Extended massless phase and the Haldane phase in a spin-1 isotropic antiferromagnetic chain

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We study the phase diagram of isotropic spin-1 models in the vicinity of the Uimin-Lai-Sutherland (ULS) model. This is done with the help of a level-one SU(3) Wess-Zumino-Witten model with certain marginal perturbations. We find that the renormalization-group flow has infrared stable and unstable trajectories divided by a critical line on which the ULS model is located. The infrared unstable trajectory produced by a marginally relevant perturbation generates an exponential mass gap for the Haldane phase, and thus the universality class of the transition from the massless phase to the Haldane phase at the ULS point is identified with the Berezinskiĭ-Kosterlitz-Thouless type. Our results support recent numerical studies by Fáth and Sólyom. In the massless phase, we calculate logarithmic finite-size corrections of the energy for the SU(ν)-symmetric and asymmetric models in the massless phase. [S0163-1829(97)02213-3]

I. INTRODUCTION

The phase diagram of isotropic spin-1 chains has not yet been understood sufficiently. The characteristics of ground states can change drastically depending on a coupling constant of the model.[1] Even though there are many rigorous[2,3] and exact[4–7] works at several isolated points, one encounters unconformable issues in a certain region, especially in the nonintegrable region around an integrable point of the Uimin-Lai-Sutherland (ULS) model.

The general form of the spin-1 Hamiltonian which consists of nearest-neighbor interactions with rotational symmetry is

$$H(\theta)=\sum_{j=1}^{L}\left[\cos\theta(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})+\sin\theta(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})^2\right], \quad (1.1)$$

where the coupling constant is controlled by one parameter $\theta\in[0,2\pi)$. It is our main concern to understand the macroscopic behavior in the vicinity of the ULS point $\theta=\pi/4$.[4,5] It is known that the ULS model has massless excitations described by the Wess-Zumino-Witten (WZW) model. The region $|\theta|<\pi/4$, which contains the standard Heiseberg antiferrromagnet ($\theta=0$), is believed to be in the Haldane phase which has only massive excitations, as suggested by some numerical works[8,9] and rigorous studies at $\theta=\tan^{-1}(1/3)$.[2,3] On the other hand, the nature of the model in the region $\pi/2>\theta>\pi/4$ is theoretically less understood.

In this paper employing a renormalization-group (RG) method in a continuum field theory, we show that the region $\theta>\pi/4$ near $\theta=\pi/4$ is a massless phase, and that the phase transition from this massless phase to the Haldane phase at ULS point belongs to the Berezinskiĭ-Kosterlitz-Thouless-(BKT) type universality class. This result is consistent with a numerical study obtained by Fáth and Sólyom.[10] For this purpose, we map the ULS model to the $SU(3)_1$ WZW model, which reproduces some exact results obtained from the Bethe ansatz.[4,5,11–16] In the nonintegrable region around the ULS point, we show that the $SU(3)_1$ WZW model is perturbed by adding a SU(3)-breaking marginal operator which causes the BKT transition. We observe several nontrivial behaviors as in some other conformal field theory (CFT) deformed by marginal operators.[16–19] Despite a number of studies on the BKT transition and the logarithmic corrections in SU(2) systems, those concerned with SU(ν) symmetry for $\nu>2$ have been seldom discussed. Here, we study the BKT transition and the logarithmic correction in the $\nu>2$ case and we find its different nature from $\nu=2$ case. The obtained continuum theory enables us to calculate the logarithmic finite-size correction in the energy of the ground state and the first excited states in the region of the massless phase. Following Ludwig and Cardy,[20,21] the finite-size correction to the ground-state energy of the model in a strip space with the width L is

$$\mathcal{E}_{\rm GS}=\varepsilon_\infty L-\frac{\pi v}{6L}c(L),$$

$$c(L)=c_{\rm vir}+\frac{d_{\rm GS}}{(\ln L)^3}+O\left(\frac{\ln(\ln L)}{(\ln L)^4},\frac{1}{(\ln L)^4}\right), \quad (1.2)$$

where ε_∞ is the nonuniversal bulk contribution to the ground-state energy depending on cutoff scale. The minimal energy of an excited state related to a certain primary field with conformal weight $x_n/2$ is given by

$$\mathcal{E}_n=\mathcal{E}_{\rm GS}+\frac{2\pi v}{L}\gamma_n(L),$$

$$\gamma_n(L)=x_n+\frac{d_n}{\ln L}+O\left(\frac{\ln(\ln L)}{(\ln L)^2},\frac{1}{(\ln L)^2}\right), \quad (1.3)$$

where d_n is a coefficient of a certain three-point function. We calculate these universal coefficients of the logarithmic corrections by the obtained continuum field theory.

The outline of this paper is as follows. In Sec. II, a strong-coupling Abelian gauge theory is introduced as a critical field theory of the ULS model, which allows us to evaluate exact values of universal quantities. We show the equivalence of this critical theory to the level-one SU(3) WZW model. This argument can be generalized to a certain SU(ν) symmetric spin model which includes the ULS model

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in the $\nu=3$ case. In Sec. III, we discuss an extended nonintegrable spin model with SU(ν)-asymmetric interaction on the basis of the level-one SU(ν) WZW model with an asymmetric perturbation. We pin down the marginal operator $\Sigma_{A=1}^{\nu^2-1}\mathcal{J}_{L\alpha\beta}^{A}(z)\mathcal{J}_{R\alpha\beta}^{A}(\overline{z})$ in the SU$(\nu)_1$ WZW model as the SU(ν)-asymmetric interaction in the original spin model. The logarithmic corrections of its energy in the massless phase are evaluated and the difference between the SU(3)-symmetric and asymmetric model is indicated. Finally, we discuss the universality class of the transition from the massless phase to the Haldane phase which belongs to the BKT type.

II. CRITICAL THEORY OF THE SU(ν) SPIN CHAIN

To begin with, we extend a QED_2 description for the SU(2) spin model[22] to the SU(ν) one. The spin chain is mapped to the WZW model with some perturbations by this method.

We redefine the Hamiltonian (1.1) near $\theta=\pi/4$ as

$$H(\gamma)\equiv\frac{1}{\cos\theta}H(\theta)=\sum_{j=1}^{L}\left[(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})+\gamma(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})^2\right] \tag{2.1}$$

with $\gamma=\tan\theta$. We use fermion operators $c_{j\alpha},c_{j\alpha}^{\dagger}$ for the spin variables

$$\boldsymbol{S}_j=\sum_{\alpha,\beta=1}^{3}c_{j,\alpha}^{\dagger}(\boldsymbol{L})_{\alpha\beta}c_{j,\beta}, \tag{2.2}$$

where L^x, L^y, and L^z are spin-1 matrices. In this case, Eq. (2.1) can be expressed in the fermions[23]

$$H(\gamma)=\sum_{j=1}^{L}\left[c_{j,\alpha}^{\dagger}c_{j,\beta}c_{j+1,\beta}^{\dagger}c_{j+1,\alpha}+(\gamma-1)c_{j,\alpha}^{\dagger}c_{j,\beta}c_{j+1,\alpha}^{\dagger}c_{j+1,\beta}\right], \tag{2.3}$$

in which a trivial constant is neglected. Here the local constraint, $\Sigma_{\alpha=1}^{3}c_{j,\alpha}^{\dagger}c_{j,\alpha}=1$, is imposed in order to restrict the dimension of physical space to three at each lattice site. Due to this constraint, empty, double, and triple occupancy states of the fermions are forbidden at each lattice site. The first term in Eq. (2.3) is an exchange operator between nearest-neighbor sites and the second one is a projector onto a singlet bond. Equation (2.3) has the local U(1) symmetry, a translational symmetry by one lattice site for all values of γ. In addition, a global SU(3) symmetry appears at the ULS point ($\gamma=1$) where the Hamiltonian (2.3) consists of only exchange operators, and the model becomes Bethe-ansatz solvable. This fermion expression can be extended to Bethe-ansatz solvable model with higher spin, when spin-S, $2S+1$ kinds of fermion on each lattice site are introduced, and the constraint on each site is given by $\Sigma_{\alpha=1}^{2S+1}c_{j,\alpha}^{\dagger}c_{j,\alpha}=1$. In general, bond interactions of an isotropic spin-S chain are represented by a polynomial of $X=\boldsymbol{S}_i\cdot\boldsymbol{S}_j$. These integral families for higher spin chains are classified by Batchelor, Yung, and Kennedy.[24] One of those is the ULS model with an arbitrary spin. We give the expressions of the exchange operator in terms of spin matrix for $S\leqslant 2$ in Table I. Hereafter we discuss the fermionized Hamiltonian (2.3) as ν species ($\nu=2S+1$).

TABLE I. Expressions of the exchange operator for $S\leqslant 2$.

S	$\mathcal{P}$
$\frac{1}{2}$	$2X+\frac{1}{2}$
1	X^2+X-1
$\frac{3}{2}$	$\frac{2}{9}X^3+\frac{11}{18}X^2-\frac{9}{8}X-\frac{67}{32}$
2	$\frac{1}{36}X^4+\frac{1}{6}X^3-\frac{7}{12}X^2-\frac{5}{2}X-1$

The Euclidean action is written by introducing Lagrangian multiplier χ for the local constraint and the Hubbard-Stratonovich transformation

$$\mathcal{A}_{\gamma}=\int_0^{\beta}d\tau\sum_{j=1}^{L}\left[c_{j,\alpha}^{\dagger}\partial_{\tau}c_{j,\alpha}+i\chi_j(c_{j,\alpha}^{\dagger}c_{j,\alpha}-1)+\mathcal{H}_{j,j+1}(\gamma)\right], \tag{2.4}$$

where β is an inverse temperature. The Hamiltonian for one bond interaction is

$$\begin{aligned}\mathcal{H}_{j,j+1}(\gamma)=&\mathcal{Q}_{j,j+1}^{*}\mathcal{Q}_{j,j+1}-c_{j,\alpha}^{\dagger}\mathcal{Q}_{j,j+1}c_{j+1,\alpha}\\&-c_{j+1,\alpha}^{\dagger}\mathcal{Q}_{j,j+1}^{*}c_{j,\alpha}\\&+(\gamma-1)c_{j,\alpha}^{\dagger}c_{j,\beta}c_{j+1,\alpha}^{\dagger}c_{j+1,\beta}.\end{aligned}$$

The complex auxiliary fields $\{\mathcal{Q}_{j,j'}\},\{\mathcal{Q}_{j,j'}^{\dagger}\}$ are introduced to decompose the two-body fermion interaction into a single body. A local U(1) gauge transformation

$$c_{j,\alpha}\rightarrow e^{\mathrm{i}\varphi_j}c_{j,\alpha},\quad \chi_j\rightarrow\chi_j-\partial_{\tau}\varphi_j,$$

$$\mathcal{Q}_{j,j'}\rightarrow e^{\mathrm{i}\varphi_j}\mathcal{Q}_{j,j'}e^{-\mathrm{i}\varphi_{j+1}}, \tag{2.5}$$

preserves Eq. (2.4).

First, we study SU(ν)-symmetric point $\gamma=1$ by the mean-field theory without taking into account the local constraint $\Sigma_{\alpha=1}^{\nu}c_{j,\alpha}^{\dagger}c_{j,\alpha}=1$. We shall treat the local constraint later. In the mean-field theory, the auxiliary field $Q_{j,j'}$ is a constant R_0, and then dispersion relation becomes $\varepsilon(k)=-R_0\cos ka$, where a is a lattice spacing. The ground state is given by the Fermi sea filled up to Fermi level $\pm k_F$ with $k_F=\pi/\nu a$. The low-energy physics can be described in terms of ψ_L and ψ_R which is the lattice fermion operator $c_{j,\alpha}$ only around the Fermi surface with a certain low-energy cutoff $\Lambda(\ll k_F)$ $\pm k_F$ as

$$\frac{1}{\sqrt{a}}c_{j,\alpha}\simeq\psi_{L\alpha}(x)\exp(-ik_Fx)+\psi_{R\alpha}(x)\exp(ik_Fx),\quad x\equiv ja. \tag{2.6}$$

In this representation, the local gauge transformation Eq. (2.5) corresponds to the U(1) vector transformation

$$\psi_{L\alpha}(x)\rightarrow e^{\varphi(x)}\psi_{L\alpha}(x),$$

$$\psi_{R\alpha}(x)\rightarrow e^{\varphi(x)}\psi_{R\alpha}(x).$$

A translation by one site on the original lattice space,

$$c_{j,\alpha}\to c_{j+1,\alpha}\exp(ik_F a),$$

corresponds to a chiral $\boldsymbol{Z}_\nu$ transformation:

$$(\psi_{L\alpha},\psi_{R\alpha})\to(\psi_{L\alpha},\psi_{R\alpha}\exp(2ik_F a)).$$

As far as the translational symmetry is not broken, the effective field theory becomes chiral $\boldsymbol{Z}_\nu$ invariant.

Now, we take into account the deviation from the mean-field approximation. In the following parametrization of the auxiliary field

$$\mathcal{Q}_{j,j'}=R\left(\frac{j+j'}{2}\right)\exp\left\{i|j-j'|A_1\left(\frac{|j+j'|}{2}\right)\right\},$$

the deviation of $\mathcal{Q}_{j,j'}$ becomes

$$\mathcal{Q}_{j,j'}\simeq R_0+\delta R\left(\frac{j+j'}{2}\right)+i|j-j'|A_1\left(\frac{|j+j'|}{2}\right). \quad (2.7)$$

The local constraint is expressed as

$$\psi^\dagger_{L\alpha}\psi_{L\alpha}(x)=0,\quad \psi^\dagger_{R\alpha}\psi_{R\alpha}(x)=0, \quad (2.8a)$$

$$\psi^\dagger_{L\alpha}\psi_{R\alpha}(x)+\psi^\dagger_{R\alpha}\psi_{L\alpha}(x)=0. \quad (2.8b)$$

We obtain a chiral $\boldsymbol{Z}_\nu$-invariant effective Lagrangian in terms of low-energy variables:

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\rm int},$$

where

$$\mathcal{L}_0=2\psi^\dagger_{L\alpha}(\overline{\partial+iA})\psi_{L\alpha}+2\psi^\dagger_{R\alpha}(\partial+iA)\psi_{R\alpha}, \quad (2.9)$$

$$\mathcal{L}_{\rm int}={\rm const}\times\psi^\dagger_{L\alpha}\psi_{L\beta}\psi^\dagger_{R\beta}\psi_{R\alpha}.$$

Here, the gauge field A_0 is a low-energy variable corresponding to the Lagrangian multiplier $\chi=aA_0$ and $A=A_0+iA_1,\overline{A}=A_0-iA_1$. This effective theory is a perturbed Abelian gauge-field theory with a sound velocity $v=R_0 a\sin(k_F a)$, here v is set to unity. In deriving the Lagrangian $\mathcal{L}$ we have picked up the terms to $O(a^2)$ and neglected the highly oscillating terms and higher derivative terms. The four-Fermi interaction $\mathcal{L}_{\rm int}$ is induced by performing the Gaussian integration over the δR_0 field and also by the second constraint (2.8b). This interaction can be expressed in the form

$$\frac{1}{\nu}j_L(z)j_R(\overline{z})+2\sum_{A=1}^{\nu^2-1}\mathcal{J}^A_L(z)\mathcal{J}^A_R(\overline{z}),$$

where $j_{L(R)}=\psi^\dagger_{L(R)\alpha}\psi_{L(R)\alpha}$ and $\mathcal{J}^A_{L(R)}=\psi^\dagger_{L(R)\alpha}T^A_{\alpha\ \beta}\psi_{L(R)\beta}$. In Appendix A, the SU($\nu$) basis is summarized. We should define a regularization for the U(1) current which preserves the local $U_V(1)$ gauge symmetry $\psi_L\to\exp(i\alpha)\psi_L$, $\psi_R\to\exp(-i\alpha)\psi_R$. An arbitrary local composite operator should be defined in the gauge-invariant regularization. The current operators are defined in this way as well. We shall discuss the importance of the marginal perturbation $\mathcal{L}_{\rm int}$ on the basis of a RG calculation later. To solve this system $\mathcal{L}_0$, we take a gauge fixing condition $A=0$ and a parametrization of $\overline{A}$ with a scalar field $\phi(z,\overline{z})$

$$\overline{A}=\overline{\partial}\phi(z,\overline{z}),\quad \psi_{L\alpha}(z)=\widetilde{\psi}_{L\alpha}(z)\exp(-i\phi). \quad (2.10)$$

The gauge-invariant regularization defines the unique Jacobian for the chiral transformation $\psi_L\to\widetilde{\psi}_L$ which induces the kinetic term of the scalar field ϕ.[22] integral, we calculate a fermionic determinant (for the left moving part) Then, the two-dimensional Abelian gauge theory with global symmetry SU(ν) is expressed as a decoupled free field Lagrangian:

$$\mathcal{A}_*=\int\frac{d^2z}{2\pi}(\mathcal{L}_{\rm matter}+\mathcal{L}_{\rm gauge}+\mathcal{L}_{\rm ghost}) \quad (2.11)$$

with

$$\mathcal{L}_{\rm matter}=2\widetilde{\psi}^\dagger_{L\alpha}\overline{\partial}\widetilde{\psi}_{L\alpha}+2\psi^\dagger_{R\alpha}\partial\psi_{R\alpha},$$

$$\mathcal{L}_{\rm gauge}=-\nu\partial\phi\overline{\partial}\phi, \quad (2.12)$$

$$\mathcal{L}_{\rm ghost}=2\overline{\eta}\partial\overline{\epsilon}+2\eta\overline{\partial}\epsilon,$$

where the last term is the Fadeev-Popov ghost originating from the measure of the gauge-field parametrized in terms of ϕ. Hereafter, the tilde of the left moving fermions will be omitted for simplicity. The operator product expansions (OPE) between free fields are

$$\psi^\dagger_{L\alpha}(z)\psi_{L\beta}(\omega)\sim\frac{\delta_{\alpha,\beta}}{z-\omega}+\cdots,$$

$$\psi^\dagger_{R\alpha}(\overline{z})\psi_{R\beta}(\overline{\omega})\sim\frac{\delta_{\alpha,\beta}}{\overline{z}-\overline{\omega}}+\cdots,$$

$$e^{i\phi(z,\overline{z})}e^{-i\phi(\omega,\overline{\omega})}\sim|z-\omega|^{2/\nu}+\cdots, \quad (2.13)$$

which allow us to calculate OPE for energy-momentum tensors and read off the central charges

$$c_{\rm matter}=\nu,\quad c_{\rm gauge}=1,\quad c_{\rm ghost}=-2.$$

As expected, the total central charge $c_{\rm total}$ equals $\nu-1$ which agrees with the Bethe ansatz's result.[25] The negative sign in the Lagrangian $\mathcal{L}_{\rm gauge}$, Eq. (2.12), suggests that the U(1) degrees of freedom freeze in the asymptotic behaviors of the spin system. Since the conformal weight of the vertex operator becomes negative, Eq. (2.13) shows unphysical infrared behavior, and thus it should not appear by itself. Actually, the U(1) current regularized in the gauge-invariant way

$$j_L(z)=\,:\widetilde{\psi}^\dagger_{L\alpha}(z)\widetilde{\psi}_{L\alpha}(z). \quad (2.14)$$

Equation (2.14) has no Goto-Imamura-Schwinger term in the U(1) Kac-Moody algebra. Therefore, this U(1) Kac-Moody algebra has only a trivial representation $j_L=0,j_R=0$. According to the bosonization formula in the $U(\nu)=U(1)\times{\rm SU}(\nu)$-invariant free Dirac theory, degrees of freedom of U(1) (charge) and SU(ν) (spin) are separated into a scalar boson and a SU(ν) WZW theory. In our case, the U(1) degree of freedom is killed by the gauge field which is represented by the scalar boson with negative norm. Equation (2.11) can be identified with the level-one SU(ν) WZW model.[26] The Wess-Zumino primary field $G(z,\overline{z})$ is given by

$$G_{\alpha\beta}(z,\overline{z})\propto\psi^\dagger_{L\alpha}(z)e^{i\phi(z,\overline{z})}\psi_{R\beta}(\overline{z}),\quad G(z,\overline{z})\in{\rm SU}(\nu\ )$$

and the conformal weight is $(\nu-1)/2\nu$. To compute the asymptotic behavior of the spin-correlation function for the bulk, it is enough to replace the spin-one operators (2.2) by the continuum fields in terms of Eqs. (2.6) and (2.10):

$$a^{-1}\boldsymbol{S}_j \simeq \boldsymbol{J}_L(r)+\boldsymbol{J}_R(r)+[e^{i\phi(r)}\psi^\dagger_{L\alpha}(r)\boldsymbol{L}_{\alpha\beta}\psi_{R\beta}(r)\exp(2ik_F r) + \text{H.c.}],$$

where $\boldsymbol{J}_{L(R)}(r)=\psi^\dagger_{L(R)\alpha}(r)\boldsymbol{L}_{\alpha\beta}\psi_{L(R)\beta}(r)$. Using this, we obtain the typical correlation function of Tomonaga-Luttinger liquids[27]

$$\langle \boldsymbol{S}_r\cdot\boldsymbol{S}_0\rangle \propto \frac{1}{r^2}+\text{const}\times\frac{\cos(2k_F r)}{r^{2x}} \tag{2.15}$$

with scaling dimension $x=1-1/\nu$, in which the second term is dominant as $r\to\infty$ and the momentum distribution shows a power-law singularity near the Fermi momentum k_F. The appearance of the oscillating factor is a reflection of the chiral $\boldsymbol{Z}_\nu$ symmetry in the antiferromagnet.

III. THE ROLE OF MARGINAL OPERATORS

We have neglected the marginal operators so far. One of them is the SU(ν) current interaction in $\mathcal{L}_{\text{int}}$ which gives logarithmic finite-size corrections. Besides, there is another operator $\Sigma_{j=1}^{L} c^\dagger_{j,\alpha}c_{j,\beta}c^\dagger_{j+1,\alpha}c_{j+1,\beta}(x)$ which breaks the global SU(ν) symmetry except at the ULS point. The continuum form of the SU(ν)-asymmetric interaction is given in terms of Eq. (2.6) by

$$c^\dagger_{j,\alpha}c_{j,\beta}c^\dagger_{j+1,\alpha}c_{j+1,\beta}\simeq\psi^\dagger_{L\alpha}\psi_{L\beta}\psi^\dagger_{R\alpha}\psi_{R\beta}+\cdots$$

which is also chiral $\boldsymbol{Z}_\nu$ invariant. The corresponding field theory is expressed by the WZW model with these marginal perturbations without global SU(ν) symmetry. This SU(ν)-breaking operator becomes marginally relevant for the coupling constant $\gamma<1$, and thus a dynamical mass generation is expected.

We consider a perturbed CFT with the following action:

$$\mathcal{A}=\mathcal{A}_{\text{SU}(\nu)_1}+\sum_{i=1}^{2} g_i\int\frac{d^2z}{2\pi}\Phi^{(i)}(z,\overline{z}), \tag{3.1}$$

where

$$\Phi^{(1)}(z,\overline{z})=\frac{2}{\sqrt{\nu^2-1}}\mathcal{J}^A_L(z)\mathcal{J}^A_R(\overline{z}),$$

$$\Phi^{(2)}(z,\overline{z})=\frac{4T^A_{\alpha\beta}T^B_{\alpha\beta}}{\sqrt{\nu^2-1}}\mathcal{J}^A_L(z)\mathcal{J}^B_R(\overline{z}). \tag{3.2}$$

There are no other relevant or marginal operators with rotational and chiral $\boldsymbol{Z}_3$ symmetry. The coupling constant g_2 is proportional to $\gamma-1$ with a positive coefficient in the case of $\nu=3$ $(S=1)$. The unperturbed action $\mathcal{A}_{\text{SU}(\nu)_1}$ is given by Eq. (2.11) and the marginal operators obey the OPE algebra

$$\Phi^{(1)}(z,\overline{z})\Phi^{(1)}(0,0)\sim\frac{1}{|z|^4}-\frac{b}{|z|^2}\Phi^{(1)}(0,0)+\cdots,$$

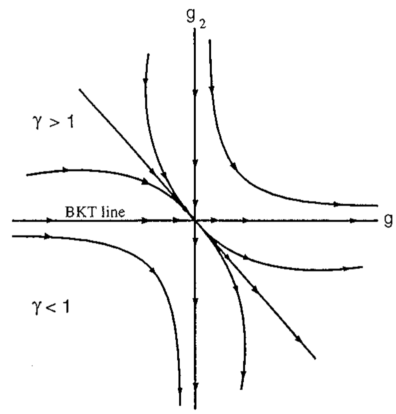

FIG. 1. The renormalization-group trajectory for the $\nu>2$ model, where $\nu=2S+1$. The coupling g_2 is defined in the vicinity of the ULS model. The BKT line $g_2=0$, $g_1\leqslant 0$ corresponds to the pure ULS model.

$$\Phi^{(2)}(z,\overline{z})\Phi^{(2)}(0,0)\sim\frac{1}{|z|^4}+\frac{b}{|z|^2}\Phi^{(2)}(0,0)+\cdots,$$

$$\Phi^{(1)}(z,\overline{z})\Phi^{(2)}(0,0)\sim\frac{1}{(\nu+1)}\frac{1}{|z|^4}-\frac{\tilde{b}}{|z|^2}[\Phi^{(1)}(0,0)-\Phi^{(2)}(0,0)]+\cdots,$$

where

$$b=\frac{2\nu}{\sqrt{\nu^2-1}},\quad \tilde{b}=\frac{2}{\sqrt{\nu^2-1}}.$$

This algebra gives the following one-loop β functions:

$$\beta_1(g_1,g_2)\equiv\frac{dg_1}{dl}=\frac{b}{2}g_1{}^2+\tilde{b}g_1g_2+O(g_1{}^3,g_2{}^2),$$

$$\beta_2(g_1,g_2)\equiv\frac{dg_2}{dl}=-\frac{b}{2}g_2{}^2-\tilde{b}g_1g_2+O(g_2{}^3,g_1{}^2), \tag{3.3}$$

where $e^l=a$. These coupled differential equations can be solved in an integral form thanks to a conservation law. An arbitrary trajectory in the coupling constant space (g_1,g_2) obeys the following equation:

$$X^2-Y^2=C|Y|^{(\nu-2)/\nu} \tag{3.4}$$

with $X=g_1-g_2$ and $Y=-g_1-g_2$, where C is an arbitrary real constant. (See Fig. 1.) The sign of the initial value of g_1 should be chosen to be negative in order to agree with the result of the Bethe ansatz. The running coupling constant g_1 is renormalized to be zero in the infrared limit. Therefore this model is in the second region $g_1<0,g_2>0$ or the third one $g_1<0,g_2<0$ in the coupling constant space. In the $\nu=2$ case, this perturbed CFT describes the well-known spin-1/2 XXZ chain and Eq. (3.4) shows a hyperbolic trajectory where the BKT transition occurs beyond the SU(2) symmetric line $X\pm Y=0$. There is one-parameter family of fixed points in $c_{\text{vir}}=1$, that is a fixed line $g_1+g_2=0$.[28] Note that the topology of the flow diagram in the case of $\nu\neq 2$ differs from that in $\nu=2$. The only fixed point is $g_1^*=g_2^*=0$ except for $\nu=2$.

Here, we show main results of the RG flow which will be illustrated in the remaining part of this paper. The RG argument classifies the coupling constant space with $g_1<0$ into the following three cases:

(i) $g_2=0$; SU(ν) symmetric and asymptotically nonfree, (ii) $g_2>0$; SU(ν) asymmetric and asymptotically nonfree, (iii) $g_2<0$; SU(ν) asymmetric and asymptotically free.

Since the interaction $\Phi^{(2)}$ is marginally relevant for $g_2<0$, and is marginally irrelevant for $g_2>0$, the trajectory along $g_2=0$ becomes the BKT transition line. In a finite system in the asymptotically nonfree region $g_2\geqslant 0$, thermodynamic quantities acquire some corrections due to the presence of marginally irrelevant operators, while in an infinite volume limit there is no influence from them. We indicate the difference of the finite-size corrections between SU(ν)-symmetric and asymmetric models. In the third case of $g_2<0$, the marginally relevant interaction $\Phi^{(2)}$ can generate a mass gap which might be interpreted as the Haldane gap.

First, following Ludwig and Cardy,[20,21] we calculate finite-size corrections in the SU(ν) symmetric model ($g_2=0$) with $g_1<0$. The finite-size corrections to the ground-state energy of the SU(ν)-symmetric models in Eq. (1.2) are calculated as

$$c_{\mathrm{vir}}=\nu-1,\quad d_{\mathrm{GS}}=\frac{\nu^2-1}{2\nu^2}, \tag{3.5}$$

where $\nu=2S+1$ for the spin-S.

The finite-size corrections to the low-lying excited energies are calculated from the most relevant primary field

$$\mathcal{O}^A(z,\overline{z})=\psi^{\dagger}_{L\alpha}(z)T^A_{\alpha\beta}\psi_{R\beta}(\overline{z})e^{i\phi(z,\overline{z})}+\text{H.c.},$$

where T^A's for $A=1,\ldots,\nu^2-1$ are the SU(ν) basis, $T^0\equiv I/\sqrt{2\nu}$, and they are also normalized as $\mathrm{Tr}[T^AT^B]=\delta^{AB}/2$. The primary states, $|\mathcal{O}^A_{\mathrm{in}}\rangle\equiv\lim_{z,\overline{z}\to 0}\mathcal{O}^A(z,\overline{z})|0\rangle$, become eigenstates of Virasoro's charge L_0 ($\overline{L}_0$) with an eigenvalue $x/2$. Their OPE are given by

$$\mathcal{O}^A(z,\overline{z})\mathcal{O}^B(0,0)\sim\frac{\delta^{AB}}{|z|^{2-2/\nu}}+\cdots,$$

$$\mathcal{O}^A(z,\overline{z})\Phi^{(1)}(0,0)\sim-\frac{b_A}{|z|^2}\mathcal{O}^A(0,0)+\cdots$$

with the OPE coefficients

$$b_A=\frac{1}{\nu\sqrt{\nu^2-1}}\times\begin{cases}\nu^2-1 & \text{for } A=0,\\ -1 & \text{for } A=1,\ldots,\nu^2-1.\end{cases}$$

We obtain the universal quantities in Eq. (1.3)

$$x_A=1-1/\nu,\quad d_A=\frac{2b_A}{b}=\begin{cases}1-1/\nu^2 & \text{for } A=0\\ -1/\nu^2 & \text{for } A=1,\ldots,\nu^2-1.\end{cases} \tag{3.6}$$

TABLE II. Finite-size corrections for the spin-1/2 Heisenberg chain.

	c_{vir}	x_t	x_s	d_{GS}	d_t	d_s	σ_t
$\mathrm{SU}(2)_1$ WZW	1	1/2	1/2	3/8	−1/4	3/4	1/2
BA	1	1/2	1/2	0.3433	−1/4	3/4	

These ν^2 states are classified by the total spin. As shown in Appendix A, the state $A=0$ describes the singlet excitation and other ν^2-1 primary states are higher spin states with spin up to $(\nu-1)/2$. In the finite-size corrections up to the logarithmic size dependence, the singlet excitation is not favored compared to those with higher spin.

The effect of the marginal operators for the spin-correlation function (2.15) is obtained immediately from the information on the excited energy:[18,29,30]

$$\langle \boldsymbol{S}_r\cdot\boldsymbol{S}_0\rangle\approx\cos(2k_Fr)\mathcal{G}_A(g_1(r),r),\quad \mathcal{G}_A(g_1(r),r)=\frac{(\ln r)^{\sigma_A}}{r^{2x_A}}, \tag{3.7}$$

where

$$\sigma_A=-2d_A=\frac{2}{\nu^2},$$

except for $A=0$. Our results for $\nu=2$ listed in Table II agrees with the Bethe ansatz's ones in Refs. 16 and 31. The leading finite-size corrections c_{vir} and x_A in the SU(ν) symmetric model agree with Bethe ansatz[12,25], as well.

Now we consider the second case $g_2>0$, where there is a marginally irrelevant SU(ν)-asymmetric interaction. The situation is crucial whether $\nu=2$ or not. Even though the action describing the ultraviolet theory has no SU(ν) symmetry due to the SU(ν)-breaking interaction, the SU(ν)-breaking interaction have no effect on the leading terms of the finite-size correction except in the $\nu=2$ case. The RG indicates that the SU(ν) symmetry appears dynamically for the macroscopic scale even though the g_2 term in Eq. (3.2) is switched into the fixed-point action. The difference between the SU(ν)-symmetric and asymmetric model appears in the logarithmic correction term.

To calculate the logarithmic correction, we note that the RG flow Eq. (3.3) with an initial condition $g_1<0$ and $g_2>0$ is absorbed into the fixed point along the line

$$g_1=-g_2. \tag{3.8}$$

The macroscopic property of the system is determined by the scale $l\gg 1$, and we can estimate the deviation from the line as $|g_1(l)+g_2(l)|\sim O(l^{-2\nu/(\nu-2)})$ with the help of the integral curve Eq. (3.4) for an arbitrary solution with an initial condition in the second region. Therefore we can calculate the logarithmic correction by assuming that the marginally irrelevant flow for $g_2>0$ is described by the action

$$\mathcal{A}=\mathcal{A}_{\mathrm{SU}(\nu)_1}+g_1\int\frac{d^2z}{2\pi}\Psi(z,\overline{z})$$

TABLE III. Finite-size corrections for the spin-1 chains.

	c_{vir}	x_q	x_t	x_s	d_{GS}	d_q	d_t	d_s	σ_q
$SU(3)_1$ WZW ($\gamma=1$)	2	2/3	2/3	2/3	4/9	−1/9	−1/9	8/9	2/9
$SU(3)_1$ WZW ($\gamma\neq 1$)	2	2/3	2/3	2/3	6	−1	1	2	2
BA	2	2/3	2/3	2/3					

with $\Psi(z,\overline{z})=\sqrt{\nu+1/2\nu}[\Phi^{(1)}(z,\overline{z})-\Phi^{(2)}(z,\overline{z})]$ which is normalized by

$$\Psi(z,\overline{z})\Psi(0,0)\sim\frac{1}{|z|^4}-\frac{B}{|z|^2}\Psi(0,0),$$

where the OPE coefficient B is

$$B=\sqrt{\frac{\nu+1}{2\nu}}(b-2\widetilde{b}).$$

This assumption might hold, since the current of the RG would spend a fair time near the fixed point with dilatation. As in the discussions of the symmetric model, we can evaluate the coefficients of the finite-size energy correction from the one-loop renormalization which obeys $dg_1/dl=(B/2)g_1^2$. For the ground-state energy, we obtain

$$c_{\mathrm{vir}}=\nu-1,\quad d_{\mathrm{GS}}=\frac{\nu(\nu-1)}{(\nu-2)^2},$$

in which the logarithmic coefficient is different from Eq. (3.5). The three-point function in the expression of the excited energy is given by using the OPE

$$\mathcal{O}^A(z,\overline{z})\widetilde{\Phi}(0,0)\sim-\frac{B_A}{|z|^2}\mathcal{O}^A(0,0)+\cdots.$$

Here the coefficient B_A takes three different values according to the symmetric properties of the matrices $\{T^A\}$ under the matrix transposition. These are given by

$$B_A=\frac{1}{\sqrt{2\nu(\nu-1)}}\times\begin{cases}\nu-1 & \text{for } A=0\\ 1 & \text{for } A(\neq 0) \text{ with } {}^t(T^A)=-T^A\\ -1 & \text{for } A(\neq 0) \text{ with } {}^t(T^A)=T^A.\end{cases}$$

As a result, we have the universal coefficients in the anomalous dimension (1.3)

$$x_A=1-1/\nu,$$

$$d_A=\frac{1}{\nu-2}\times\begin{cases}\nu-1 & \text{for } A=0\\ 1 & \text{for } A(\neq 0) \text{ with } {}^t(T^A)=-T^A\\ -1 & \text{for } A(\neq 0) \text{ with } {}^t(T^A)=T^A.\end{cases}\tag{3.9}$$

The OPE coefficents B and B_A give the exponents $\sigma_{A\neq 0}$ characterizing logarithmic distance dependence in Eq. (3.7)

$$\sigma_A=\frac{2}{(\nu-2)}.$$

The primary states with $A\neq 0$ in the symmetric model are degenerate even if we consider the logarithmic correction, those in the asymmetric model split to two levels. As shown in Appendix A, the difference of the OPE coefficients because of the symmetric and antisymmetric properties of SU(ν) Lie algebra basis is classified by the total spin. In particular, for the SU(3) problems, the primary with the identity matrix ($A=0$) is spin-singlet, three primaries with the antisymmetric matrices are spin-triplet, and the remainder with symmetric ones are spin-quintuplet. The universal coefficients characterizing the SU(3)-symmetric and asymmetric model are shown in Table III.

Let us now consider the third case $g_2<0$, which corresponds to $\theta<\theta_c\equiv\pi/4$ in the $S=1$ model. The theory is asymptotically free, then we expect the mass generation which can be identified with the Haldane gap in the $S=1$ case. One can estimate the mass gap by solving the renormalization group Eq. (3.3). The conservation law Eq. (3.4) enables us to reduce the simultaneous equation for the two unknown functions g_1 and g_2 to that for the one unknown $Y=-g_1-g_2$

$$\frac{dY}{dl}=\pm\nu\sqrt{\nu^2-1}Y^2\sqrt{1+CY^{-(\nu+2)/\nu}},\tag{3.10}$$

where the sign of the right-hand side is identical to that of $X=g_1-g_2$. Let us set the initial condition of the running coupling constants near the transition point

$$g_1(0)=-a_1,\quad g_2(0)\simeq a_2(\theta-\theta_c),$$

where a_1 and a_2 are positive constants. This condition sets the integral constant as $C\simeq a_3(\theta-\theta_c)$ in Eq. (3.4) with a positive constant a_3. The renormalization-group equation (3.10) is immediately integrated under this condition

$$\left(-\int_{Y(0)}^{|C|^\sigma}+\int_{|C|^\sigma}^{Y(l)}\right)\frac{dY}{Y^2\sqrt{1+CY^{-1/\sigma}}}=\nu\sqrt{\nu^2-1}\ln l,\tag{3.11}$$

where $\sigma\equiv\nu/(\nu+2)$. This gives us the order of the scale m^{-1} which makes the running coupling constant diverge $g_2(\ln m^{-1})=\infty$. This scale m is the energy gap

$$m=\exp(-A|C|^{-\sigma})\simeq\exp[-c(\theta_c-\theta)^{-\sigma}],\tag{3.12}$$

where

$$A=\frac{2}{\nu\sqrt{\nu^2-1}}\int_1^\infty\frac{dy}{y^2\sqrt{1-y^{-1/\sigma}}},$$

and $c=a_3A$ is positive. Therefore we conclude that the phase transition is infinite order. This result agrees with the recent numerical studies of the $S=1$ model by Fáth and Sólyom.[10] To see this, one should check their obtained energy gap directly rather than the one-parameter β function estimated from it, since we have a two-parameter β function (3.3). Their numerical data of the energy gap fit the function

Eq. (3.12) with the universal constant $\sigma=0.8\pm0.2$. This is consistent with our result $\sigma=\nu/(\nu+2)=0.6$ at $\nu=3$.

IV. DISCUSSION AND OPEN PROBLEMS

We have investigated the isotropic spin-1 model to clarify the phase diagram around the Uimin-Lai-Sutherland (ULS) point. The low-energy theory of the ULS model is described by a strong-coupling Abelian gauge theory which can be regarded as the critical level-one SU(3) WZW model. We have shown a mechanism of the dynamical mass generation in the $S=1$ Haldane phase in the presence of the SU(3) breaking interaction with dimension 2. We have shown that the dimension 2 operator makes the massless phase $\theta\leqslant\pi/4$ and the massive phase $\theta<\pi/4$ around the ULS point $\theta=\pi/4$ in the model Eq. (1.1). This nature can be understood by the level $k=1$ WZW theory, which has neither a relevant operator with the chiral $\boldsymbol{Z}_3$ invariance nor the tensored operator of the WZ matrices but merely marginal operators. Therefore, the Haldane phase has the exponential mass gap as a result of the BKT transition. The region $\pi/2\leqslant\theta\leqslant\pi/4$ is concluded to be massless from this analysis and the numerical study.[10] Here, we indicate the difference of the phase transitions at the ULS point and at another integrable point $\theta=-\pi/4$ of the Takhatajan-Babujian (TB) model. In an alternative field-theoretical approach for understanding the Haldane massive phase, Affleck and Haldane investigated the relevant deformation of the $S=1$ TB model.[32] The universality class of this TB model is the level-two SU(2) WZW model, where the one-site translation corresponds to the chiral $\boldsymbol{Z}_2$ transformation. In the level-k theory with $k>1$, one can make the chiral $\boldsymbol{Z}_2$ invariant relevant operator in terms of tensoring of the SU(2) WZ matrices $G(z,\overline{z})$, for example $(\mathrm{Tr}[G])^2$. Therefore the transition from that massless point to the Haldane phase becomes second order, and the mass gap opens obeying the power law. In this case, the TB point $\theta=-\pi/4$ is isolated as a massless point in the massive region, namely the Haldane phase $\theta>-\pi/4$ and the dimer phase $\theta<-\pi/4$.

The renormalization-group flow given by Eq. (3.3) has a unique fixed point in the $\nu>2$ case, while that in $\nu=2$ case has a fixed line. Contrary to the $\nu=2$ case, the logarithmic corrections appears in the massless phase for $\nu>2$ even if there is a SU(ν) symmetry-breaking interaction. We have calculated coefficients of logarithmic corrections to the energies of the ground state and some excited states both in SU(ν) symmetric and asymmetric models. We find the different coefficients in these two cases from their numerical data of the energy gap as in the form Eq. (3.12). The nature of this model with $\nu>2$ suggests Cardy's argument that a natural irreducible CFT with one parameter should have the central charge $c_{\mathrm{vir}}=1$.[33] Nonetheless, no one has ever succeeded in classifying CFT with $c_{\mathrm{vir}}>1$, and therefore to search CFT with a fixed line (or surface) might be worth attempting. Since we need to spread the coupling constant space at least, the simplest candidate is a model with anisotropic parameters or q deformation of the Lie algebra SU(ν). This program is now in progress.

Here we present some conjectures deduced from the CFT kinematics. We note that $\boldsymbol{J}_L(z)$ and $\boldsymbol{J}_R(\overline{z})$, which are in a subalgebra of $\mathrm{SU}(3)_1$ Kac-Moody algebra, except the normalization, satisfy the level-four SU(2) Kac-Moody algebra. The representation of $\mathrm{SU}(3)_1$ is involved in that of $\mathrm{SU}(2)_4$. The central charge of both theories are $c_{\mathrm{vir}}=2$ and the conformal weight of the primary field with spin-j is $\triangle^{(j)}=j(j+1)/6$ with $0\leqslant j\leqslant 2$.[34] If we neglect primaries with half-odd-integer spin in the $\mathrm{SU}(2)_4$ WZW model, we obtain those in the $\mathrm{SU}(3)_1$ WZW model. The $\mathrm{SU}(2)_4$ WZW model can be regarded as a critical theory of the spin-2 TB model, and therefore we can expect the following prediction:

Conjecture 1: *There is a crossover flow from the spin-*2 *Takhtajan-Babujian model to the spin-*1 *Uimin-Lai-Sutherland model.*

As recognized in the studies of the SU(2) spin chains, coefficients d_j in the logarithmic correction to the excited states with total spin-j satisfy the following sum rule:[35] $3d_t+1d_s=0$, where $d_{s(t)}$ is the universal coefficient for the singlet (triplet) excitation(s) and the prefactor is the dimension of the spin representation. We have seen that such a similar rule exists in the spin-1 models discussed above, as well. That is $5d_q+3d_t+d_s=0$.[36] Therefore, we are led to the following conjecture:

Conjecture 2: *There exists a sum rule among the coefficients* $\{d_j\}$ *of the leading logarithmic correction term in the excited energy with total spin-j; i.e.,*

$$\sum_{j=0}^{2S}(2j+1)d_j=0.$$

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APPENDIX

The fundamental representation of the SU(ν) Lie algebra $[T^A,T^B]=f^{AB}{}_C T^C$ is summarized as follows. The SU(ν) exchange operator is decomposed in terms of the SU(ν) basis as

$$\mathcal{P}=\frac{1}{\nu}I\times I+2\sum_{A=1}^{\nu^2-1}T^A\times T^A.$$

This basis is normalized as $\mathrm{Tr}[T^AT^B]=(1/2)\delta^{AB}$ or $\sum_{A=1}^{\nu^2-1}T^AT^A=(\nu^2-1)/2\nu$. The structure constant f^{ABC} has the quadratic Casimir of the adjoint representation: $\sum_{A,B=1}^{\nu^2-1}f^{ABC}f_{ABD}=-\nu\delta^C{}_D$. Another expression of the exchange operator is available when the spin chains are studied. On a space $\boldsymbol{C}^{2S+1}\times\boldsymbol{C}^{2S+1}$, it is given by

$$\mathcal{P}=(-1)^{2S}\sum_{j=0}^{2S}(-1)^j\mathcal{P}^{(j)},$$

where $\mathcal{P}^{(j)}$ is the projector onto a space of spin-i conforming to an identity $I\times I=\mathcal{P}^{(0)}+\cdots+\mathcal{P}^{(2S)}$. The projector $\mathcal{P}^{(j)}$ on a spin-j space is represented using the spin operators with the magnitude S as follows:

$$\mathcal{P}^{(j)}=\prod_{\substack{k=0\\(\neq j)}}^{2S}\left[\frac{X-x_k}{x_j-x_k}\right],\quad X=\sum_{a=1}^{3}S^a\times S^a,$$

where $x_k=[k(k+1)-2S(S+1)]/2$. The expressions of the exchange operator in terms of the spin operator are shown in Table I.

In particular, the representation of SU(3) is realized by Gell-Mann matrices

$$\lambda_1=\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \lambda_8=\begin{pmatrix}1/\sqrt{3}&0&0\\0&1/\sqrt{3}&0\\0&0&-2/\sqrt{3}\end{pmatrix},$$

where $T^A=\lambda_A/2$. Here $\lambda_{A=2,5,7}$ are antisymmetric matrices and the remainder of them are symmetric.

The primary states $\{|\mathcal{O}^A_{\rm in}\rangle\}$ can be classified by total spin-j. The total spin operator is given by

$$\boldsymbol{S}_{\rm tot}=\int_0^L dx\ \ \boldsymbol{S}(x)=\boldsymbol{J}_{L,0}+\boldsymbol{J}_{R,0},$$

where SU(2) charge operators are $\boldsymbol{J}_{L,0}=\oint(dz/2\pi i)\boldsymbol{J}_L(z)$ and $\boldsymbol{J}_{R,0}=\oint(d\overline{z}/2\pi i)\boldsymbol{J}_R(\overline{z})$. The magnitude of total spin of the primary states takes values 0,1, or 2 from a synthesis of two fermions with spin 1. Acting $\boldsymbol{S}_{\rm tot}$ on the primary fields, we obtain the OPE

$$\boldsymbol{S}_{\rm tot}\mathcal{O}^{A=0}(z,\overline{z})=0,$$

$$\boldsymbol{S}_{\rm tot}\mathcal{O}^{A\neq 0}(z,\overline{z})=4\mathcal{O}^{A}(z,\overline{z})+2T^A_{\beta\alpha}(\psi^\dagger_{L\alpha}(z)\psi_{R\beta}(\overline{z})e^{i\phi(z,\overline{z})}+\text{H.c.}).$$

Here we have used the properties of the SU(3) basis. Using the symmetric and asymmetric properties of the Gell-Mann matrices, we obtain

$$(\boldsymbol{S}_{\rm tot})^2|\mathcal{O}^A_{\rm in}\rangle=j(j+1)|\mathcal{O}^A_{\rm in}\rangle,$$

where $j=0,1,2$. The primary with identity matrix $(A=0)$ is the singlet state $(j=0)$. Three antisymmetric ones $(A=2,5,7)$ in the Gell-Mann matrices give spin-triplet states $(j=1)$. The remainders which are symmetric matrices, become quintuplet states $(j=2)$.

[1] U. Schollwöck, Th. Jolicœur, and T. Garel, Phys. Rev. B **53**, 3304 (1996).
[2] I. Affleck, T. Kennedy, E. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987); Commun. Math. Phys. **115**, 477 (1988); S. Knabe, J. Stat. Phys. **52**, 627 (1988).
[3] H. Tasaki, Phys. Rev. Lett. **66**, 798 (1991); T. Kennedy and H. Tasaki, Phys. Rev. B **45**, 304 (1992).
[4] G. V. Uimin, JETP Lett. **12**, 225 (1970); C. K. Lai, J. Math. Phys. **15**, 1675 (1974); B. Sutherland, Phys. Rev. B **12**, 3795 (1975).
[5] P. P. Kulish and N. Yu. Reshetikhin, Sov. Phys. JETP **53**, 108 (1981).
[6] L. A. Takhtajan, Phys. Lett. **87A**, 479 (1982); H. M. Babujian, *ibid.* **90A**, 479 (1982).
[7] M. N. Barber and M. T. Batchelor, Phys. Rev. B **40**, 4621 (1989); M. T. Batchelor and M. N. Barber, J. Phys. A **23**, L15 (1990).
[8] M. P. Nightingale and H. W. Blöte, Phys. Rev. B **33**, 659 (1986).
[9] K. Nomura, Phys. Rev. B **40**, 2421 (1989).
[10] G. Fáth and J. Sólyom, Phys. Rev. B **44**, 11 836 (1991); **47**, 872 (1993).
[11] H. Johannesson, Phys. Lett. A **116**, 133 (1986); Nucl. Phys. B **270**, 235 (1986).
[12] S. V. Pokrovskiĭ and A. M. Tsvelik, Sov. Phys. JETP **66**, 1275 (1987).
[13] K. Lee and P. Schlottmann, Phys. Rev. B **36**, 466 (1987).
[14] P. Schlottmann, Phys. Rev. B **45**, 5293 (1992).
[15] L. Mezincescu, R. I. Nepomechie, P. K. Townsend, and A. M. Tsvelik, Nucl. Phys. B **406**, 681 (1993).
[16] I. Affleck, D. Gepner, H. J. Schulz, and T. Ziman, J. Phys. A **19**, 511 (1989).
[17] S. Eggert, I. Affleck, and M. Takahashi, Phys. Rev. Lett. **73**, 332 (1994).
[18] K. A. Hallberg, P. Horsch, and G. Martínez, Phys. Rev. **52**, R719 (1995).
[19] S. Eggert, Phys. Rev. B **54**, R9612 (1996).
[20] A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. B **285**, 687 (1987).
[21] J. L. Cardy, J. Phys. A **19**, L1093 (1986); **20**, 5039(E) (1987).
[22] C. Itoi and H. Mukaida, J. Phys. A **27**, 4695 (1994).
[23] I. Affleck, Nucl. Phys. B **265**, 409 (1986); **305**, 582 (1988).
[24] M. T. Batchelor and C. M. Yung (unpublished); T. Kennedy, J. Phys. A **25**, 2809 (1992).

[25] H. J. de Vega, J. Phys. A **20**, 6023 (1987); **21**, L1089 (1988).
[26] A. M. Polyakov, in *Two Dimensional Quantum Gravity. Superconductivity at High* T_c, 1988 Les Houches Lecture Notes, edited by E. Brezin and J. Zinn-Justin (North-Holland, Amsterdam, 1990).
[27] N. Kawakami and S.-K. Yang, J. Phys. Condens. Matter. **3**, 5983 (1991); Phys. Lett. A **148**, 359 (1990).
[28] A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).
[29] A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986); Sov. J. Nucl. Phys. **46**, 1090 (1987).
[30] T. Giamarchi and H. J. Schulz, Phys. Rev. B **39**, 4620 (1989).
[31] F. Woynarovich and H.-P. Eckle, J. Phys. A **20**, L97 (1987); C. J. Hamer, M. N. Barber, and M. T. Batchelor, J. Stat. Phys. **52**, 679 (1988).
[32] I. Affleck and F. D. Haldane, Phys. Rev. B **36**, 5291 (1987).
[33] J. L. Cardy, J. Phys. A **20**, L891 (1987).
[34] A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP **43**, 657 (1986).
[35] T. Ziman and H. J. Schulz, Phys. Rev. Lett. **59**, 140 (1987).
[36] A. Kitazawa, K. Nomura, and K. Okamoto (private communication).