Complementary variational theorems for inhomogeneous superconductors

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We present complementary variational theorems for an inhomogeneous London superconductor, in which *both* the London penetration length $\lambda_L(\mathbf{r})$ and permeability $\mu(\mathbf{r})$ vary randomly. A characteristic feature here is the explicitly self-consistent coupling between the magnetic and supercurrent polarization effects due to these inhomogeneities. Our results may be important to composite systems containing magnetic (normal) *and* superconducting components. Applications of the theorems to such systems and their relationship to nonlinear composites will also be discussed. [S0163-1829(97)00713-3]

An inhomogeneous *local* superconductor is well known to be described by the free energy (in the London gauge div $\mathbf{A}=0$) (Ref. 1) as

$$\overline{U}=\frac{1}{8\pi}\int d\mathbf{r}\left[-\frac{4\pi}{c}\mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})+\mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})\right] , \qquad (1)$$

where $\mathbf{J}=-[c/4\pi\lambda_L^2(\mathbf{r})]\mathbf{A}$ is the local diamagnetic supercurrent, while $\mathbf{B}=\text{curl }\mathbf{A}=\mu(\mathbf{r})\mathbf{H}$ is the local magnetic induction field. We wish to emphasize at the outset the random, spatially varying character of *both* the material parameters λ_L and μ, and the coupling of the two terms in Eq. (1), which is in fact quite general. The only assumption is the validity of London-Maxwell electrodynamics[1] and it applies equally to inhomogeneous superconductors or superconductor/normal composites. It can of course be made specific by assuming a granular system, for example. This is strictly not necessary for the purpose of our paper, but it will be useful later on for illustrating its application to physical systems.

In this case we shall have superconducting grains α with a $\lambda_{L\alpha}$ and normal grains i with a μ_i, all of which are uniform within each grain. By virtue of a well-known vector identity [see Eq. (18) below], Eq. (1) for a granular model is then

$$\overline{U}=\frac{1}{8\pi}\sum_\alpha \int \mathbf{n}\cdot[\mathbf{H}_\alpha(\mathbf{r})\times\mathbf{A}_\alpha(\mathbf{r})]\ dS + \sum_i \int d\mathbf{r}\,\mathbf{B}_i(\mathbf{r})\cdot\mathbf{H}_i(\mathbf{r}) , \qquad (2)$$

where the first integral, over the surface of each superconducting grain is the London energy and the second integral over the volume of each normal grain is the magnetic energy. The complexity of this (random) boundary value problem Eq. (2), i.e., to find a solution for $\mathbf{A}(\mathbf{r})$ that will satisfy the boundary conditions on *each* grain, highlights the usefulness of our variational methods. Note also that the traditional form Eq. (1) has also some mathematical conveniences, for upon the standard variation with respect to $\mathbf{A}(\mathbf{r})$, we easily obtain the well-known London equation [Eq. (3)] as the Euler-Lagrange equation for the system, which is less transparent via Eq. (2).

For the oxide high-T_c superconductors, the coherence length ξ is known to be small, typically several Å, and the order parameter $\psi(\mathbf{r})$ is gradient free: $\nabla\psi(\mathbf{r})=0$, which is the characteristic of a local (London) superconductor. On macroscopic scales of several thousand Å, where μ and λ_L can vary either due to impurities or to stoichiometric variations in the microstructure, we cannot ignore the coupling between *both* magnetic and superconducting inhomogeneities in our boundary value problem. In fact this is an important question for experiments, since the effective parameters $\overline{\mu}$ and $\overline{\lambda_L}$ are accessible quantities, via the magnetization, microwave absorption, or μSR measurements.[2] A similar situation exists, no doubt, in other contexts; as in the electromagnetic properties of composites where the magnetic and dielectric properties vary macroscopically. Here the coupling is a direct consequence of the displacement current term in the Maxwell equations.[3,4]

A superior approach that can treat this coupling self-consistently and has the potential to go beyond effective-medium theory[5] is that based on variational theorems, the most well known being that derived by Hashin and Shtrikman[6] and its later generalizations, e.g., to elastic composites.[7] These theorems are rigorous and they provide additional insights on variationally optimal approximation schemes when detailed statistical information on the microstructure is absent. A revival of interest in variational methods is seen in recent works on dynamical problems, as in porous-elastic solids[8] and in strongly nonlinear composites.[9,10]

The appropriate mathematical tool for a variational treatment of coupled fields with Lagrangian densities like Eq. (1) is no doubt that formulated via a canonical Hamiltonian variational principle for complementary bounds,[11] which incidentally has interesting nonlinear generalizations. As far as we know, its application to inhomogeneous superconductors in a formulation involving generalized polarization fields[6] has not been obtained. Unfortunately, without the latter concepts, these variational methods[11] do not offer new physical insights as the optimum fields $\mathbf{H}$ and here also $\mathbf{A}$ are impossible to guess.[12] Bounds obtained as such using naive trial field functions $\mathbf{H}$ and $\mathbf{A}$ are generally inferior to an effective-medium approximation. In this paper we shall report on the derivation of two complementary variational theorems for the inhomogeneous London superconductor Eq. (1) which will (i) provide the foundations underlying earlier effective-medium approximations,[5] and (ii) open a path to study similar coupled systems and their dynamics. We shall first state these theorems, then provide the proofs for the superconductor Eq. (1) and later show their applications, which in appropriate limiting cases recover earlier results.[5]

The magnetic induction **B** and field intensity **H** for the superconductor Eq. (1), obey the London equation,[1]

$$\mathrm{curl}(\lambda_L^2(\mathbf{r})\mathrm{curl}\ \ \mathbf{H}) = -\mathbf{B}. \tag{3}$$

We first consider a homogeneous *finite* superconducting reference body with μ_0 and λ_{L0} and of volume V and surface S. This body is subjected to a prescribed surface potential $\psi_0(S)$ such that $\mathbf{H}_0(S) = -\mathrm{grad}\ \ \psi_0(S)$ on the infinitesimal outer boundary of S. In the bulk of this sample Eq. (3) is obeyed together with $\mathbf{B}_0 = \mu_0\mathbf{H}_0$, in which μ_0 and λ_{L0} are constants. We now replace this reference system by our inhomogeneous superconductor Eq. (1), *without in any way changing the prescribed surface potential*.

Theorem 1. With this reference system for which μ_0 and λ_{L0} are *homogeneous* but in which $\mathbf{H}_0(\mathbf{r})$ *is not necessarily uniform*,[12,13] the functional

$$\begin{aligned} U_\Phi = {} & \lambda_{L0}^2\langle \mathrm{curl}\ \ \mathbf{H}_0, \mathrm{curl}\ \ \mathbf{H}_0\rangle - \left\langle \mathbf{\Phi}_s, \frac{\mathbf{\Phi}_s}{\lambda_L^2 - \lambda_{L0}^2}\right\rangle \\ & + 2\langle \mathbf{\Phi}_s, \mathrm{curl}\ \ \mathbf{H}_0\rangle + \langle \mathbf{\Phi}_s, \mathrm{curl}\ \ \mathbf{H}'\rangle + \mu_0\langle \mathbf{H}_0, \mathbf{H}_0\rangle \\ & - \left\langle \mathbf{\Phi}_m, \frac{\mathbf{\Phi}_m}{\mu - \mu_0}\right\rangle + 2\langle \mathbf{\Phi}_m, \mathbf{H}_0\rangle + \langle \mathbf{\Phi}_m, \mathbf{H}'\rangle \quad , \end{aligned} \tag{4}$$

is stationary for arbitrary variations of the supercurrent polarization field $\mathbf{\Phi}_s$ and magnetic polarization field $\mathbf{\Phi}_m$, respectively, subject to the subsidiary conditions:

$$\mathrm{curl}\ \ \mathbf{\Phi}_s + \lambda_{L0}^2\ \ \mathrm{curl}\ \ \mathrm{curl}\ \ \mathbf{H}' = -\mathbf{\Phi}_m - \mu_0\mathbf{H}' \quad , \tag{5}$$

and the boundary conditions (for the parallel and normal components):

$$\mathbf{H}'_\parallel(S) = 0 \Rightarrow \mathrm{curl}_\perp\ \ \mathbf{H}'(S) = 0 \quad , \tag{6}$$

if

$$\mathbf{\Phi}_s = (\lambda_L^2 - \lambda_{L0}^2)\mathrm{curl}\ \ \mathbf{H} \tag{7}$$

and

$$\mathbf{\Phi}_m = (\mu - \mu_0)\mathbf{H} \quad . \tag{8}$$

Here $\mathbf{H}'$ is the magnetic perturbation field due to the inhomogeneities, i.e., $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$ and we have used a compact scalar product notation[11] in which $\langle \mathbf{f}, \mathbf{g}\rangle = (1/8\pi)\int d\mathbf{r}\ (\mathbf{f}\cdot\mathbf{g})$. Moreover the stationary functional is the physical energy functional of Eq. (1) which we can rewrite as

$$\overline{U_\Phi} = \langle \lambda_L^2\,\mathrm{curl}\ \ \mathbf{H}, \mathrm{curl}\ \ \mathbf{H}\rangle + \langle \mu\mathbf{H}, \mathbf{H}\rangle. \tag{9}$$

The complementary theorem that we can similarly prove requires an alternative viewpoint. Here the magnetic vector potential **A** and superconducting current **J** for the superconductor Eq. (1), obey the equivalent London equation[1]

$$\mathrm{curl}(\tilde{\mu}(\mathbf{r})\mathrm{curl}\ \ \mathbf{A}) = \frac{4\pi}{c}\mathbf{J}, \tag{10}$$

where $\tilde{\mu} = \mu^{-1}$. Our homogeneous *finite* superconducting reference body of volume V and surface S is defined by $\tilde{\mu}_0 = \mu_0^{-1}$ and $\tilde{\lambda}_{L0} = \lambda_{L0}^{-1}$. This body is now subjected to a prescribed surface magnetic vector potential $\mathbf{A}_0(S)$ [equivalently a superconducting current $\mathbf{J}_0(S)$] on the infinitesimal inner boundary of S. In the bulk of this sample Eq. (10) is obeyed together with $\mathbf{J}_0 = -(c/4\pi)\tilde{\lambda}_{L0}^2\mathbf{A}_0$, in which $\tilde{\mu}_0$ and $\tilde{\lambda}_{L0}$ are constants. We now replace this reference system by our inhomogeneous superconductor Eq. (1), *without in any way changing the prescribed surface vector potential* $\mathbf{A}_0(S)$.

Theorem 2. With this reference system for which $\tilde{\mu}_0$ and $\tilde{\lambda}_{L0}$ are *homogeneous* but in which $\mathbf{A}_0(\mathbf{r})$ *is not necessarily uniform*,[12,14] the functional:

$$\begin{aligned} U_\Theta = {} & \tilde{\mu}_0\langle \mathrm{curl}\ \ \mathbf{A}_0, \mathrm{curl}\ \ \mathbf{A}_0\rangle - \left\langle \mathbf{\Theta}_m, \frac{\mathbf{\Theta}_m}{\tilde{\mu} - \tilde{\mu}_0}\right\rangle \\ & + 2\langle \mathbf{\Theta}_m, \mathrm{curl}\ \ \mathbf{A}_0\rangle + \langle \mathbf{\Theta}_m, \mathrm{curl}\ \ \mathbf{A}'\rangle + \tilde{\lambda}_{L0}^2\langle \mathbf{A}_0, \mathbf{A}_0\rangle \\ & - \left\langle \mathbf{\Theta}_s, \frac{\mathbf{\Theta}_s}{\tilde{\lambda}_{L0}^2 - \tilde{\lambda}_L^2}\right\rangle + 2\langle \mathbf{\Theta}_s, \mathbf{A}_0\rangle + \langle \mathbf{\Theta}_s, \mathbf{A}'\rangle, \end{aligned} \tag{11}$$

is stationary for arbitrary variations of the supercurrent polarization field $\mathbf{\Theta}_s$ and magnetic polarization field $\mathbf{\Theta}_m$, respectively, subject to the subsidiary conditions:

$$\mathrm{curl}\ \ \mathbf{\Theta}_m + \tilde{\mu}_0\ \ \mathrm{curl}\ \ \mathrm{curl}\ \ \mathbf{A}' = -\mathbf{\Theta}_s - \tilde{\lambda}_{L0}^2\ \ \mathbf{A}' \quad , \tag{12}$$

and the boundary conditions:

$$\mathbf{A}'_\parallel(S) = 0 \Rightarrow \mathrm{curl}_\perp\ \ \mathbf{A}'(S) = 0 \quad , \tag{13}$$

if

$$\mathbf{\Theta}_m = (\tilde{\mu} - \tilde{\mu}_0)\mathrm{curl}\ \ \mathbf{A} \tag{14}$$

and

$$\mathbf{\Theta}_s = (\tilde{\lambda}_L^2 - \tilde{\lambda}_{L0}^2)\mathbf{A} \quad . \tag{15}$$

Here $\mathbf{A}'$ is the supercurrent perturbation field due to the inhomogeneities, i.e., $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}'$ and the stationary functional is the physical energy functional of Eq. (1) which can also be rewritten as

$$\overline{U_\Theta} = \langle \tilde{\mu}\ \ \mathrm{curl}\ \ \mathbf{A}, \mathrm{curl}\ \ \mathbf{A}\rangle + \langle \tilde{\lambda}_L^2\mathbf{A}, \mathbf{A}\rangle \quad . \tag{16}$$

In addition these stationary energy functionals Eqs. (9) and (16) are absolute maximum, i.e., $\overline{U_\Phi} \geqslant U_\Phi$ (convexity) if $\lambda_L > \lambda_{L0}$ and $\mu > \mu_0$ or absolute minimum, i.e., $\overline{U_\Phi} \leqslant U_\Phi$ (concavity) if $\lambda_L < \lambda_{L0}$ and $\mu < \mu_0$, the situation being reversed for the complementary functionals $\overline{U_\Theta}$ and U_Θ, accordingly. Thus $U_\Phi \leqslant \overline{U_\Phi} = \overline{U_\Theta} \leqslant U_\Theta$ (for $\lambda_L > \lambda_{L0}$ and $\mu > \mu_0$) and $U_\Theta \leqslant \overline{U_\Phi} = \overline{U_\Theta} \leqslant U_\Phi$ (for $\lambda_L < \lambda_{L0}$ and $\mu < \mu_0$), are the two-sided (rather tight) bounds in this paper. Equality of these bounds occur if the polarization fields $\mathbf{\Phi}$'s and $\mathbf{\Theta}$'s are exact.

Before discussing the proofs we emphasize that these theorems are nontrivial extensions of the classic Hashin-

Shtrikman theorems.[6] This is because (i) here the reference fields $\mathbf{H}_0$ and $\mathbf{A}_0$ are not homogeneous in general,[12] (ii) the appropriate choice of conjugate fields, curl $\mathbf{H}'$ and curl $\mathbf{A}'$ is not obvious, complicating the task (iii) for finding the appropriate subsidiary conditions Eqs. (5) and (12). The latter are dictated by the canonical Hamilton equations of motion,[11] in order that *both* polarization fields $\boldsymbol{\Phi}_m$, $\boldsymbol{\Phi}_s$ *and* their respective conjugates $\mathbf{H}'$, curl $\mathbf{H}'$ can be subjected to independent arbitrary variations. In turn this poses certain difficulties in extending the classic arguments[6] that must be resolved. The proofs proceed in three steps: (a) proof of stationarity (b) proof of convexity or concavity, and finally (c) proof that the extremums are indeed the physical energy functionals. We show these steps for Theorem 1.

(a) This step is by far the easiest. The first-order variation δU_Φ, upon use of Eqs. (7) and (8) is easily shown to be

$$\delta U_\Phi = \langle \boldsymbol{\Phi}_s, \mathrm{curl}\ \ \delta\mathbf{H}' \rangle - \langle \delta\boldsymbol{\Phi}_s, \mathrm{curl}\ \ \mathbf{H}' \rangle + \langle \boldsymbol{\Phi}_m, \delta\mathbf{H}' \rangle - \langle \delta\boldsymbol{\Phi}_m, \mathbf{H}' \rangle \ . \tag{17}$$

The last two terms, which are similar but *not* equivalent to Ref. 6, must here be reduced using the variation on Eq. (5), i.e., curl $\delta\boldsymbol{\Phi}_s + \lambda_{L0}^2$ curl curl $\delta\mathbf{H}' = -\delta\boldsymbol{\Phi}_m - \mu_0 \delta\mathbf{H}'$ and a well-known vector identity:

$$\langle \mathbf{h}, \mathrm{curl}\ \ \mathbf{A} \rangle = \langle \mathbf{A}, \mathrm{curl}\ \ \mathbf{h} \rangle + \int \mathbf{n}\cdot(\mathbf{h}\times\mathbf{A})\, dS, \tag{18}$$

where the surface terms vanish by virtue of Eq. (6). Then

$$\langle \boldsymbol{\Phi}_m, \delta\mathbf{H}' \rangle = -\mu_0 \langle \mathbf{H}', \delta\mathbf{H}' \rangle - \langle \boldsymbol{\Phi}_s, \mathrm{curl}\ \ \delta\mathbf{H}' \rangle - \lambda_{L0}^2 \langle \mathrm{curl}\ \ \mathbf{H}', \mathrm{curl}\ \ \delta\mathbf{H}' \rangle \ . \tag{19}$$

Similarly the last term in Eq. (17) can also be shown to be

$$\langle \delta\boldsymbol{\Phi}_m, \mathbf{H}' \rangle = -\mu_0 \langle \delta\mathbf{H}', \mathbf{H}' \rangle - \langle \delta\boldsymbol{\Phi}_s, \mathrm{curl}\ \ \mathbf{H}' \rangle - \lambda_{L0}^2 \langle \mathrm{curl}\ \ \delta\mathbf{H}', \mathrm{curl}\ \ \mathbf{H}' \rangle, \tag{20}$$

so that stationarity $\delta U_\Phi = 0$ is thus proved.

(b) The proof of convexity is also straightforward[6] using the method outlined above, and we arrive at

$$\delta U_\Phi^2 = -\left\langle \delta\boldsymbol{\Phi}_s, \frac{\delta\boldsymbol{\Phi}_s}{\lambda_L^2 - \lambda_{L0}^2} \right\rangle - \left\langle \delta\boldsymbol{\Phi}_n, \frac{\delta\boldsymbol{\Phi}_m}{\mu - \mu_0} \right\rangle - \mu_0 \langle \delta\mathbf{H}', \delta\mathbf{H}' \rangle - \lambda_{L0}^2 \langle \mathrm{curl}\ \ \delta\mathbf{H}', \mathrm{curl}\ \ \delta\mathbf{H}' \rangle, \tag{21}$$

hence proving an absolute maximum condition, i.e., $\delta U_\Phi^2 < 0$ if $\lambda_L > \lambda_{L0}$ and $\mu > \mu_0$. The proof of concavity is however not as straightforward. First the equation: div $\boldsymbol{\Phi}_m = -\mu_0$div $\mathbf{H}'$ as satisfied trivially by Eq. (5), leads to the variation: $\delta\boldsymbol{\Phi}_m = -\mu_0 \delta\mathbf{H}' + \delta\mathbf{C}_n$, where $\mathbf{C}_n$ is a divergence free function. Equation (5) itself can be suitably manipulated into the form $\boldsymbol{\Phi}_s + \lambda_{L0}^2$curl $\mathbf{H}' = \mathbf{C}_s$, where $\mathbf{C}_s$ is however not divergence free. Upon variation and substitutions of $\delta\mathbf{H}'$ and curl $\delta\mathbf{H}'$ thus obtained into Eq. (21), then

$$\delta U_\Phi^2 = -\left\langle \delta\boldsymbol{\Phi}_s, \delta\boldsymbol{\Phi}_s \left[\frac{1}{\lambda_L^2 - \lambda_{L0}^2} + \frac{1}{\lambda_{L0}^2} \right] \right\rangle - \left\langle \delta\boldsymbol{\Phi}_n, \delta\boldsymbol{\Phi}_n \left[\frac{1}{\mu - \mu_0} + \frac{1}{\mu_0} \right] \right\rangle + \frac{1}{\lambda_{L0}^2} \langle \delta\mathbf{C}_s, \delta\mathbf{C}_s \rangle + \frac{1}{\mu_0} \langle \delta\mathbf{C}_n, \delta\mathbf{C}_n \rangle \ , \tag{22}$$

thereby proving the absolute minimum property, i.e., $\delta U_\Phi^2 > 0$ if $\lambda_L < \lambda_{L0}$ and $\mu < \mu_0$.

(c) No new steps are required for the proof of the physical energy apart from the judicious use of the vector identity Eq. (18) and the London equation for $\mathbf{H}_0$. The algebra, however, is lengthy and will not be reproduced here. We can easily arrive at the result Eq. (9). The proof for Theorem 2 is similar and shall not be repeated here.

We shall now consider the application of these theorems in the light of earlier results.[5] The key to applications is the choice of the reference system (μ_0, λ_{L0}), which represents our lack of detailed statistical information on the inhomogeneities. Considerable care must be exercised in defining the effective material parameters $\overline{\lambda_L}$ and $\overline{\mu}$ by the use of the energy expressions Eqs. (9) or (16). This is in contrast to the simple expression:

$$\overline{\mu} = \frac{8\pi \overline{U_\Phi}}{V\mathbf{H}_0^2} \tag{23}$$

[see Eq. (3.1) in Ref. 6], since our total energy Eq. (9) contains two parts which in general *are coupled*. As such any thermodynamic measurement will in general involve *both* $\overline{\lambda_L}$ and $\overline{\mu}$. A full discussion of this point will have to be deferred elsewhere.[16] It suffices here to note that physically one *must* choose the reference parameters *in accordance with the experimental situation*. In the following we shall restrict ourselves to a binary granular superconductor/normal composite model Eq. (2), in the two limiting cases $T \to T_c$ and $T \to 0$, respectively, where the simple expression Eq. (23) or its analog for superconductors

$$\overline{\tilde{\lambda}_L^2} = \frac{8\pi \overline{U_\Theta}}{V\mathbf{A}_0^2}, \tag{24}$$

is valid.

We shall consider for this system, a normal component $(\mu = \mu_g)$, while the other component is superconducting but nonmagnetic $(\mu = 1, \lambda_L = \lambda_{Ls})$. The statistics of this composite is assumed to be random, characterized solely by the volume fractions of the respective materials, which in analogy with Ref. 6 defines an equivalent concentric shell model of two components. Here we shall choose $\lambda_{L0} \to \infty$, (i.e., $T \to T_c$), so that the field is assumed to penetrate the sample entirely; *then* $\mathbf{H}_0$ *is now uniform* while $\boldsymbol{\Phi}_s$ can be taken to be zero. The latter follows because we can now *replace* the superconducting component by an equivalent magnetic component whose permeability is μ_s. This equivalent permeability can be easily determined, in this case by equating the magnetic dipole polarizability of a superconducting sphere with a normal sphere of permeability μ_s, as in Ref. 5:

$$\mu_s=\mu_0\left(\frac{1-Z_L(a/\lambda_{Ls})}{1+\frac{1}{2}Z_L(a/\lambda_{Ls})}\right) \quad , \tag{25}$$

where a is the radius of the sphere and $Z_L(a/\lambda_{Ls})$ is related to the Langevin function.[5] Then the equivalent ansatz[6] for our two-phase system, one with permeability μ_g and another with permeability μ_s is in this case

$$\mathbf{\Phi}_m=\sum_{i\in\substack{\text{normal}\\\text{phase}}}\mathbf{T}_g\theta_i(\mathbf{r})+\sum_{j\in\substack{\text{superconducting}\\\text{phase}}}\mathbf{T}_s\theta_j(\mathbf{r}) \quad , \tag{26}$$

where now the magnetic dipole polarizations $\mathbf{T}_g$ and $\mathbf{T}_s$ are uniform.[15] The evaluation of the integrals in Eq. (4) using the above ansatz now follows essentially the same route as Ref. 17. After minimizing with respect to $\mathbf{T}_g$ and $\mathbf{T}_s$ and using the definition Eq. (23), we obtain the Hashin-Shtrikman-type bound:

$$\overline{\mu}\geqslant\mu_0+\mu_0 v_g\left(\frac{\mu_g-\mu_0}{\mu_g+2\mu_0}\right)-\mu_0 v_s\frac{1}{2}Z_L(a/\lambda_{Ls}) \quad , \tag{27}$$

where v_g and v_s are the volume fractions of normal and superconductor grains, respectively. The effective-medium approximation $\overline{\mu}=\mu_0$ now recovers the result Eq. (14a) of Refs. 5 and 18. A shortcoming of the latter theory is now clearly seen here, since at percolation $\mu_0=\overline{\mu}\to 0$, which is incompatible with the assumption $\lambda_{L0}\to\infty$ and a uniform reference field $\mathbf{H}_0$.

For the complementary case, we shall choose for the reference system a superconductor with $\tilde{\mu}_0\to\infty$ and $\mathbf{A}_0$ *is now uniform*, appropriate to the same sample but near $T=0$. Theorem 2 is then applicable in the same way as the above example to yield an effective $\overline{\lambda}_L$ using the definition Eq. (24):[19]

$$\tilde{\lambda}_L^2\leqslant\tilde{\lambda}_{L0}^2+\tilde{\lambda}_{L0}^2 v_s\left(\frac{\tilde{\lambda}_{Ls}^2-\tilde{\lambda}_{L0}^2}{\tilde{\lambda}_{Ls}^2+2\tilde{\lambda}_{L0}^2}\right)-\tilde{\lambda}_{L0}^2 v_g\frac{1}{2} \quad . \tag{28}$$

A more sophisticated ansatz than Eq. (26) will have to be investigated in a future study. Here we shall point out an interesting reformulation of the above problem by noting that Eq. (9) can also be written as $U_\Phi=\langle\mathbf{B},\mathbf{H}\rangle$ with the nonlinear magnetic induction:

$$\mathbf{B}=\mu\mathbf{H}\left[1+\frac{\lambda_L^2}{\mu}\frac{(\text{curl }\mathbf{H})^2}{\mathbf{H}^2}\right], \tag{29}$$

thereby showing that our coupled system can be viewed as a nonlinear inhomogeneous composite.[9] This relation offers a potentially useful *linear* perspective for nonlinear systems, by subsuming the nonlinearities of the permeability into the properties of a canonically conjugate field. We note that for computations, Eqs. (4) and (11) furnish ideal Monte Carlo-type algorithms hitherto uninvestigated, especially for superconductors.

In conclusion, we have derived complementary theorems for an inhomogeneous superconductor where the coupling due to inhomogeneities in λ_L and μ are treated self-consistently. Further insight is offered by the choice of the reference system to the experimental situation, as well as an alternate perspective for nonlinear composites. Generalizations to include weak links can also be achieved and will introduce new coupled fields with additional parameters.

[1] See, e.g., M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975), p. 105. The energy scale here is measured relative to the condensate.

[2] See, e.g., H. Jiang *et al.*, Phys. Rev. B **49**, 9924 (1994); V. Fesenko *et al.*, Physica C **211**, 343 (1993).

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[10] H.-C. Lee, K.W. Yu, and G.Q. Gu, J. Phys. Condens. Matter **7**, 8785 (1995).

[11] A.M. Arthurs, *Complementary Variational Principles* (Clarendon Press, Oxford, 1980).

[12] The field inside a superconducting sphere in an external field is generally nonuniform, unlike a dielectric or magnetic sphere; see, e.g., Refs. 5,14.

[13] This aspect of the reference system differs from all previous works with the exception of Ref. 9 in which μ_0 is not homogeneous, while $\mathbf{H}_0$ is, thereby complementing our case.

[14] Solutions for the magnetic vector potential inside a superconductor can be found in many textbooks dating back to F. London, *Superfluids* (John Wiley, New York, 1950), Vol. I.

[15] In general these polarizations, even if assumed to be identical for each phase (Ref. 12) are however not uniform, except for this limiting case, by construction.

[16] $\overline{\lambda}_L$ can be accessed independently, by microwave or μSR measurements (Ref. 2).

[17] D.J. Bergman, Phys. Rev. B **14**, 1531 (1976).

[18] The need to avoid double counting in effective-medium theories is now transparent here; see H. Rauh, A.M. Stoneham, and T.C. Choy, *Materials Modelling: From Theory to Technology* (Institute of Physics, London, 1992), p. 201.

[19] The equations div $\mathbf{J}=0$, curl $\mathbf{A}=0$, and $\mathbf{J}=-(c/4\pi\lambda_L^2)\mathbf{A}$ are now the direct analogs of div $\mathbf{B}=0$, curl $\mathbf{H}=0$, and $\mathbf{B}=\mu\mathbf{H}$.