\Pi_{\Sigma,N}^{(k)}-\Pi_{\Sigma,N-2}^{(k+1)}). \tag{54}$$

V. DETERMINATION OF $G_{\mu,\mu}$

Periodic approximants of the GFC have the property that the words $w_n^{s_1\cdots s_n}$ with given $\mathcal{N}_n$, but different $\mathcal{R}_n$, are equivalent to one another. Thus we distinguish only those chains which can be described by words of the form

$$w_\infty(L,S)=\lim_{n\to\infty}\lim_{p\to\infty}(w_n^{0\cdots 0}(L,S))^p \tag{55}$$

[and $\hat{\boldsymbol{\mathcal{P}}}(w_\infty(L,S))$ respectively], defining the limiting process mentioned in Sec. II.

Let $\rho=n-k+1$. $\hat{\boldsymbol{\mathcal{X}}}_{M_k-s_k,s_k}^{-1}$ induces on $w_\rho^{r_\rho\cdots r_1}(L,S)$ a *deflation operation* $L^{M_k-s_k}SL^{s_k}\to L, L\to S$ such that $w_\rho^{r_\rho\cdots r_1}\to w_{\rho-1}^{r_{\rho-1}\cdots r_1}\in \boldsymbol{F}_2^+$ will be accomplished. Thus one may interpret $w_{\rho-1}^{r_{\rho-1}\cdots r_1}$ as a sublattice of $w_\rho^{r_\rho\cdots r_1}$.

The transformation $\hat{\boldsymbol{\mathcal{X}}}_{\xi^{(n)}}^{-1}$ eliminates in the dual word $\Sigma_n^{r_n\cdots r_1}=\sigma_0^{(n)}\sigma_1^{(n)}\cdots\sigma_{F_n-1}^{(n)}$, corresponding to $w_n^{r_n\cdots r_1}$, all symbols except for its first one $\sigma_0^{(0)}$. According to the periodic boundary condition (i.e., $\sigma_\mu^{(\rho)}=\sigma_{\mu+F_\rho}^{(\rho)}$) these transformations eliminate in the dual word, corresponding to $w_n^{0\cdots 0}$ (!), all symbols except $\sigma_{\xi^{(n)}}$. Note that we set $\sigma_{\xi^{(k)}}\equiv\sigma_{\xi^{(k)}}^{(k)}$ (cf. Fig. 3).

We now come back to the relation between the Dyson equation and the associated dual words. Let the index ν in the Green function $G_{\mu,\nu}$ equal $\nu=\xi\equiv\xi^{(n)}$. By so doing, we guarantee that the RT ${}^R\hat{\boldsymbol{\mathcal{X}}}_\xi^{-1}$ eliminates all equations, associated with the unit cell, except the equation $(z-\epsilon'_\xi)G_{\xi\xi}=1+\cdots$, containing the inhomogeneous part of Eq. (8) corresponding to the symbol σ_ξ. Thus the RT ${}^R\hat{\boldsymbol{\mathcal{X}}}_\xi^{-1}\in \boldsymbol{RG}_{\downarrow x,y}$ maps Eq. (8) to the Dyson equation of a chain,

$$[z-\epsilon'(z)]G_{\zeta\xi}(z)=\delta_{\zeta\xi}+t'(z)[G_{\zeta+F_n,\xi}(z)+G_{\zeta-F_n,\xi}(z)], \tag{56}$$

with only *one* site energy $\epsilon'(z)=\epsilon_{\alpha|\hat{\mathcal{X}}_\xi^{-1}}$, and a *single* hopping integral $t'(z)=t_{L|\hat{\mathcal{X}}_\xi^{-1}}$ (now *dependent* upon the energy), which are for given ξ *in*dependent from $\zeta\equiv\xi(\mathrm{mod}F_n)$. The solution of Eq. (56) is well known:[21]

$$G_{\zeta,\zeta}^+(E)=\lim_{\eta\to 0^+}\frac{1}{\sqrt{[z-\epsilon'(z)]^2-4t'^2(z)}}=\lim_{\eta\to 0^+}\frac{|\Pi_{\Sigma,F_n-1}^{(1)}|}{\sqrt{t_{-1}^2(\Pi_{\Sigma,F_n}^{(0)}-\Pi_{\Sigma,F_n-2}^{(1)})^2-4}}. \tag{57}$$

$$
\begin{aligned}
{}^R\hat{\mathcal{X}}_{4,2}^{-1} &= {}^R\hat{\mathcal{U}}^{-2}\,{}^R\hat{\mathcal{A}}^{-4}\,{}^R\hat{\mathcal{P}} \\
&= {}^R\hat{\mathcal{U}}^{-1}\,{}^R\hat{\mathcal{A}}^{-1}\,{}^R\hat{\mathcal{U}}^{-1}\,{}^R\hat{\mathcal{A}}^{-3}\,{}^R\hat{\mathcal{P}} \\
&= \ldots = {}^R\hat{\mathcal{A}}^{-4}\,{}^R\hat{\mathcal{U}}^{-2}\,{}^R\hat{\mathcal{P}}
\end{aligned}
$$

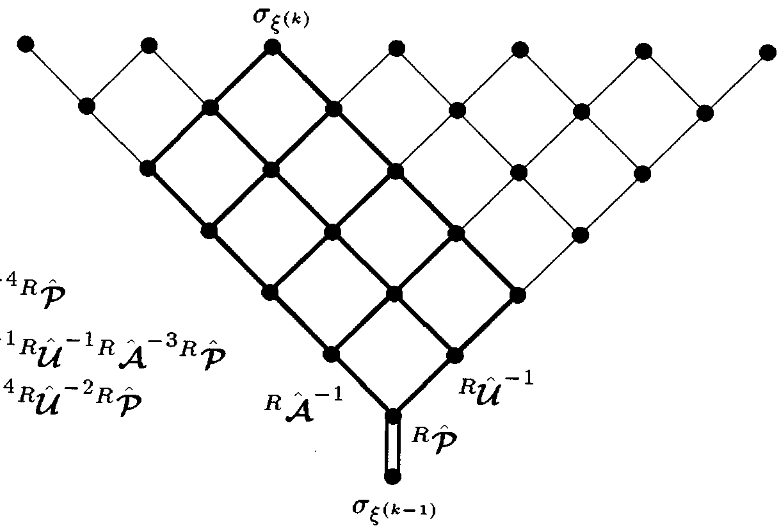


FIG. 4. The subgraph (bold) of $\boldsymbol{G}(V,\boldsymbol{E})$ which connects the points ${}^R\hat{\mathcal{X}}_{\xi(k)}^{-1}$ and ${}^R\hat{\mathcal{X}}_{\xi(k-1)}^{-1}$ corresponds to the edge ${}^R\hat{\mathcal{X}}_{4,2}^{-1}$ [cf. Eq. (58)]: ${}^R\hat{\mathcal{X}}_{\xi(k)}^{-1} = {}^R\hat{\mathcal{X}}_{4,2}^{-1} \bigcirc {}^R\hat{\mathcal{X}}_{\xi(k-1)}^{-1}$.

Thus the exact diagonal elements of the Green function have been found.

As an example we consider the fifteenth generation Fibonacci lattice [i.e., $\mathcal{N}_{15}=(1,\ldots,1)$], of length $|w_{15}|=F_{15}=1597$. Numerical results for the LDOS are given in Fig. 10 where $\xi\equiv\xi^{(15)}=888=\Sigma_{k=1}^{15}s_kF_{k-1}=F_2+F_4+F_7+F_{11}+F_{13}$. The Fibonacci numbers F_k have been determined via the recursion formula (3). Iterating the algorithm (28) with the initial value $\mu=\xi^{(15)}=888$ yields all s_k. Thus the inverse substitutions $\hat{\mathcal{X}}_{\xi=888}^{-1}$ have been fixed and we eventually get the desired RT [cf. Eqs. (5) and (17)]: ${}^R\hat{\mathcal{X}}_{\xi=888}^{-1} = {}^R\hat{\mathcal{X}}_{1,0}^{-1} \circ {}^R\hat{\mathcal{X}}_{0,1}^{-1} \circ \cdots \circ {}^R\hat{\mathcal{X}}_{0,1}^{-1} \circ {}^R\hat{\mathcal{X}}_{1,0}^{-2}$.

VI. A GRAPH THEORETICAL APPROACH— PATH RENORMALIZATION

Let us study in some detail the intertwined connections between the single GFC and the accompanying elements of the renormalization group. For this purpose we represent the group structure of $\boldsymbol{\Gamma}_2$, and $\boldsymbol{RG}_{\downarrow x,y}$, respectively, by a graph, i.e., a set $\boldsymbol{G}(V,\boldsymbol{E})$ with two types of elements which are designated as *points* $\sigma\in V$, and *edges* $\boldsymbol{e}\in\boldsymbol{E}$, respectively. From the statements made in Sec. III it is clear that the semigroups $\boldsymbol{\Gamma}_2^+,\boldsymbol{\Phi}_2^-$, and $\boldsymbol{RG}_{\downarrow x,y}^-$ (analogously defined as $\boldsymbol{\Phi}_2^-$) are isomorphic to each other. Let $V=\boldsymbol{RG}_{\downarrow x,y}^-\cong\boldsymbol{\Phi}_2^-\cong\boldsymbol{\Gamma}_2^+$. The different points $\sigma\in V$ corre-

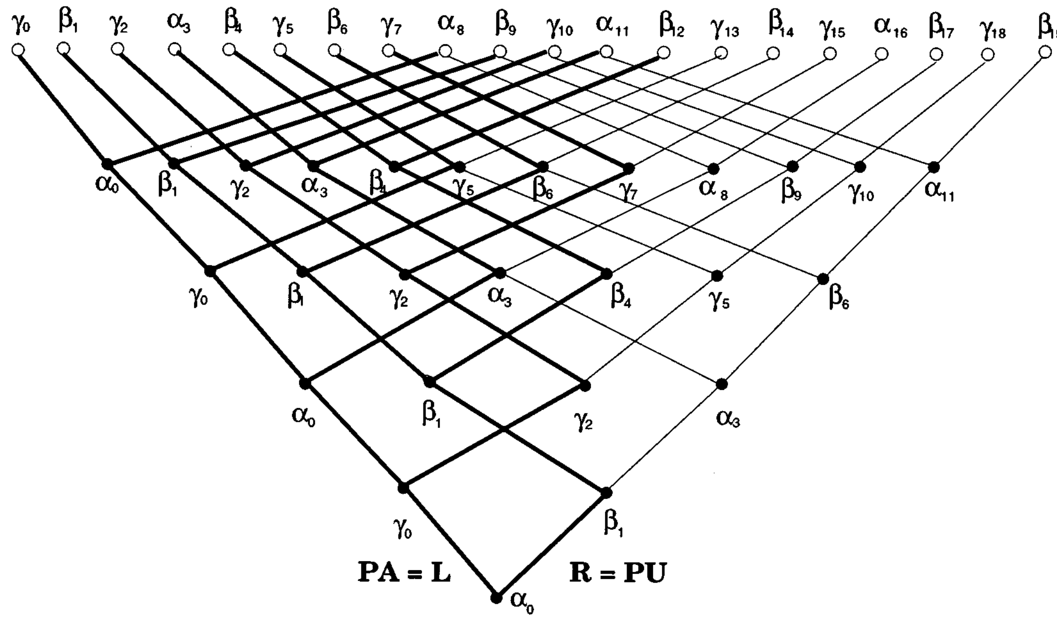
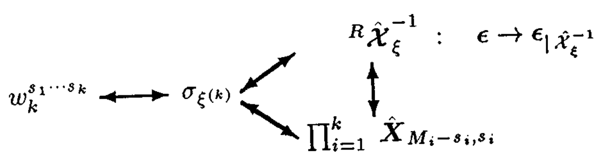


FIG. 5. $\boldsymbol{G}(W_{\mathcal{N}_5},\boldsymbol{E}_{\mathcal{N}_5})$ for the case of a fifth generation standard Fibonacci lattice [i.e., $\mathcal{N}_5=(1,1,1,1,1)$]. Set of bold edges: Tree $\boldsymbol{T}(\overline{W}_{\mathcal{N}_5},\overline{\boldsymbol{E}}_{\mathcal{N}_5})\subset\boldsymbol{G}(W_{\mathcal{N}_5},\boldsymbol{E}_{\mathcal{N}_5})$

spond directly to the elements of $\boldsymbol{RG}^{-}_{\downarrow x,y}$. We link a generator of $\boldsymbol{RG}^{-}_{\downarrow x,y}$ with each edge. This means, that any tuple $({}^{R}\hat{\mathcal{B}}_1^{-1},{}^{R}\hat{\mathcal{B}}_2^{-1};{}^{R}\hat{\mathcal{G}}^{-1})$ can connect with each edge $\boldsymbol{e}$, where ${}^{R}\hat{\mathcal{B}}_i^{-1}\in\boldsymbol{RG}^{-}_{\downarrow x,y}$, $i=1,2$ are the initial point, and end point of the edge respectively, and ${}^{R}\hat{\mathcal{G}}^{-1}\in\{{}^{R}\hat{\mathcal{P}},{}^{R}\hat{\mathcal{A}}^{-1},{}^{R}\hat{\mathcal{U}}^{-1}\}$ is the respective generator of $\boldsymbol{RG}^{-}_{\downarrow x,y}$ for which ${}^{R}\hat{\mathcal{B}}_2^{-1}=\hat{\mathcal{G}}^{-1}\circ{}^{R}\hat{\mathcal{B}}_1^{-1}$ is valid. A sequence of edges $\boldsymbol{\Pi}=\boldsymbol{e}_n\boldsymbol{e}_{n-1}\ldots\boldsymbol{e}_1$ where the end point σ' of $\boldsymbol{e}_\mu$ for $\mu=1,\ldots,n-1$, is connected with the initial point σ of $\boldsymbol{e}_{\mu+1}$ is called a *path* in a graph. Since a presentation of $\boldsymbol{RG}^{-}_{\downarrow x,y}$ results immediately from our presentation stated in Eq. (15), the construction of $\boldsymbol{G}(V,\boldsymbol{E})$ is basically trivial.

A common factor is seen when considering the following subgraphs. We define for any given $\mathcal{N}_n$ the following point set: $V_{\mathcal{N}_n}=\cup_{k=1}^{n}\{{}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k)}}|\forall\xi^{(k)}\in I_k\}\subset V$ and the set $\boldsymbol{E}_{\mathcal{N}_n}$ of all tuples $({}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k)}}={}^{R}\hat{\mathcal{X}}^{-1}_{M_k-s_k,s_k}\circ{}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k-1)}},{}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k-1)}})$ of $V_{\mathcal{N}_n}$, for which $s_k=0,\ldots,M_k$ and $k=1,\ldots,n$ is valid. Note that we identify the subgraph of $\boldsymbol{G}(V,\boldsymbol{E})$ which connects both points ${}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k-1)}}$ and ${}^{R}\hat{\mathcal{X}}^{-1}_{\xi^{(k)}}$ with one edge ${}^{R}\hat{\mathcal{X}}^{-1}_{M_k-s_k,s_k}\in\boldsymbol{E}_{\mathcal{N}_n}$ (cf. Fig. 4) in the "new" graph $\boldsymbol{G}(V_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ (cf. Fig. 5).

The procedure above itself suggests the following correspondences [cf. (24), (5), and (27) as well as (2)]:



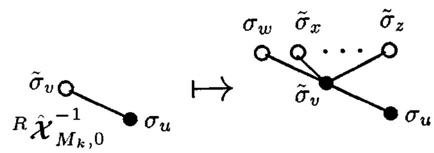
(58)

with $\xi^{(k)}\in I_k$. The meaning of $\sigma_{\xi^{(k)}}\in W_{\mathcal{N}_n}=\cup_{p=1}^{n}\{\sigma_{\xi^{(p)}}|\forall\xi^{(p)}\in I_p\}$ is manifold. It represents the corresponding elements of $\boldsymbol{RG}^{-}_{\downarrow x,y}$ (which *decimate* the lattice $w_k^{s_1\ldots s_k}$), and $\boldsymbol{\Gamma}_2^{+}$ (which *generate* the lattice $w_k^{s_1\ldots s_k}$), respectively, as points in the graph $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ which is obviously isomorphic to $\boldsymbol{G}(V_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$. But it also represents the $\xi^{(k)}$th symbol of the dual word pertaining to $w_k^{0\ldots0}=\hat{\mathcal{T}}^{-\xi^{(k)}}_{F_k}(w_k^{s_1\ldots s_k})$.[22] Finally, one may identify $\sigma_{\xi^{(k)}}$ with the renormalized parameters $\boldsymbol{\epsilon}_{|\hat{\mathcal{X}}_\xi^{-1}},\xi=\xi^{(k)}$. For sake of ease, we will unite all these objects in the symbol $\sigma_{\xi^{(k)}}$.

The importance of the graphs $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ results from their hierarchical structure which manifests itself through the following *recursive construction law*. Let

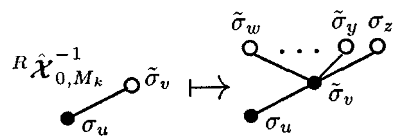
$$\xi^{(k)}=s_kF_{k-1}+\xi^{(k-1)},\quad \xi^{(1)}=s_1F_0, \tag{59}$$

[cf. Eq. (24)] and let the map $\boldsymbol{\Upsilon}_{M_k},M_k\in\mathbb{N}^{+}$ with $s_k=0,\ldots,M_k$, be defined as follows:



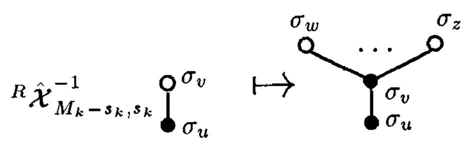
(60)

with $(\sigma,\widetilde{\sigma})\in\{(\alpha,\gamma),(\gamma,\alpha)\}$,



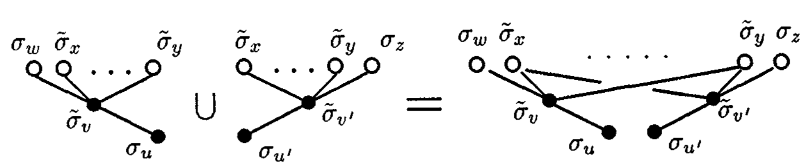
(61)

with $(\sigma,\widetilde{\sigma})\in\{(\alpha,\beta),(\beta,\alpha)\}$ and



(62)

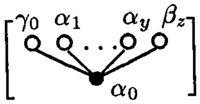
with $\sigma\in\{\alpha,\beta,\gamma\}$. We have used the abbreviations $u=\xi^{(k-1)}$, $v=\xi^{(k)}$, $w=\xi^{(k)}$, $x=\xi^{(k)}+F_k,\ldots,$ $y=\xi^{(k)}+(M_{k+1}-1)F_k$, and $z=\xi^{(k)}+M_{k+1}F_k$. Each defining relation of the presentation of $\boldsymbol{RG}_{\downarrow x,y}$ corresponds to a closed path in the graph $\boldsymbol{G}(V,\boldsymbol{E})$. In the now-to-be-generated graph $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ we consider these throughout the process, in which we still must attach to the map $\boldsymbol{\Upsilon}_{M_k}$ an additional condition



(63)

Here we have used the abbreviations $u=\xi^{(k-1)}$, $u'=\xi^{(k-1)}+s_{k-1}F_{k-2}$, with $s_{k-1}=1,\ldots,M_{k-1}$, and $v=\xi^{(k-1)}$, $v'=u'+M_kF_{k-1}$, as well as $w=\xi^{(k-1)}$, $x=\xi^{(k-1)}+F_k,\ldots,$ $y=\xi^{(k-1)}+(M_{k+1}-1)F_k$, and $z=\xi^{(k-1)}+M_{k+1}F_k$. As a result the graph $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ is generated as follows:

$$\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})=\prod_{k=2}^{n}\Upsilon_{M_k}\left[\quad\right] \tag{64}$$



where $y=M_1-1$ and $z=M_1$ (cf. Fig. 5).

It may be instructive to compare our general results presented here with similar considerations about the "*genealogy of a Fibonacci lattice*" as presented in Ref. 6. Moreover, the so-called *key sites*, corresponding to a power of an arbitrary (but finite) product of $\hat{\boldsymbol{\mathcal{P}}},\hat{\boldsymbol{\mathcal{A}}}$, and $\hat{\boldsymbol{\mathcal{U}}}$, introduced in Ref. 5 are found within our approach as special paths in the graph $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n}),\mathcal{N}_n=(1,1,1,\ldots)$.

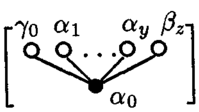
We stated in Sec. IV that $\hat{\boldsymbol{\mathcal{B}}}_1\neq\hat{\boldsymbol{\mathcal{B}}}_2$ always implies ${}^R\hat{\boldsymbol{\mathcal{B}}}_1\neq{}^R\hat{\boldsymbol{\mathcal{B}}}_2$. But as it has been stressed in Sec. III, $\hat{\boldsymbol{\mathcal{B}}}\neq\hat{\boldsymbol{\mathcal{B}}}'\in\boldsymbol{\Phi}_2$ does *not* necessarily imply $\hat{\boldsymbol{\mathcal{B}}}(L)\neq\hat{\boldsymbol{\mathcal{B}}}'(L)\in\boldsymbol{F}_2$, i.e., the two lattices described by $\hat{\boldsymbol{\mathcal{B}}}(L)$ and $\hat{\boldsymbol{\mathcal{B}}}'(L)$ are not necessarily distinct from each other. This is a direct consequence of Eq. (22). Its pendant in the renormalization group is Eq. (42). From Eq. (57) it is obvious that in the calculation of $G_{\mu\mu}$ only $\epsilon_{\alpha|\hat{\mathcal{X}}_\xi^{-1}}$ and $t_{L|\hat{\mathcal{X}}_\xi^{-1}}$ enter. Thus ${}^R\hat{\boldsymbol{\mathcal{X}}}_\xi^{-1}$ *and* ${}^R\hat{\boldsymbol{\mathcal{X}}}_{\xi+F_n}^{-1}$ yield actually the same result for $G_{\mu\mu},\mu=\xi \bmod F_n$.

It suggests itself to build this fact into the construction law of appropriate chosen subgraphs of $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$, which we want to generate as follows. Let $\tilde{\Upsilon}_{M_k}$ be the map which one obtains from Eqs. (60)–(62) omitting the edge connecting $\tilde{\sigma}_v$ with σ_z in Eq. (61), as well as the edge connecting σ_v with σ_z in Eq. (62). The correspondence (58) is extended by

$$\begin{aligned}\boldsymbol{\epsilon}_{|\hat{\mathcal{X}}_\xi^{-1}}&\leftrightarrow\boldsymbol{\varepsilon}_{\xi^{(k)}}\\&\equiv(x_{\alpha|\hat{\mathcal{X}}_\xi^{-1}},y_{L|\hat{\mathcal{X}}_\xi^{-1}})\\&\leftrightarrow\left(x_{\alpha|\hat{\mathcal{X}}_\xi^{-1}},G_{\xi\xi}=\frac{|y_{L|\hat{\mathcal{X}}_\xi^{-1}}|}{\sqrt{(x_{\alpha|\hat{\mathcal{X}}_\xi^{-1}})^2-4}}\right),\end{aligned} \tag{65}$$

with $0\leqslant\xi\equiv\xi^{(k)}<F_k$. Then,

$$\boldsymbol{T}(\overline{W}_{\mathcal{N}_n},\overline{\boldsymbol{E}}_{\mathcal{N}_n})=\prod_{k=2}^{n}\tilde{\Upsilon}_{M_k}\left[\quad\right] \tag{66}$$



is a *connected circuit-free* [i.e., there are no closed paths in the tree $\boldsymbol{T}(\overline{W}_{\mathcal{N}_n},\overline{\boldsymbol{E}}_{\mathcal{N}_n})$] subgraph ("*tree*," set of bold edges in Fig. 5) from $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$ with the *center* (root) α_0 where $\overline{W}_{\mathcal{N}_n}=\cup_{p=1}^{n}\{\sigma_{\xi^{(p)}}|0\leqslant\xi^{(p)}<F_p\}\subset W_{\mathcal{N}_n}$. $\overline{\boldsymbol{E}}_{\mathcal{N}_n}\subset\boldsymbol{E}_{\mathcal{N}_n}$ is defined analogously to $\boldsymbol{E}_{\mathcal{N}_n}$.

In $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$, and therefore in the tree $\boldsymbol{T}(\overline{W}_{\mathcal{N}_n},\overline{\boldsymbol{E}}_{\mathcal{N}_n})$, each path

$$\Pi_{\alpha_0^{(0)}\to\sigma_{\xi^{(k)}}}\leftrightarrow{}^R\hat{\boldsymbol{\mathcal{X}}}_{M_k-s_k,s_k}^{-1}\circ\cdots\circ{}^R\hat{\boldsymbol{\mathcal{X}}}_{M_1-s_1,s_1}^{-1} \tag{67}$$

corresponds respectively to a sequence of RT which uniquely determines the parameters of the Dyson equation (8) (*path renormalization*). By this means, the diagonal elements of the Green function are determined. It may be instructive to compare from this point of view the example considered at the end of Sec. V and Fig. 5. Note that the procedure introduced here offers itself as a very efficient basis for the numerical calculation of $G_{\mu\mu}$.

In the remainder of this section we will confine ourselves to RT corresponding to Eq. (16). To lighten the notation we abbreviate $y_{|\hat{\mathcal{X}}_{(r_n\cdots r_1)}^{-1}}$ by $y_{|s_n\cdots s_1}$ and $y_{|\hat{\mathcal{X}}_\xi^{-1}}$ by $y_{|\xi^{(k)}}$, etc. We will determine the tuple $\boldsymbol{\varepsilon}_{|\xi^{(k)}}$ (65) recursively on the path $\Pi_{\alpha_0^{(0)}\to\sigma_{\xi^{(k)}}}$ in the graph $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$.

For given $\mathcal{N}_n$ and $s_k=0,\ldots,M_k,k=1,\ldots,n$, one gets from Eq. (45)

$$x_{\alpha|s_k\cdots s_1}=x_{\alpha|M_k\cdots M_1}\equiv x_\alpha^{(k)}. \tag{68}$$

${x'}_\alpha^{(k)}$ possesses the same property. Thus the iteration of $x_\alpha^{(k)}$ is independent of the choice of the path in $\boldsymbol{G}(W_{\mathcal{N}_n},\boldsymbol{E}_{\mathcal{N}_n})$.

From Eqs. (45) and (47) one obtains, with $x=x_\alpha^{(k-1)}/2$ and $2\mathcal{T}_k=\mathcal{U}_{k+1}-\mathcal{U}_{k-1}$,[17] for given $\mathcal{N}_n$,

$$x_\alpha^{(k)}={x'}_\alpha^{(k-1)}\mathcal{U}_{M_k}(x)-x_\alpha^{(k-2)}\mathcal{U}_{M_k-1}(x), \tag{69}$$

$$\begin{aligned}y_{L|\xi^{(k)}}=&\,y_{L|\xi^{(k-1)}}R_{s_{k-1}}+y_{L|\xi^{(k-2)}}\mathcal{T}_{M_k-2s_k}(x)\\&+\Delta_{|\xi^{(k-1)}}\mathcal{U}_{M_k-2s_k}(x),\end{aligned} \tag{70}$$

with the initial conditions $x_\alpha^{(-1)}=x_\delta,x_\alpha^{(0)}=x_\alpha$, as well as y_S,y_L. $\Delta_{|\xi^{(k)}}$ and ${x'}_\alpha^{(k)}$ are recursively defined via

$$\begin{aligned}\Delta_{|\xi^{(k)}}\equiv\pm\sqrt{\chi^2_{|\xi^{(k)}}-4\,\eta_{|\xi^{(k)}}}=&\left[\frac{1}{2}y_{L|\xi^{(k-1)}}Q_{k-1}\right.\\&\left.-(x^2-1)y_{L|\xi^{(k-2)}}\right]\mathcal{U}_{M_k-2s_k}(x)-\Delta_{|\xi^{(k-1)}}\mathcal{T}_{M_k-2s_k}(x)\end{aligned} \tag{71}$$

and

$${x'}_\alpha^{(k)}={x'}_\alpha^{(k-1)}\mathcal{U}_{M_k+1}(x)-x_\alpha^{(k-2)}\mathcal{U}_{M_k}(x), \tag{72}$$

with ${x'}_\alpha^{(0)}=x'_\alpha$. We have used

$$R_{s_k}=Q_{k-1}\mathcal{U}_{M_k-s_k}(x)\mathcal{U}_{s_k}(x)+\frac{x_\alpha^{(k-2)}}{2}\mathcal{U}_{M_k}(x) \tag{73}$$

and

$$Q_k={x'}_\alpha^{(k)}-\frac{x_\alpha^{(k)}x_\alpha^{(k-1)}}{2}. \tag{74}$$

Taking into account the invariant mentioned in Sec. IV, it can be shown that Eqs. (69) and (70) are actually two-term

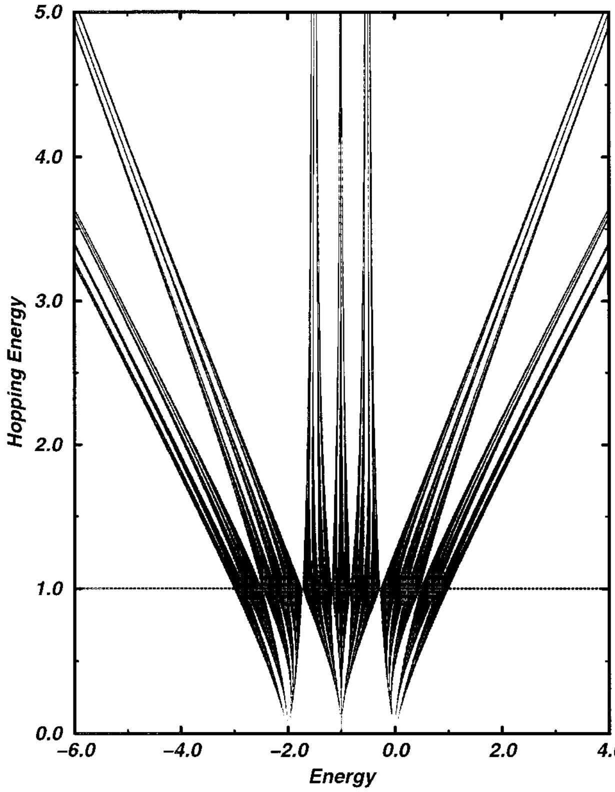


FIG. 6. The region $|x_\alpha^{(8)}| \leq 2$ (*spectrum*) for the standard Fibonacci lattice w_8 as function of the hopping energy t_L ($t_S = 1$) for the parameter tuple: $\epsilon_\mu = -1$. (Periodic case is obtained for $t_L = 1$.)

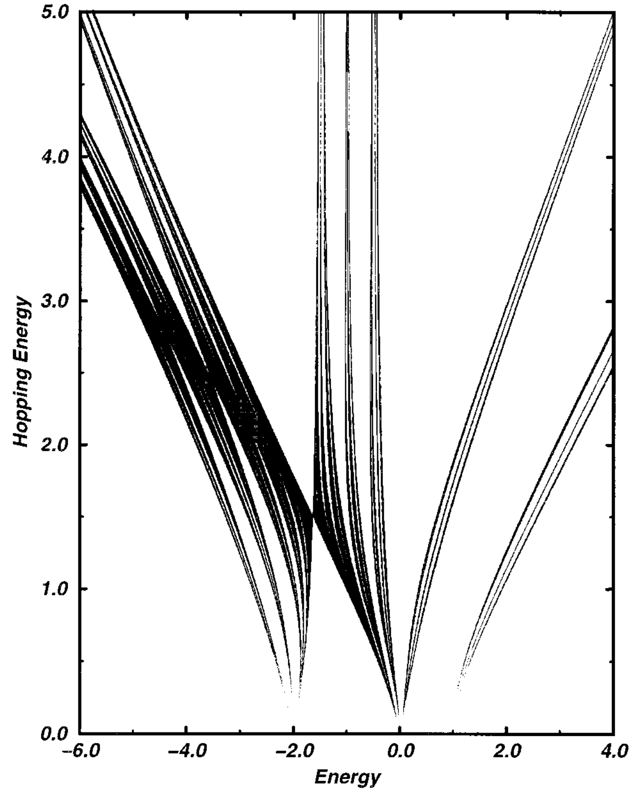


FIG. 7. The region $|x_\alpha^{(8)}| \leq 2$ (*spectrum*) for the standard Fibonacci lattice w_8 as function of the hopping energy t_L ($t_S = 1$) for the parameter tuple: $\epsilon_\alpha = -\epsilon_\beta = -\epsilon_\gamma = 1$.

recursion relations $\Upsilon_{M_k}: (\boldsymbol{\epsilon}_{|\xi^{(k-2)}}, \boldsymbol{\epsilon}_{|\xi^{(k-1)}}) \rightarrow (\boldsymbol{\epsilon}_{|\xi^{(k-1)}}, \boldsymbol{\epsilon}_{|\xi^{(k)}})$ as we have stated in Eqs. (60)–(62).

VII. NUMERICAL RESULTS AND DISCUSSION

It may be instructive to consider first the *periodic case*, i.e., $t_L = t_S$ and $\epsilon_\alpha = \epsilon_\beta = \epsilon_\gamma$. In the following (cf. Ref. 17), we set $\xi = \xi^{(n)}$. It can be shown that $x_\alpha^{(n)}(E) = 2\mathcal{T}_{F_n}(x_\alpha/2) = 2\cos(F_n k)$, and $x_\alpha'^{(n)}(E) = 2\mathcal{T}_{F_n + F_{n-1}}(x_\alpha/2)$, respectively, with $\cos(k) = (E - \epsilon_\alpha)/2t_L$. Moreover, one obtains $y_{L|\xi}(E) = y_L \mathcal{U}_{F_n}(x_\alpha/2)$ and $y_{\beta|\xi}(E) = y_L \mathcal{U}_{F_n + F_{n-1}}(x_\alpha/2) = y_{\gamma|\xi}(E)$. It is obvious that Eq. (57) yields the well known result for the periodic tight-binding model (cf., e.g., Ref. 21).

In contrast to $y_{L|\xi}$ the iteration of $x_\alpha^{(n)}(E)$ is *in*dependent of the concrete choice of the path in $\boldsymbol{G}(W_{\mathcal{N}_n}, \boldsymbol{E}_{\mathcal{N}_n})$ [cf. Eq. (68)]. Hence, for a given $\mathcal{N}_n$ the factor $1/\sqrt{4 - (x_\alpha^{(n)})^2}$ in Eq. (57) is the same for all $G_{\xi\xi}$, *in*dependent from the index ξ. It determines the spectrum of the different GFC (55) (cf. Fig. 6 and Fig. 7) which is a Cantor set.[23]

The *local* properties, as described by the LDOS, are caused through the modulation of this root factor by the orthogonal polynomials $y_{L|\xi}(E) = \Pi_{\Sigma, F_n - 1}^{(\xi+1)}$. The zeros $E_\mu^{(\xi)}, \mu = 1, 2, \ldots, F_n - 1$, of $y_{L|\xi}$ (which corresponds to the *poles* of the renormalized hopping element) are all *real*, *single*, and lay in the gaps of the spectrum or coincides with band edges. These properties are an immediate consequence of the orthogonality of $y_{L|\xi}(E)$. The zeros $E_\mu^\pm, \mu$

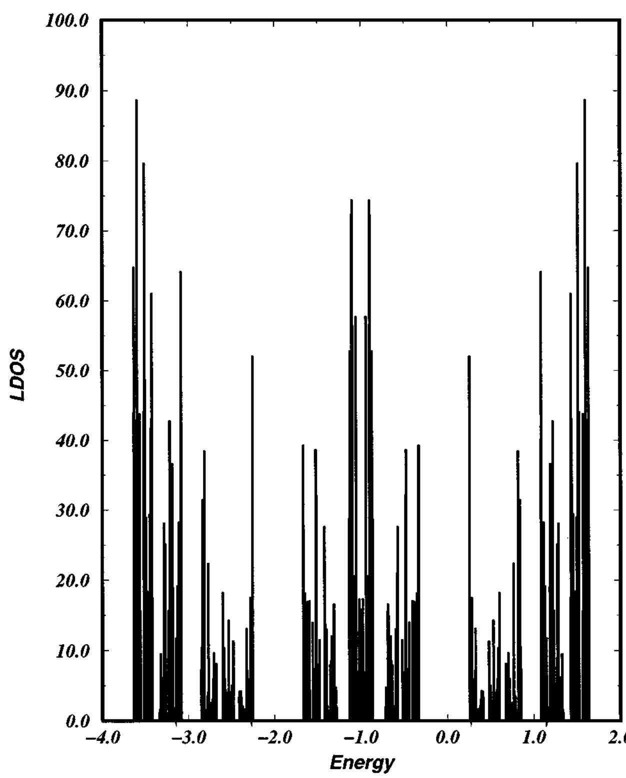


FIG. 8. LDOS $\rho_\xi(E)$, at site $\xi = 111$, for the standard Fibonacci lattice w_{15} with parameters $t_L = 1.5$, $t_S = 1$, and $\epsilon_\mu = -1$.

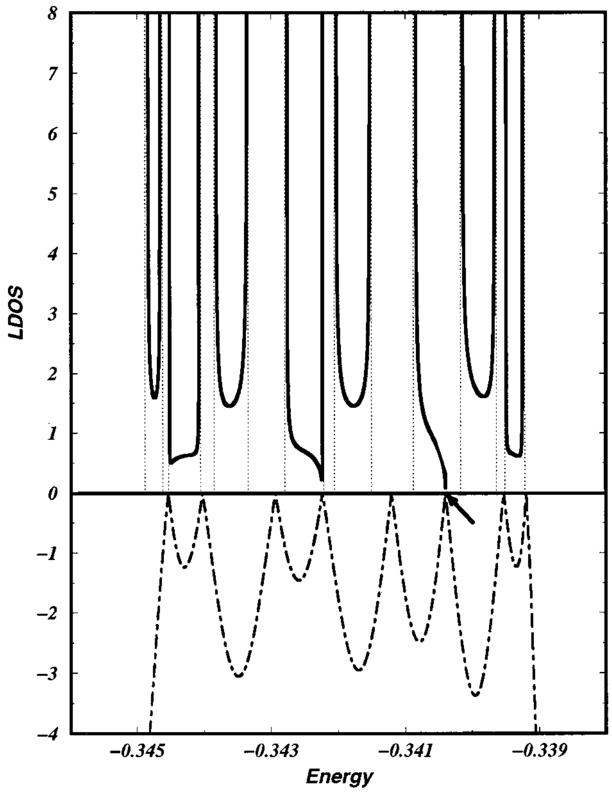


FIG. 9. Enlarged part of the LDOS presented in Fig. 8 (bold line). Dot-dashed line: The corresponding renormalized inverse hopping energy $-|y_{L|\xi}(E)|$.

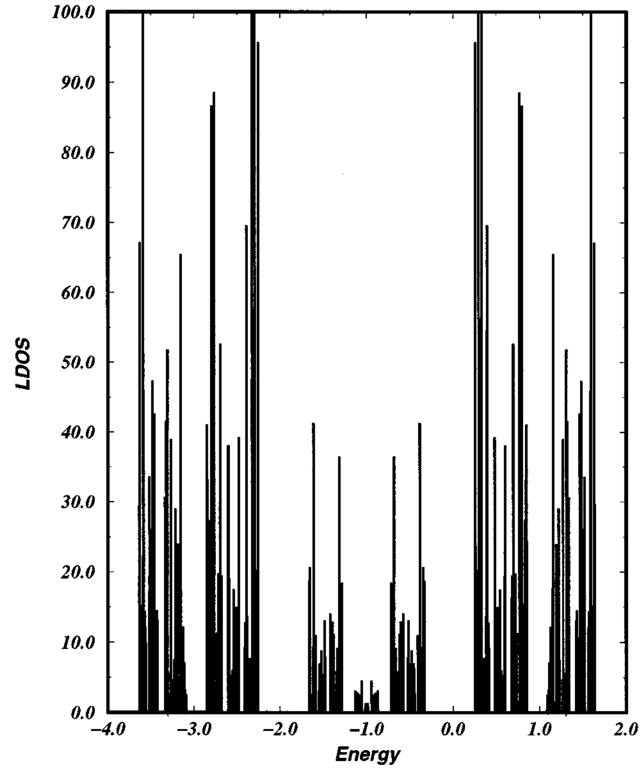


FIG. 10. LDOS $\rho_\xi(E)$, at site $\xi=888$, for the standard Fibonacci lattice w_{15} with parameters $t_L=1.5$, $t_S=1$, and $\epsilon_\mu=-1$.

$=1,2,\ldots,F_n$, of $x_\alpha^{(n)}\pm 2$ are also real but may be single or double [cf. Ref. 20(b), pp. 46–48]. A zero $E_\mu^\pm$ can only be double (i.e., band degeneracy) if $E_\mu^+=E_{\mu-1}^{(\xi)}$ or $E_\mu^-=E_{\mu-1}^{(\xi)}$. In general, the reversal is obviously not true. Let $E_\mu^\pm$ be a single zero coinciding with a zero of $y_{L|\xi}(E)$. In this case, the LDOS vanishes at the band edge (as indicated by an arrow in Figs. 8 and 10). If $E_\mu^\pm$ is double, then the LDOS possesses a finite value (i.e., it is not singular). In Figs. 8–11 we present the LDOS for two different sites ξ. Note that the support of $\rho_{111}(E)$ coincides with the support of $\rho_{888}(E)$.

We remark that it is apparent from Eqs. (52) and (57) that $|x_\alpha^{(n)}|\to\infty$ rather than the limit $t_{L|\xi}\to 0$ implies $G_{\xi\xi}\to 1/|E-\epsilon_{\alpha|\xi}|$. Thus the investigation of the convergence of the renormalization procedure seems to be much more complicated than in the standard belief. A serious investigation of the asymptotic properties of the renormalized hopping element $t_{L|\xi}(E)$ and of the LDOS for any infinite succesion of RT, however, is beyond the scope of this article and is left for further studies.

We wish to point out a further relation between $x_\alpha^{(k)}$ and $y_{L|\xi^{(k)}}$. Using Eq. (66), one gets

$$\sum_{\xi^{(k)}=0}^{F_k-1} y_{L|\xi^{(k)}}=\sum_{s_k=0}^{M_k-1}\sum_{\xi^{(k-1)}=0}^{F_{k-1}-1} y_{L|s_k,\xi^{(k-1)}} + \sum_{\xi^{(k-2)}=0}^{F_{k-2}-1} y_{L|M_k,s_{k-1}^{(0)},\xi^{(k-2)}} \tag{75}$$

$(s_k^{(0)}=0,\forall k)$. Thus, by induction,

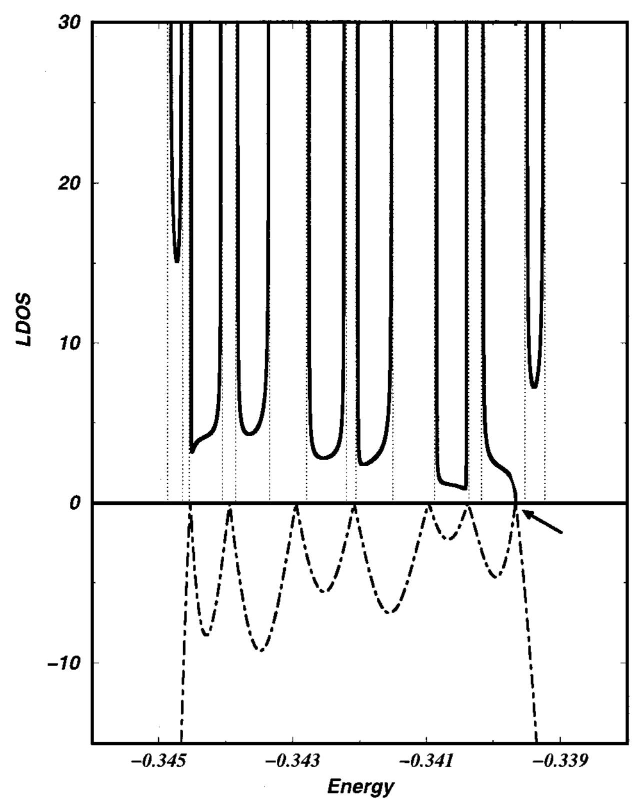


FIG. 11. Enlarged part of the LDOS presented in Fig. 10 (bold line). Dot-dashed line: The corresponding renormalized inverse hopping energy $-|y_{L|\xi}(E)|$.

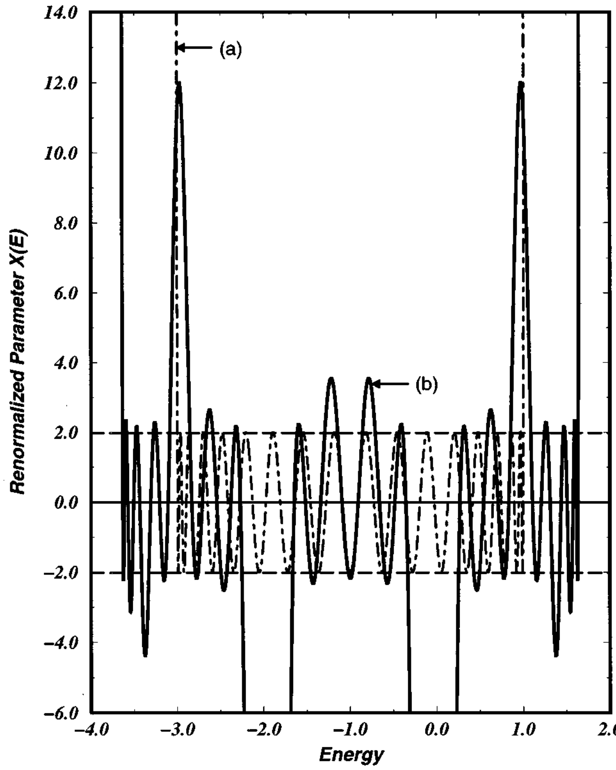


FIG. 12. The polynomial $X(E) \equiv x_\alpha^{(7)}$ in the cases (a) $t_L = t_S = 1$, and (b) $t_L = t_S, t_L = 1.5$ ($\epsilon_\mu = 0$).

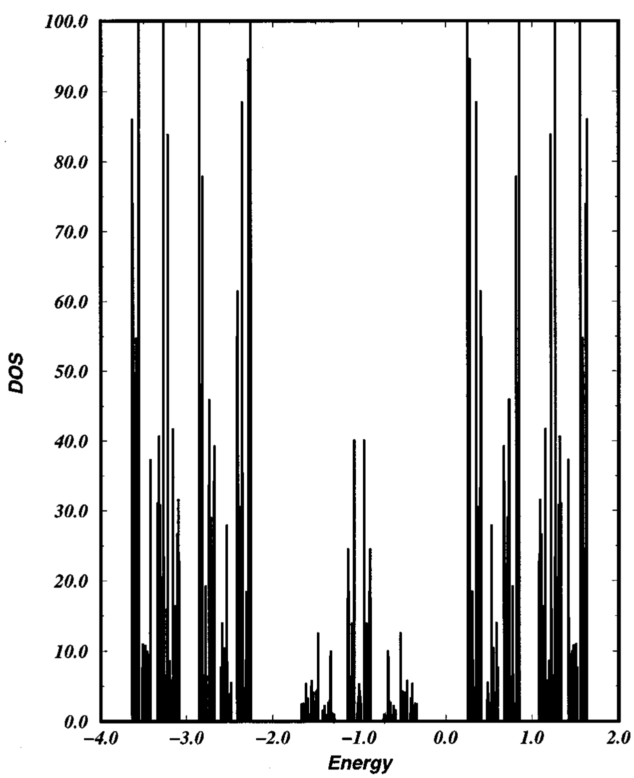


FIG. 13. Total density of states (DOS) for the standard Fibonacci lattice w_{15} ($t_L = 1.5, t_S = 1, \epsilon_\mu = -1$).

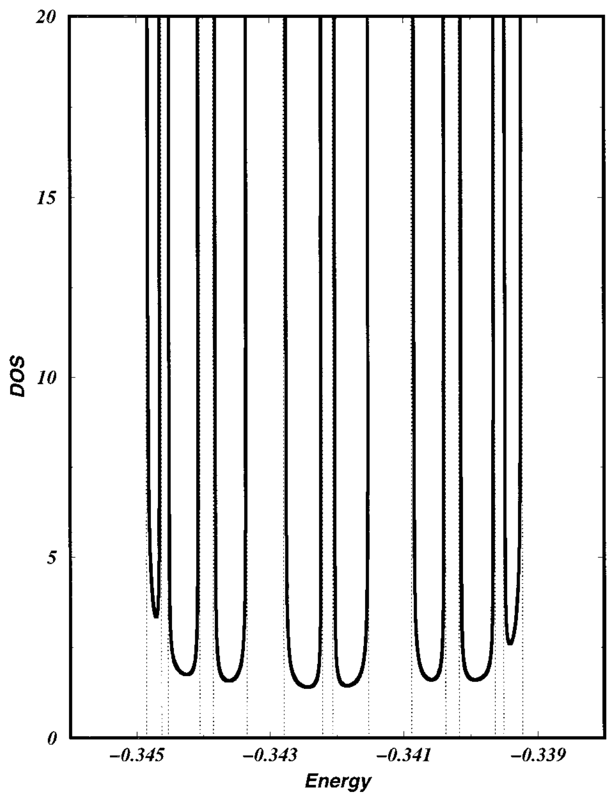


FIG. 14. Enlarged part of the DOS displayed in Fig. 13.

$$\frac{\partial x_\alpha^{(k)}}{\partial z} = \sum_{\xi=0}^{F_k - 1} y_{L|\xi^{(n)}}. \tag{76}$$

Since the polynomials $y_{L|\xi^{(k)}}$ have the same sign within the energy spectrum of $\mathcal{H}$ for all $\xi^{(k)}$, $0 \leqslant \xi^{(k)} < F_k$, the density of states (DOS) is given by

$$\mathcal{N}(E) = -\frac{1}{\pi F_n} \sum_\mu \operatorname{Im} G_{\mu,\mu}(E + i0^+)$$

$$= \frac{1}{\pi F_n \sqrt{4 - (x_\alpha^{(n)})^2}} \left| \frac{\partial x_\alpha^{(n)}}{\partial E} \right|, \tag{77}$$

where

$$\cos[F_n K(E)] = \frac{1}{2} x_\alpha^{(n)} \tag{78}$$

(cf. Fig. 12).

$K(E)$ represents the integrated DOS $[\int^\infty \mathcal{N}(E')dE' = 1]$

$$K(E) = \pi \int_{-\infty}^{E} \mathcal{N}(E')dE'. \tag{79}$$

Numerical results are displayed in Figs. 13 and 14.

VIII. SUMMARY

Starting with the presentation of $\mathbf{\Phi}_2 = \boldsymbol{Aut}(\boldsymbol{F}_2)$ introduced by Nielsen,[7] we found a complete system of defining relations for the semigroup $\mathbf{\Phi}_2^+ \subset \mathbf{\Phi}_2$ whose elements generate

$E(K)$, DOS, IDOS $\qquad$ LDOS

$$\boldsymbol{\Pi}_x \quad \subset \quad \boldsymbol{\Pi}_{x,y}$$

$$| \qquad\qquad |$$

$$\boldsymbol{F}_2^A = \mathbb{Z} \oplus \mathbb{Z} \quad \leftarrow \quad \boldsymbol{F}_2 = \langle A, B \rangle$$

$$\boldsymbol{PGL}_2(\mathbb{Z}) \cong \boldsymbol{RG}_{\downarrow x} \;\leftarrow\; \boldsymbol{Aut}(\boldsymbol{F}_2) \cong \boldsymbol{RG}_{\downarrow x,y}$$

FIG. 15. Summary of the algebraic properties of the RT ${}^R\hat{\boldsymbol{\mathcal{B}}} \in \boldsymbol{RG}_{\downarrow x,y}$ and their connection to the substitutions introduced in Sec. III. Note that the first line presents the relevant physical quantities, which are determined through the corresponding RT.

all cyclic permutations of the generalized Fibonacci chains (GFC).

The renormalization group $\boldsymbol{RG}_{\downarrow x,y} < \boldsymbol{RG}(\boldsymbol{F}_2)$ was introduced, which is generated by three elementary renormalization transformations (RT) ${}^R\hat{\boldsymbol{\mathcal{P}}}, {}^R\hat{\boldsymbol{\mathcal{O}}}$, and ${}^R\hat{\boldsymbol{\mathcal{U}}}$. We found that $\boldsymbol{RG}_{\downarrow x,y} \cong \boldsymbol{Aut}(\boldsymbol{F}_2)$. We presented an algorithm to find a possible succession of the RT ${}^R\hat{\boldsymbol{\mathcal{P}}}, {}^R\hat{\boldsymbol{\mathcal{O}}}$, and ${}^R\hat{\boldsymbol{\mathcal{U}}}$, to calculate the *diagonal* elements $G_{\mu\mu}$ of the Green function for all indices μ. Several characteristics of the RT were examined. Figure 15 summarizes the relationship between the geometric structure of the GFC, the renormalization group, and the corresponding physical quantities.

Based upon our presentation of $\boldsymbol{\Phi}_2^+$, we constructed a recursive scheme for all subgraphs of the graph $\boldsymbol{G}(V,\boldsymbol{E})$ of the renormalization group $\boldsymbol{RG}_{\downarrow x,y}$ which are relevant for the calculation of the diagonal elements $G_{\mu\mu}$. Each path in $\boldsymbol{G}(V,\boldsymbol{E})$ defines a succession of individual renormalization steps, converting the Dyson equation for any diagonal element $G_{\mu\mu}$ to the Dyson equation of a periodic tight-binding model with one renormalized site energy and one renormalized hopping element each dependent upon the energy (*path renormalization*).

Finally, we presented some numerical results.

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[10] Suppose the word $w(L,S) = \Pi_{i=1}^k L^{p_i} S^{q_i} \in \boldsymbol{F}_2, p_i, q_i \in \mathbb{Z}$ to be freely reduced. The *length* of $w(L,S)$ is defined through $|w(L,S)| = \Sigma_{i=1}^k (p_i + q_i) \in \mathbb{Z}$. The *norm* of $w(L,S)$ is defined through $\|w(L,S)\| = \Sigma_{i=1}^k (|p_i| + |q_i|)$.

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[12] $\Pi_{k=1}^n X_k \equiv X_n X_{n-1} \cdots X_1$. For definiteness recall the following notation. Let $f: x \mapsto y$ and $g: y \mapsto z$. Then $h: x \mapsto z$ with $h(x) = g \circ f(x) = g[f(x)]$.

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[16] Equivalently, let the symbol σ_ξ of the dual word Σ_k correspond to that equation in the system of equations (8) containing the inhomogeneous part. We require that the substitutions $\hat{\boldsymbol{\mathcal{B}}} \in \boldsymbol{\Phi}_2$, and the inverse reduction process, respectively, considered here do *not* create the *types* $\sigma \in \{\alpha_\mp^\pm, \delta_\mp^\pm\}$ pertaining to the symbol σ_ξ.

[17] The Chebyshev polynomials of the first kind of degree n are defined as $\mathcal{T}_n(x) = \cos[n \arccos(x)]$. They obey the recursion formula $\mathcal{T}_{n+1}(x) = 2x\mathcal{T}_n(x) - \mathcal{T}_{n-1}(x), \mathcal{T}_0(x) = 1, \mathcal{T}_1(x) = x$. The Chebyshev polynomials of the second kind are defined by $\mathcal{U}_n(x) = \sin[n \arccos(x)]/\sin[\arccos(x)]$. They obey the recursion relations $\mathcal{U}_{n+1}(x) = 2x\mathcal{U}_n(x) - \mathcal{U}_{n-1}(x)$ with $\mathcal{U}_0(x) = 0$, $\mathcal{U}_1(x) = 1$.

[18] Special cases of this invariant were found in the context of real-space rescaling theory (Ref. 5): J.A. Ashraff *et al.* (case $M_k = 1, \forall k$) and A. Chakrabarti *et al.* (case $M_k = 2, \forall k$).

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