Critical-current diffraction pattern of annular Josephson junctions

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A derivation of the exact analytical expressions for the critical current versus magnetic-field-diffraction pattern of ''electrically'' small annular Josephson junctions is presented. These formulas have been recently used to fit experimental data [N. Martucciello and R. Monaco, Phys. Rev. B **54**, 9050 (1996)]. They include, as a special case, the approximate analytical results previously published [N. Martucciello and R. Monaco, Phys. Rev. B **53** 3471 (1996)]. [S0163-1829(97)01602-0]

There has been renewed interest very recently in the physics of annular Josephson tunnel junctions.[1,3] Annular junctions offers a unique chance to study fluxon dynamics because of the absence of fluxon colliding boundaries.[1] Also, the nonconnected geometry plays a special role in determining the behavior of such a junction when a magnetic field is present. Martucciello and Monaco[2] (MM) have considered the static configurations of the phase inside an annular (external radius R, internal radius r) Josephson tunnel junction in the presence of an externally applied magnetic field parallel to the plane of the junction. In Ref. 2 MM report a detailed experimental investigation of the dependence on the magnetic field of the critical current for different annular geometries (mainly determined by different fabrication techniques).

For rings having a circumference shorter than the Josephson penetration depth, and in the limit of vanishing ring width $[(R-r)\rightarrow 0]$ [so that the phase $\phi=\phi(\rho,\theta)$, which normally depends both on ρ and θ, depends only on θ] MM derive a simple analytical result for the lobe structures of the critical current versus magnetic field which is in excellent agreement with the experimental data at low-field values.

For large field values, '' . . . discrepancies in the amplitudes of the secondary maxima show up,'' with apparently no explanation. In this report I present the diffraction pattern formulas of small annular junctions which were calculated by dropping the approximation of a vanishing ring width. These formulas allow one to fit experimental data on a small annular junction very satisfactorily in the whole field range[3] showing that the agreement between theory and experiment can be indeed fully recovered, when the finite width $R-r$ of the ring is correctly considered. In this report the modifications due to the presence of trapped fluxons are also briefly considered and the corresponding formulas for the diffraction pattern are provided.

Figure 1 shows schematically the ring-shaped Josephson barrier lying in the x-y plane, the polar-cylindrical coordinate used, and the externally applied magnetic field in the plane of the junction. ''Small'' junctions only are considered. More specifically it is assumed that $2\pi R<\lambda_j$, where R is the external radius of the ring and λ_j is the Josephson penetration depth, $\lambda_j=[\Phi_0/(2\pi\mu_0 d_{\rm eff}J_0)]^{1/2}$. In this case the self-field effect can be neglected and, moreover, trapped fluxons are not localized.

The equation relating the phase difference φ between the two junction electrodes at the point (ρ,ϕ) to the external magnetic field **H** is[4]

$$\nabla_{\rho,\phi}\varphi=\frac{2\pi\mu_0 d_{\rm eff}}{\Phi_0}\mathbf{H}\times\mathbf{u}_z, \tag{1}$$

where the left side is the gradient operator in polar-cylindrical coordinates, that is, $\nabla_{\rho,\phi}\equiv(\mathbf{u}_\rho\partial\varphi/\partial_\rho+\mathbf{u}_\phi 1/\rho\partial\varphi/\partial\phi)$, $\mathbf{u}_z$ is the unit vector in the z direction, $d_{\rm eff}=d_n+2\lambda_s$ is the effective thickness of the field-penetrated region, and d_n and λ_s are the thickness of the barrier and the London penetration depth, respectively. If the externally applied magnetic field **H** is uniform, parallel to the x-y junction plane, and at an angle θ with respect to the positive x axis, its components in the chosen polar coordinates are

$$H_r(\phi)=H\cos(\theta-\phi), \tag{2}$$

$$H_\phi(\phi)=H\sin(\theta-\phi). \tag{3}$$

Then Eq. (1), written in its two components, gives

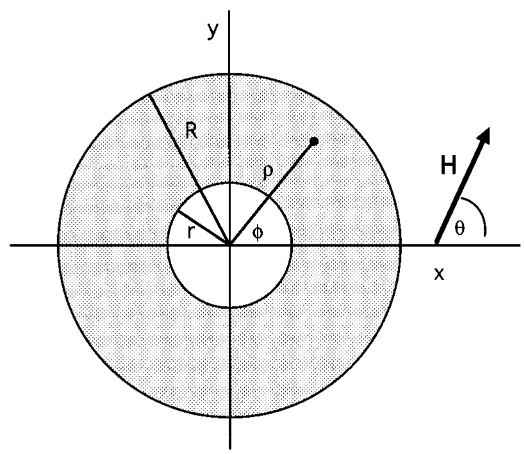

FIG. 1. A ring-shaped Josephson barrier. Indicated are the essential geometric parameters and coordinates used in the calculations.

$$\frac{\partial\varphi}{\partial\rho}=k\sin(\theta-\phi),\quad \frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}=-k\cos(\theta-\phi), \tag{4}$$

$$k=\frac{2\pi\mu_0 d_{\mathrm{eff}}H}{\Phi_0}, \tag{5}$$

which by integration yield

$$\varphi(\rho,\phi)=k\rho\sin(\theta-\phi)+\varphi_0. \tag{6}$$

The constant φ_0 may be identified with the phase difference along the direction of the magnetic field, $\phi=\theta$. Thus, in the presence of a magnetic field, the phase difference may be written as

$$\varphi(\rho,\phi)=\varphi_0+2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}, \tag{7}$$

where $\Phi(\rho,\phi)=d_{\mathrm{eff}}\rho\mu_0 H$ represents the flux of **H** enclosed between the z axis and the polar coordinate (ρ,ϕ) in the insulating barrier plus the penetration layers. Now, the Josephson current density in the z direction can be expressed in the form[5]

$$J_z(\rho,\phi)=J_0\sin\left(\varphi_0+2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right) \tag{8}$$

and the Josephson current across the area S of the junction can be obtained by integrating Eq. (8) over $S=\pi R^2-\pi r^2$. Assuming a uniform maximum Josephson current density J_0 one obtains

$$I=I_0\left(\left\langle \sin 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle\cos\varphi_0+\left\langle \cos 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle\sin\varphi_0\right), \tag{9}$$

where the symbol $\langle\ \rangle$ denotes the spatial average over the junction area and $I_0=J_0S$ is the maximum Josephson critical current.

By maximizing Eq. (9) with respect to φ_0 we obtain the dependence of the critical current on the external magnetic field, or the diffraction patter[6]

$$I_c=I_0\left[\left\langle \sin 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle^2+\left\langle \cos 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle^2\right]^{1/2}. \tag{10}$$

Once Eqs. (6) and (7) have been substituted in Eq. (10), one has to evaluate the two integrals

$$\left\langle \sin 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle=\frac{\int_r^R\rho d\rho\int_0^{2\pi}d\phi\sin[k\rho\sin(\theta-\phi)]}{\pi(R^2-r^2)}, \tag{11}$$

$$\left\langle \cos 2\pi\frac{\Phi(\rho,\phi)}{\Phi_0}\right\rangle=\frac{\int_r^R\rho d\rho\int_0^{2\pi}d\phi\cos[k\rho\sin(\theta-\phi)]}{\pi(R^2-r^2)}. \tag{12}$$

The integration can be easily done by using the Fourier-Bessel expansions[7]

$$\sin(a\sin b)=2\sum_{j=0}^{\infty}J_{2j+1}(a)\sin[2j+1)b],$$

$$\cos(a\sin b)=J_0(a)+2\sum_{j=0}^{\infty}J_{2j}(a)\cos(2jb). \tag{13}$$

Here $J_i(a)$ is the ith Bessel function of integer order.

The first integral, Eq. (11) gives zero contribution, no term surviving to the integration on ϕ. Only the first term contributes in the second integral yielding, after the integration on ρ has been performed too,

$$I_c(k)=2I_0\left|\frac{kRJ_1(kR)-krJ_1(kr)}{k^2R^2-k^2r^2}\right|. \tag{14}$$

Equation (14) determines the critical-current diffraction pattern of a ''small'' annular tunnel junction, for any $(R-r)$ value. Equation (14) reduces to the simple result obtained by MM in Ref. 2,

$$I_c(k)=I_0|J_0(kR)|. \tag{15}$$

(J_0 is now the zero-order Bessel function) in the limit $r\to R$ of vanishing width of the ring barrier.

Of course, in the opposite limit, that is $r\to 0$, from Eq. (14) the well- known result for the circular junction (disk barrier junction) is recovered,[4]

$$I_c(k)=I_0\left|\frac{J_1(kR)}{1/2(kR)}\right|. \tag{16}$$

It is interesting to introduce the normalized field dependence of the critical current.

$$I_c=2I_0\left|\frac{J_1(\pi h)-\varepsilon J_1(\varepsilon\pi h)}{\pi h(1-\varepsilon^2)}\right|, \tag{17}$$

where $\varepsilon=r/R$ and h is the normalized total flux across the junction or equivalently the normalized magnetic field

$$kR\equiv\pi h=\pi\frac{\Phi_{\mathrm{tot}}}{\Phi_0}=\pi\frac{H}{H_0}, \tag{18}$$

where

$$\Phi_{\mathrm{tot}}=\mu_0H(2R)d_{\mathrm{eff}},\quad H_0=\frac{\Phi_0}{(2R)\mu_0 d_{\mathrm{eff}}}. \tag{19}$$

Φ_{tot} is the total flux across the junction and H_0 is the ''characteristic field'' that is the minimum field required for introducing one flux quantum. When fluxons are trapped into the barrier, which is a common experimental occurrence, the critical current pattern is modified because now the magnetic field of the fluxon is to be added to the externally applied magnetic field. The trapping modality in the annular junction is, in some sense, simple. It is determined by the nonconnected geometry.

A ring junction with a trapped fluxon (or ''antifluxon'' depending on the field lines direction considered) in the upper electrode, threading the barrier, is represented very schematically in Fig. 2. In principle, an arbitrary number of fluxons (or antifluxons) can be independently trapped in each electrode. Each flux quantum contributes to the current circulating in the electrode in one of the two possible directions (clockwise or anticlockwise).

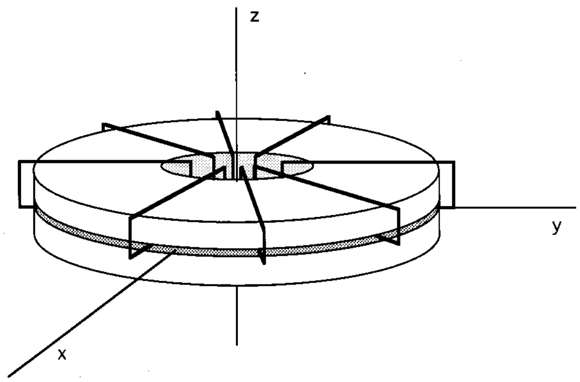

FIG. 2. Schematic representation of an annular junction with a flux quantum trapped in the upper electrode.

The central hole in the two electrodes plays the role of a major ''pinning center'' and I will not consider other possible conditions of trapping in this report. When the circumference of the junction is small with respect to λ_j, we can assume that the fluxon (the field lines) is uniformly spread all over the circle. Then the presence of a fluxon amounts to add a radial component to the external field. This component can be easily calculated. At a distance ρ the magnetic field crosses a surface $2\pi\rho d_{\mathrm{eff}}$. Then the magnetic field produced by n trapped fluxons (or antifluxons) in the junction barrier is given by

$$H_n=\frac{n\Phi_0}{2\pi\rho\mu_0 d_{\mathrm{eff}}}. \qquad (20)$$

Adding this component to the radial part of the external field, Eq. (2), the equations determining the phase become

$$\frac{\partial\varphi}{\partial\rho}=k\sin(\theta-\phi); \quad \frac{\partial\varphi}{\partial\phi}=-k\rho\cos(\theta-\phi)-n, \qquad (21)$$

from which by integration

$$\varphi(\rho,\phi)=k\rho\sin(\theta-\phi)-n\phi+\varphi_0. \qquad (22)$$

It is not difficult to show now that the critical-current diffraction pattern generalizing Eq. (14) when n fluxons is present reads

$$\frac{I_c}{I_0}=\left|\frac{2}{(kR)^2-(kr)^2}\int_{kr}^{kR}k\rho J_n(k\rho)dk\rho\right| \qquad (23)$$

or

$$\frac{I_c}{I_0}=\frac{2}{(1-\varepsilon^2)}\int_{\varepsilon}^{1}xJ_n(x\pi h)dx, \qquad (24)$$

where $\varepsilon=r/R$ and $n=0,1,2,...$; when $(R-r)\to 0$ then

$$\frac{I_c}{I_0}=|J_n(\pi h)|, \qquad (25)$$

which is the approximate result appearing in Ref. 3.

It is interesting to observe that the characteristic effect of the presence of trapped fluxons is to set at zero the critical current. This is easily seen from Eqs. (11) and (12), since, in the absence of an external magnetic field, $\varphi(\rho,\phi)=\varphi_0+n\phi$, and both the integrals in Eqs. (11) and (12) give no contribution. This property is preserved in the presence of an external field as shown by Eq. (24), for I_c vanishes at $h=0$, unless $n=0$. In conclusion the analytical expression [Eq. (14)] of the critical-current dependence on the magnetic field for a small annular junction has been constructed without any approximation. This formula contains as a special case the approximate result previously calculated,[2] and allows one to fix[3] the previous observed discrepancies. For the case in which fluxons are trapped in the junction electrodes, the corresponding exact analytical result (valid for any width of the junction ring width) is given in integral form by Eq. (23) or Eq. (24).

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