Diagrammatic analysis of the two-state quantum Hall system with chiral invariance

S. Hikami and K. Minakuchi
Department of Pure and Applied Sciences, University of Tokyo, Meguro-ku, Komaba, Tokyo 153, Japan

(Received 30 October 1996)

The quantum Hall system in the lowest Landau level including the Zeeman term is studied by a two-state model, which has a chiral invariance. Using a diagrammatic analysis, we examine this two-state model with random impurity scattering and obtain the exact value of the conductivity at the Zeeman energy $E=\Delta$. We further study the conductivity at another extended state $E=E_1$ ($E_1>\Delta$). We find that the values of the conductivities at $E=\Delta$ and $E=E_1$ do not depend upon the value of the Zeeman energy Δ. We discuss also the case where the Zeeman energy Δ becomes a random field. [S0163-1829(97)04112-X]

I. INTRODUCTION

The critical behavior around the extended state in the two-dimensional quantum Hall system has been studied by various methods. Recently, the spin-degenerate case has attracted interest. The spin-up state and the spin-down state are almost degenerate when the Zeeman energy is small. It is considered that these two states can be mixed by impurity scattering.

Hikami, Shirai, and Wegner[1] considered a two-state model in the lowest Landau level, in which impurity scattering occurs only between different spin states. This model corresponds to the strong spin-orbit scattering limit, in which the spin should be changed at each impurity scattering. Remarkably, there are three extended states in this model: one is at the band center $E=0$ and the other two at $E=\pm E_1$. The conductivity at $E=0$ has been obtained exactly by a diagrammatic analysis and is $\sigma=e^2/2\pi^2\hbar$. This model has chiral invariance; the energy eigenvalues always appear in positive and negative pairs. The state at $E=0$ becomes a special state whose wave functions at different points can be hybridized due to this chiral invariance.[2] The density of states near $E=0$ shows a resonating behavior; it is enhanced and could be singular. At $E=0$, all higher-order scattering effects are canceled out for the conductivity and the localization effect is smeared out. This cancellation occurs not only for the Gaussian white-noise distribution but also for a general local non-Gaussian random distribution.[1]

This model has been examined further by numerical method.[3,4] The localization length exponent at $E=0$ is different from the usual quantum Hall system and belongs to a different universality class. Two other extended states at $E=E_1$ belong to the conventional quantum Hall universality class with the localization length exponent $\nu\simeq 2.3$.

The state at $E=0$ in this model has been suggested to be relevant to the chiral Dirac-Fermi model with a random vector potential,[5] which gives a singularity for the density of states. The value of the conductivity for this random vector potential model agrees with the value of the Hikami-Shirai-Wegner (HSW) model.

In previous numerical work,[4] the effect of the Zeeman term has been investigated. It has been shown that the density of state has a gap less than the Zeeman energy Δ and the extended state shifts from $E=0$ to the Zeeman energy $E=\Delta$. Note that the Zeeman term does not break the chiral invariance.

In this paper, we consider this extended HSW model with the Zeeman term using a diagrammatic method. We evaluate the exact value of the longitudinal conductivity at $E=\Delta$. Also we will discuss the extended state at $E=\pm E_1$ ($E_1>\Delta$), which may belong to the conventional quantum Hall universality class. We show exactly that the inclusion of the Zeeman term does not alter the values of the conductivities of the extended state at $E=\Delta$ and $E=E_1$. This result may be expected, but we verify it by a diagrammatic expansion method. When the Zeeman energy becomes a random variable, the situation will change. We briefly discuss this random Zeeman energy case using the diagrammatic method.

II. DIAGRAMMATIC ANALYSIS OF THE TWO-STATE QUANTUM HALL SYSTEM

The Hamiltonian for the two-spin state may be described by 2×2 matrix[4]

$$H=\frac{1}{2m}(p-eA)^2+\begin{pmatrix}\Delta & v^\dagger(r)\\ v(r) & -\Delta\end{pmatrix}, \tag{2.1}$$

where $v^\dagger(r)$ and $v(r)$ are random potentials at the spatial point r. The constant Δ represents the Zeeman energy. In Landau quantization, the up-spin state and the down-spin state acquire Zeeman energy $\pm\Delta$, respectively. The matrix of the second term of Eq. (2.1) acts on the spin state whose eigenstates are represented by a vector of two components. The distribution of the random potential $v(r)$ is assumed to be a Gaussian white-noise distribution, i.e.,

$$\langle v(r)\rangle_{\rm av}=\langle v^\dagger(r)\rangle_{\rm av}=0, \tag{2.2}$$

$$\langle v^\dagger(r)v(r')\rangle_{\rm av}=w\,\delta(r-r'). \tag{2.3}$$

The diagrammatic expansions for the one-particle Green function and the two-particle Green function for the lowest Landau level have been investigated.[6–8] In the case of no Zeeman term, a useful expansion for the diffusion constant D was derived, by which the exact value of the conductivity was obtained.[1] Note that, although we mainly consider the

Gaussian white-noise ditribution in this paper, the same argument can be extended to any local non-Gaussian random potential, as shown in Ref. 1.

In the two-dimensional case, the Green function for the lowest Landau level is simply expressed by

$$G(r)=\left\langle \left\langle r \left| \frac{1}{E-H+\frac{i}{2}\epsilon} \right| r \right\rangle \right\rangle_{\mathrm{av}} = \frac{1}{A_1+iA_2}, \quad (2.4)$$

where $G(r)$ is translationally invariant and A_1 and A_2 are real numbers, which are independent of r. The density of state $\rho(E)$ is expressed simply by $-A_2/\pi(A_1^2+A_2^2)$.

When the two-spin state model is considered, we have two different Green functions $G_A(r)$ and $G_B(r)$. A and B are the spin-up state and the spin down state, respectively. Using the self-energy Σ for A and B, we obtain, by definition,

$$A_1=2\pi\left(E-\frac{1}{2}\hbar\omega_c-\Delta-\frac{1}{2\pi}\mathrm{Re}\Sigma_A\right), \quad (2.5)$$

$$A_2=2\pi\left(\frac{\epsilon}{2}-\frac{1}{2\pi}\mathrm{Im}\Sigma_A\right), \quad (2.6)$$

$$B_1=2\pi\left(E-\frac{1}{2}\hbar\omega_c+\Delta-\frac{1}{2}\mathrm{Re}\Sigma_B\right), \quad (2.7)$$

$$B_2=2\pi\left(\frac{\epsilon}{2}-\frac{1}{2\pi}\mathrm{Im}\Sigma_B\right). \quad (2.8)$$

The diagrammatic expansion follows the previous studies and a convenient method for obtaining the coefficients of each order may be found in previous papers.[6,7] First, let us approximate the self-energy Σ by the Green function itself. Then we have $\Sigma_A=2\pi w/(B_1+iB_2)$ and $\Sigma_B=2\pi w/(A_1+iA_2)$. It may be convenient to represent two Green functions by $G_A=C_Ae^{i\theta_A}$ and $G_B=C_Be^{i\theta_B}$, and $x=C_AC_B(2\pi w)$. From Eqs. (2.6) and (2.8), in the limit $\epsilon\to 0$, we obtain $\theta_A=\theta_B$ and $x=1$. We represent the energy $E-\frac{1}{2}\hbar\omega_c$ simply by E.

From Eqs. (2.5) and (2.7), using $x^2=4\pi^2w^2/(A_1^2+A_2^2)(B_1^2+B_2^2)=1$, we obtain

$$A_1=2\pi(E-\Delta)-\frac{1}{2\pi w}B_1(A_1^2+A_2^2)$$

$$=2\pi(E-\Delta)-\frac{1}{w}(E+\Delta)(A_1^2+A_2^2)+A_1. \quad (2.9)$$

Thus we obtain $A_1^2+A_2^2=2\pi w(E-\Delta)/(E+\Delta)$. Similarly, we get $B_1^2+B_2^2=2\pi w(E+\Delta)/(E-\Delta)$. Then Eq. (2.5) becomes

$$A_1=2\pi(E-\Delta)-B_1\frac{(E-\Delta)}{(E+\Delta)}. \quad (2.10)$$

From Eq. (2.6), we have $A_2=2\pi wB_2/(B_1^2+B_2^2)=B_2(E-\Delta)/(E+\Delta)$. Further, noting that $A_1/A_2=B_1/B_2$ and from Eq. (2.10), we obtain the solution

$$A_1=\pi(E-\Delta). \quad (2.11)$$

Similarly, we get

$$B_1=\pi(E+\Delta). \quad (2.12)$$

The imginary parts A_2 and B_2 are obtained from $A_1^2+A_2^2=2\pi w(E-\Delta)/(E+\Delta)$. They are

$$A_2=\frac{1}{2}\sqrt{\frac{E-\Delta}{E+\Delta}}\sqrt{4w-(E^2-\Delta^2)}, \quad (2.13)$$

$$B_2=\frac{1}{2}\sqrt{\frac{E+\Delta}{E-\Delta}}\sqrt{4w-(E^2-\Delta^2)}. \quad (2.14)$$

The densities of states ρ_A and ρ_B are given by $\rho_A=-A_2/\pi(A_1^2+A_2^2)$ and $\rho_B=-B_2/\pi(B_1^2+B_2^2)$. Since $A_1^2+A_2^2=2\pi w(E-\Delta)/(E+\Delta)$ and $B_1^2+B_2^2=2\pi w(E+\Delta)/(E-\Delta)$, the density of states $\rho(E)$ has a gap between $-\Delta<E<\Delta$ and the inverse square root singularity at $E=\pm\Delta$. This behavior resembles the density of state of the superconductor. Note that we fix the Zeeman energy parameter Δ. Later we will consider the average over this Δ for the density of state.

We now go beyond this approximation by expanding the self-energy in the power series of w. In this two-state model with the Zeeman term, the diagrams are the same as the two-state model without the Zeeman term. Using the notation $A_2/A_1=-\tan\theta_A$, $B_2/B_1=-\tan\theta_B$, $x=C_AC_B(2\pi w)$, $G_A=C_Ae^{i\theta_A}$, and $G_B=C_Be^{i\theta_B}$ and $(-\pi/2<\theta_A<0$ and $-\pi/2<\theta_B<0)$, we have

$$\frac{\pi\epsilon}{A_2}=1-x\frac{\sin\theta_B}{\sin\theta_A}-\frac{1}{4}x^3\frac{\sin(3\theta_B+2\theta_A)}{\sin\theta_A}-\frac{2}{5}x^4\frac{\sin(4\theta_B+3\theta_A)}{\sin\theta_A}+\cdots, \quad (2.15)$$

$$\frac{\pi\epsilon}{B_2}=1-x\frac{\sin\theta_A}{\sin\theta_B}-\frac{1}{4}x^3\frac{\sin(3\theta_A+2\theta_B)}{\sin\theta_B}-\frac{2}{5}x^4\frac{\sin(4\theta_A+3\theta_B)}{\sin\theta_B}+\cdots. \quad (2.16)$$

Up to order x,[10] the expansion coefficients are given by Eq. (4.1) in Ref. 1.

At $E=\pm\Delta$, the phases θ_A and θ_B are equal to $-\pi/2$. This is evident within the first-order approximation (2.12) and (2.14); $A_2/A_1=\tan\theta_A=0$ and $B_2/B_1=\tan\theta_B=0$, which remain true beyond this order. We have evaluated the real part of the Green function numerically by the same method as in Ref. 4 and we indeed find that the real part vanishes at $E=\pm\Delta$.

The conductivity in the lowest Landau level is obtained from the Kubo formula by diagrammatic expansion.[8] As an alternative method, we can use the Einstein relation $\sigma=e^2D\rho$, where D is a diffusion constant. Here we use this Einstein relation, since a diagrammatic expansion is simpler for the diffusion constant.[6] The diffusion constant D is defined as the coefficient of q^2 in the inverse of the two-particle correlation function $K(q)$,

$$K(q)=\int\left\langle\left\langle r\left|\frac{1}{E-H+i0}\right|r'\right\rangle\right.\times\left.\left\langle r'\left|\frac{1}{E-H-i0}\right|r\right\rangle\right\rangle_{\mathrm{av}}\times e^{-iq(r-r')}d^2r'. \tag{2.17}$$

This $K(q)$ is expanded in the power series of w. The Feynman rule for this expansion may be seen in previous literatures.[6] We have for the small momentum q

$$\frac{K(q=0)}{K(q)}=1+\frac{D}{\epsilon}q^2. \tag{2.18}$$

Since we have two different propagators G_A and G_B, the two-particle correlation function $K(q)$ is also divided into two parts $K_A(q)$ and $K_B(q)$. The diffusion constant D also is defined differently by Eq. (2.18). The denominator ϵ in Eq. (2.18) can be expressed by Eqs. (2.15) and (2.16). Finally, we obtain the following equations, which are modification of the previous expression:[1]

$$\frac{2\pi D_B}{B_2}=1-\frac{x^3}{4}[\cos(2\theta_B+2\theta_A)+\cos(\theta_A+\theta_B)]+\cdots, \tag{2.19}$$

$$\frac{2\pi D_A}{A_2}=1-\frac{x^3}{4}[\cos(2\theta_B+2\theta_A)+\cos(\theta_A+\theta_B)]+\cdots. \tag{2.20}$$

The imaginary part of G_A and G_B are proportional to the density of states ρ. The conductivity σ_{xx} is given by the Einstein relation

$$\sigma_{xx}=\frac{1}{2}(e^2D_A\rho_A+e^2D_B\rho_B). \tag{2.21}$$

At $E=\pm\Delta$, we have $A_1/A_2=B_1/B_2=0$, as explained before. Thus we have $\theta_A=\theta_B=-\pi/2$. Remarkably, all corrections cancel out in Eqs. (2.19) and (2.20) except one. These cancellations are essentially the same as the previous case without the Zeeman term.[1] The conductivity σ_{xx} at $E=\pm\Delta$ becomes $e^2/2\pi^2\hbar$, which is the same value as for the case of no Zeeman term at $E=0$.

Thus we have found the exact value of the longitudinal conductivity at $E=\pm\Delta$. In previous numerical work,[4] this value was obscure, although the numerical value suggested was similar. We find that the conductivity at $E=\Delta$ is independent of the value of the Zeeman energy Δ.

The effect of the Zeeman term on the density of states may be discussed by a matrix model. A complex block matrix model, analogous to the HSW model, has been studied.[9,10] This matrix model has chiral invariance and the universal oscillation of the density of states near $E=0$ has been obtained in the large-N limit, where N is the size of the matrix. The matrix that we are discussing is given by

$$M=\begin{pmatrix}\Delta & v^\dagger\\ v & -\Delta\end{pmatrix}, \tag{2.22}$$

where Δ is a unit matrix multiplied by Δ and $v^\dagger$ is a $N\times N$ complex matrix. It is a straightforward exercise to evaluate the density of states for finite N using the Kazakov method.[10,11] The eigenvalues appear always in a positive and negative pair. The effect of this Zeeman term Δ is just a shift of the energy E. When we take the large-N limit first in this model, the density of states coincides with Eqs. (2.13) and (2.14). However, there is a crossover[10] to the oscillatory behavior near Δ in the small region of order $1/N$.

III. EXTENDED STATE AT $E=E_1$

As pointed out by previous numerical works,[3,4] there are extended states located at $E=\pm E_1$, which is greater than Δ. It was suggested that the universality class at $E=E_1$ is the same as the conventional one, with a localization exponent $\nu\simeq 2.3$.[3,4] The shift of the conventional extended state from $E=0$ to $E=E_1$ is due to the effective magnetic-field effect of the off-diagonal random potential v.

This shift of the conventional extended state at the middle of the band to $E=E_1$ has been observed using several models. The Chalker-Coddington network model[12] was extended to include the spin scattering and the shift of the extended state is noticed with the same localization exponent.[13–15]

Since the previous work for HSW model[1] did not discuss this extended state at $E=E_1$, we first consider this state without the Zeeman term $\Delta=0$. The diagrammatic expansion for D/A_2 was given up to order x^8 and we rewrite the result[1] here. The series for the diffusion constant D without the Zeeman term becomes

$$\begin{aligned}\frac{2\pi D}{A_2}=1&-\frac{1}{4}(\cos4\theta+\cos2\theta)x^3-(0.32\cos6\theta+0.16\cos4\theta+0.16)x^4-(1.142\,791\,551\,88\cos8\theta\\&+0.715\,564\,738\,292\cos6\theta+0.180\,555\,555\,555\cos4\theta+0.751\,951\,331\,49\cos2\theta+0.144\,168\,962\,351)x^5\\&-(4.016\,042\,129\,58\cos10\theta+2.107\,802\,167\,29\cos8\theta+0.228\,564\,968\,429\cos6\theta\\&+1.658\,373\,906\,74\cos4\theta+0.613\,624\,866\,859\cos2\theta+1.092\,055\,890\,84)x^6-(16.893\,859\,425\,2\cos12\theta\\&+8.856\,696\,127\,84\cos10\theta+1.347\,981\,581\,41\cos8\theta+5.491\,807\,258\,09\cos6\theta+1.751\,176\,105\,91\cos4\theta\\&+6.748\,550\,192\,06\cos2\theta+1.104\,036\,465\,47)x^7-(79.791\,511\,842\,0\cos14\theta+40.555\,240\,802\,6\cos12\theta\\&+5.999\,393\,350\,79\cos10\theta+20.196\,866\,445\,487\cos8\theta+4.421\,537\,815\,874\,7\cos6\theta\\&+23.475\,132\,715\,585\cos4\theta+6.492\,652\,058\,833\,1\cos2\theta+12.477\,855\,103\,819)x^8+\cdots,\end{aligned} \tag{3.1}$$

where the variable x is solved by the asymptotic expansion of Eq. (2.15). Setting $\theta_A=\theta_B$ and $\epsilon=0$ in Eq. (2.15), we solve x using the series up to the third order,

$$x \simeq 1-\frac{1}{4}\frac{\sin 5\theta}{\sin\theta}. \tag{3.2}$$

This approximation shows the maximum of x at $\theta\simeq -0.9$. The maximum value of x becomes approximately 1.3 and the value of x becomes zero for $\theta\to 0$.

This is quite similar to the case of the conventional quantum Hall case: the exact value[16] of x at the band center is $x=4/\pi=1.2732$ and x becomes zero for $\theta\to 0$. Thus the point $\theta=-0.9$ for this two-state quantum Hall system corresponds to the band center of the one-state quantum Hall system. The shift appears due to the off-diagonal two-state random potential. We insert the value of Eq. (3.2) in Eq. (3.1),

$$\frac{2\pi D}{A_2}=1-\frac{1}{4}(\cos 4\theta+\cos 2\theta)\left(1-\frac{1}{4}\frac{\sin 5\theta}{\sin\theta}\right)^3. \tag{3.3}$$

The maximum of $2\pi D/A_2$ is 1.6 at $\theta=-0.9$. The conductivity σ is obtained by multiplying a factor $e^2\sin^2\theta/2\pi^2\hbar$ by the value $2\pi D/A_2$, due to the Einstein relation. We have analyzed here up to order x^3. We think that the maximum peak of the conductivity remains finite for the higher-order analysis; namely, we expect that the state at $\theta=-0.9$ corresponds to the band center of the one-state quantum Hall system and becomes extended. This is consistent with the previous numerical result,[4] which shows that there is an extended state at $E=E_1$ except $E=0$. The states of energy $0<E<E_1$ and $E>E_1$ are considered to be localized. For the investigation of localization, we need the renormalization-group analysis via the $1/N$ expansion,[6] which we do not discuss here.

For the Zeeman case $(\Delta\neq 0)$, the series (3.1) is modified as Eqs. (2.19) and (2.20), where 2θ is replaced by $\theta_A+\theta_B$. In general, $\theta_A\neq\theta_B$. The range of these angles is between $-\pi/2$ and 0. We assume that there is an extended state at $E=E_1$ for the Zeeman case. We find that if $\theta_A+\theta_B$ is the same as the critical value θ_c in Eq. (3.1), we have the same expression for Eqs. (2.19) and (2.20). Since there is one extended state, we have $\theta_A=\theta_B$ at $E=E_1$. If we have an extended state for $\theta_A\neq\theta_B$, it may contradict the assumption that there is only one extended state at $E=E_1$. There is a duality between the A state and B state. We find that the same conductivity as in the case of no Zeeman term at $E=E_1$, since $\theta_A=\theta_B=\theta_c$. The conductivity is obtained from Eq. (2.19) by multiplying a factor $\sin^2\theta$, which is $A_2^2/(A_1^2+A_2^2)$ for the case $A_1\neq 0$. Indeed our previous numerical result shows this behavior. This argument of the equivalence does not determine the absolute value of the conductivity, but it verifies that the value of the conductivity at $E=E_1$ does not depend upon the Zeeman energy Δ.

IV. RANDOM ZEEMAN ENERGY MODEL

In the previous sections, we assumed that the Zeeman energy Δ is a fixed constant. When Δ in Eq. (2.1) is a random field, which depends upon the spatial coordinate r, the situation is different. The distribution of this random field $\Delta(r)$ is Gaussian. We will discuss this random Zeeman energy model using a diagrammatic expansion method.

Instead of the Zeeman energy Δ, we represent it now by a random field $u(r)$. Then the second term of Eq. (2.1) becomes

$$V(r)=\begin{pmatrix} u(r) & v^\dagger(r)\\ v(r) & -u(r)\end{pmatrix}, \tag{4.1}$$

where r is the place of impurity scattering. This model represents the spin flip at r due to the random field v and the random Zeeman energy by $u(r)$. There is no correlation between $v(r)$ and $u(r)$. The matrix $V(r_1)$ does not commute with the matrix $V(r_2)$. We have to consider the successive operation of the random scattering at $r_1,r_2,\ldots,r_N$ on the spin eigenstate. The spin eigenstate is represented by a vector of two components. The random variable u has the average

$$\langle u(r)u(r')\rangle_{\rm av}=w'\delta(r-r'). \tag{4.2}$$

The diagrammatic expansions of Eqs. (2.15) and (2.16) become a series of scattering strengths w and w'. Note that some terms have a negative sign due to the minus sign in Eq. (4.1) in the matrix element.

In this random Zeeman energy model, the chiral invariance is broken. The scattering appears between state A and state B and also between the same spin state due to the diagonal random field $u(r)$.

We find that after averaging the multiplication of the matrix $V(r_i)$ over the random distribution, the nonvanishing diagrams can be expressed by assigning the indices A and B for the Green function. The self-energy of Σ_A becomes, by diagrammatic expansion,

$$\begin{aligned}\Sigma_A={}&w'G_A+wG_B-ww'G_AG_B^2\\&-\frac{1}{4}(w'^3G_A^5+w^3G_A^2G_B^3+3ww'^2G_A^2G_B^3)\\&-\frac{1}{3}(3w^3G_A^5+3ww'^2G_A^2G_B^3-2ww'G_AG_B^4\\&+w'w^2G_AG_B^4-4w'^2wG_A^3G_B^2+2w'w^2G_A^3G_B^2)+\cdots.\end{aligned} \tag{4.3}$$

From this equation, we have

$$\begin{aligned}\frac{\pi\epsilon}{A_2}={}&1-\frac{1}{\sin\theta_A}(w'C_A^2\sin\theta_A+wC_BC_A\sin\theta_B)\\&+ww'C_A^2C_B^2\frac{\sin(\theta_A+2\theta_B)}{\sin\theta_A}\\&-\frac{1}{4}\left(w'^3C_A^6\frac{\sin 5\theta_A}{\sin\theta_A}+(w^3+3w'^2w)\right.\\&\left.\times C_A^3C_B^3\frac{\sin(2\theta_A+3\theta_B)}{\sin\theta_A}\right)\\&-\frac{1}{3}\left(3w^3C_A^6\frac{\sin 5\theta_A}{\sin\theta_A}+3ww'^2C_A^3C_B^3\frac{\sin(2\theta_A+3\theta_B)}{\sin\theta_A}\right.\\&+(w'w^2-2ww'^2)C_A^2C_B^4\frac{\sin(\theta_A+4\theta_B)}{\sin\theta_A}\\&\left.+(2w'w^2-4w'^2w)C_A^4C_B^2\frac{\sin(3\theta_A+2\theta_B)}{\sin\theta_A}\right)+\cdots.\end{aligned} \tag{4.4}$$

By the symmetry between A and B states, we are able to set $\theta_A = \theta_B$ and $C_A = C_B$. Then, Eq. (4.4) becomes simpler.

It may be interesting to consider three different cases: (i) $w' \ll w$, (ii) $w' \sim w$, and (iii) $w' \gg w$. Case (i) corresponds to the two-state model, which we have discussed previously for $\Delta = 0$. The perturbation of the parameter w' can be obtained. Case (ii) shows the strong effect of the random field $u(r)$. When, for example, $w' = \frac{1}{2}w$, the series of Eq. (2.15) has an alternative sign, and when $\theta_A = \theta_B = -\pi/2$ at $E = 0$, the density of states is suppressed. This behavior is similar to that for the gap state for the nonvanishing Zeeman energy Δ, which we have discussed in Sec. II. Case (iii) is similar to the conventional quantum Hall sysytem since two states A and B can be decoupled completely in the limit $w \to 0$. The extended energy $E = E_1$ approaches $E = 0$.

Case (i) can be studied in the perturbation of $1/N$. We need to generalize the model to the N-orbital model. The random field $u(r)$ in Eq. (4.1) changes to $u(r)I$, where I is an $N \times N$ unit matrix. v is also a complex $N \times N$ matrix. The density of states in the $1/N$ expansion shows the logarithmic singularity[1] at order $1/N^2$ for $w' = 0$. In the lowest order of w' and for the large N, Eq. (2.15) becomes, at $E = 0$,

$$\frac{\pi\epsilon}{A_2} = 1 - wC^2 - \frac{w'}{N}\left(\frac{wC^2}{1 - wC^2}\right) + \frac{d_1}{N^2}\ln^2(1 - wC^2) + \cdots . \tag{4.5}$$

Up to order $1/N$, solving Eq. (4.5), we obtain

$$wC^2 = 1 + \frac{w'}{2Nw} \pm \sqrt{\frac{w'}{Nw}} . \tag{4.6}$$

Thus the logarithmic divergence of the density of state in Eq. (4.5) is smeared out for small w' since $\ln^2\epsilon$ changes to $\ln^2 w'$.

In the presence of w', the conductivity $\sigma = e^2/2\pi^2\hbar$ also changes. When $w' = 0$, at the band center $E = 0$, the logarithmic terms in the diffusion constant of order $1/N^2$ are canceled by the vertex corrections and the conductivity remains finite. Indeed, each diagram for the higher-order term cancels completely, not just the logarithmically divergent part, as we have seen in Eq. (3.1). When $w' \neq 0$, this cancellation does not occur at $E = 0$ and the logarithmic term exists. This logarithmic term leads to the decrease of the diffusion constant and eventually the state becomes localized. Thus there is a localization for $E = 0$ when $w' \neq 0$.

V. CONCLUSION

In this paper, we have evaluated the exact value of the conductivity of the two-state model including the fixed Zeeman term at the Zeeman energy $E = \Delta$ and we find the value $\sigma = e^2/2\pi^2\hbar$, which is independent of Δ. We also observed that this result is consistent with a previous numerical result.[4] We have discussed that the conductivity at $E = E_1$ for the case of nonvanishing Zeeman energy is the same as the conductivity at $E = E_1$ without the Zeeman energy. Thus the effect of the Zeeman term does not alter the values of the conductivities at $E = \Delta$ and $E = E_1$.

We discussed how the situation is modified when the Zeeman energy becomes a random field, which obeys the Gaussian white-noise distribution. Then, the diagrammatic expansion has two parameters w and w'. We find, in the first order of w', the cut-off of the singularity of the density of states, which leads to the localization at $E = 0$.

ACKNOWLEDGMENTS

This work was supported by a Grant-in-Aid for Scientific Reseach by the Ministry of Education, Science and Culture, and by CREST of Japan Science and Technology Corporation.

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