

## Quantum fluctuations in the equilibrium state of a thin superconducting loop

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We study the oscillatory flux dependence of the supercurrent in a thin superconducting loop, closed by a Josephson junction. Quantum fluctuations of the order parameter in the loop affect the shape and renormalize the amplitude of the supercurrent oscillations. In a short loop, the amplitude of the sinusoidal flux dependence is suppressed. In a large loop, the supercurrent shows a sawtooth dependence on flux in the classical limit. Quantum fluctuations not only suppress the amplitude of the oscillations, but also smear the cusps of the sawtooth dependence. The oscillations approach a sinusoidal form with increasing fluctuation strength. At any finite length of the loop, the renormalized current amplitude is finite. This amplitude shows a power-law dependence on the junction conductance, with an exponent depending on the low-frequency impedance of the loop. [S0163-1829(97)05106-0]

### I. INTRODUCTION

Quantum effects in ultrasmall Josephson junctions have been studied intensely now for more than a decade, both experimentally<sup>1</sup> and theoretically.<sup>2</sup> The most important manifestation of quantum fluctuations is the well-known macroscopic quantum tunneling of the phase across a current-biased junction.<sup>1</sup> This phenomenon leads to the observation of events of quantum phase slip at a bias that is relatively close to the critical current.

More generally, macroscopic quantum tunneling causes a finite voltage to appear at any finite current. The corresponding  $I$ - $V$  characteristic may be nonlinear, and in the limit of zero temperature one finds  $V=AI^\gamma$ , with an exponent  $\gamma$  which depends on the impedance of the leads. This relation may be derived in the framework of the Caldeira-Leggett model<sup>3</sup> that treats quantum transitions between neighboring minima of a tilted washboard potential in the presence of a dissipative “environment.” It has been shown in Refs. 4 and 5 that the coefficient  $A$  is proportional to the square of the “bare” (i.e., unaffected by the environment) tunnel matrix element for transitions between the two minima of the potential. The dual result for a voltage-biased junction shows that the dc current in such a junction is proportional to the square of the Josephson energy of the junction, also unrenormalized by the environment.<sup>6-8</sup>

Actually, these two complementary results follow from very similar treatments of two closely related models, the Caldeira-Leggett model,<sup>3</sup> and the electromagnetic environment model.<sup>9,10</sup> The effective boundary conditions for the quantum fluctuations of the “environment modes” in these treatments do not depend on the Josephson energy of the junction itself. This approach (adequate for most existing experiments) is absolutely legitimate in the case of weak fluctuations of the phase of the order parameter across the junction. A more cautious analysis is needed, though, if these fluctuations are strong. Indeed, quantum phase fluctuations in

a one-dimensional (1D) superconductor are known to diverge logarithmically with length. If these fluctuations would result in a diverging random phase across the junction, the Josephson energy of the system would average to zero, and neither of the quoted results would be true.

We will show that a finite renormalized Josephson energy can arise because the junction itself affects the fluctuations of the environment.<sup>11</sup> Simultaneously, the modes of the environment renormalize the plasmon oscillations in the junction. This mutual influence of different parts of the circuit makes the separation of it on two entities—the junction with a fixed capacitance and the environment—to be somewhat a matter of convention.

We consider a thin superconducting loop which contains a Josephson junction. The “environment modes” of the loop consist of propagating plasmon modes with a soundlike dispersion.<sup>12</sup> The most straightforward way to observe a possible renormalization of the Josephson energy, is to phase bias the junction by threading a flux  $\Phi$  through the loop and measure the loop magnetization, which is proportional to the Josephson current  $J=J(\Phi)$ . The renormalization leads to values of the critical current  $J_c$  which are smaller than one would expect from the mean-field result,  $J_c^0=\pi\Delta G/(2e)$ , where  $G$  is the conductance of the junction, and  $\Delta$  is the superconducting gap in the loop. For a large loop, it also changes the flux dependence  $J(\Phi)$ .

The paper is organized as follows. The model for the loop with the junction is presented in Sec. II, followed by a qualitative discussion of the dependence of  $J$  on  $\Phi$  in Sec. III. The problem of the renormalization of the Josephson coupling is treated in Sec. IV where we make use of the similarity to the problem of quantum Brownian motion in a periodic potential.<sup>13</sup> The effect of the macroscopic quantum tunneling on the phase dependence of the Josephson current for a relatively large loop is discussed in Sec. V. We employ the analogy with the problem of pinning of a 1D charge density wave,<sup>14</sup> or Wigner crystal,<sup>15</sup> and use an instanton

approach<sup>16</sup> to calculate  $J(\Phi)$ . Concluding remarks can be found in Sec. VI.

## II. THE MODEL

We consider a superconducting wire of length  $L$  and small cross-sectional area  $S = a \times a$ , which is embedded in a medium with dielectric constant  $\epsilon$ . The wire is closed by a Josephson junction to form a loop. The bare Josephson energy of the junction is  $E_J^0 \equiv \pi \Delta \hbar G / (4e^2)$ , and its charging energy is  $E_c = 4e^2 / C$ . Perpendicular to the loop, a magnetic field  $\mathcal{H}$  is applied, such that a flux  $\mathcal{H}L^2 / (4\pi)$  threads the loop. We introduce the corresponding phase  $\Phi = \mathcal{H}L^2 / (2\Phi_0)$ , where  $\Phi_0$  is the superconducting flux quantum.

The low energy excitation spectrum of this system can be described in terms of the phase of the order parameter  $\varphi(x)$  by the Lagrangian  $\mathcal{L} = K - U$ , where

$$K = \int_0^L dx \frac{\hbar^2}{2e_c} [\dot{\varphi}(x)]^2 + \frac{\hbar^2 [\dot{\varphi}(L) - \dot{\varphi}(0)]^2}{2E_c} \quad (1)$$

and

$$U = \int_0^L dx \frac{\hbar^2 n_s S}{8m} \left( \frac{\partial \varphi(x)}{\partial x} - \frac{\Phi}{L} \right)^2 - E_J^0 \cos[\varphi(L) - \varphi(0)]. \quad (2)$$

Here,  $1/e_c = [8e^2 \ln(R/a)/\epsilon]^{-1}$  is the characteristic inverse charging energy per unit length of the loop ( $R$  is the distance to a metallic screen<sup>17</sup>),  $n_s$  is the density of the superconducting condensate, and  $m$  is the electron mass.

The first term on the right-hand side of Eq. (1) together with the first term on the right-hand side of Eq. (2) describe the propagating plasma mode<sup>12</sup> along the loop.<sup>18</sup> They correspond to the electrostatic energy stored in the plasmons and to the energy associated with the supercurrent in the wire, respectively. In the latter energy, we included only the kinetic inductance, assuming the wire is thin and electrodynamic effects are weak. The plasma mode is characterized by a linear dispersion relation  $\omega(k) = v_{\text{pl}} k$  between frequency  $\omega(k)$  and wave vector  $k$ , where the plasma velocity is given by

$$v_{\text{pl}} = \sqrt{\frac{e_c n_s S}{4m}} = c \frac{a}{2\lambda_L} \sqrt{\frac{2 \ln(R/a)}{\pi \epsilon}}. \quad (3)$$

Here  $c$  is the speed of light and  $\lambda_L = \sqrt{mc^2 / (4\pi n_s e^2)}$  is the London penetration depth of the wire. At temperatures  $T$  much smaller than the superconducting gap  $\Delta$ , the plasma mode involves oscillations of the supercurrent only, and damping due to thermally excited quasiparticles is negligible. Retardation effects were neglected in the derivation of Eqs. (1)–(3), i.e., we assume that  $v_{\text{pl}} \ll c$ ; hence we require  $a \ll \lambda_L$ .

The remaining terms in Eqs. (1) and (2) refer to the Josephson junction. For simplicity, we will completely neglect the junction capacitance,  $C = 0$ , throughout this paper. This approximation corresponds to neglecting the last term in comparison with the first one in Eq. (1) for all the relevant scales of the phase variation. The shortest scale in the time dependence of  $\varphi$  is  $\hbar/\Delta$ , and correspondingly the smallest

part of the ring involved in the phase fluctuation is  $\hbar v_{\text{pl}}/\Delta$ . Therefore, we may set  $C$  to zero if

$$\frac{E_c}{\Delta} \gg \frac{m v_{\text{pl}}}{\hbar n_s S}. \quad (4)$$

An important length scale in our model is determined by the length  $L^*$  at which the energy of supercurrents in the loop and the Josephson energy of the junction are of the same order. It is given by

$$L^* \equiv \frac{\hbar^2 n_s S}{4m E_J^0} = \frac{\hbar g v_{\text{pl}}}{\pi E_J^0}, \quad (5)$$

where  $g$  is defined through

$$\frac{1}{g} = \frac{4}{\pi} \frac{m v_{\text{pl}}}{\hbar n_s S} = 8 \frac{e^2}{\hbar c} \frac{\lambda_L}{a} \sqrt{\frac{2 \ln(R/a)}{\pi \epsilon}}. \quad (6)$$

Here  $e^2/\hbar c \approx 1/137$  is the fine structure constant. Physically,  $1/g$  is the dimensionless zero-frequency impedance of the superconducting wire,<sup>19–22</sup>

$$Z(\omega=0) = \frac{1}{g} \frac{2\pi\hbar}{(2e)^2}.$$

In the absence of fluctuations, the phase varies linearly with the distance along the loop; therefore, it is convenient to introduce a new variable,  $\chi(x)$ , by the relation

$$\varphi(x) = \varphi_0 \frac{x}{L} + \chi(x), \quad (7)$$

where  $\varphi_0$  is determined such that the energy  $U$  has its minimum at  $\chi=0$ . This condition can be written in a simple form:

$$\sin \varphi_0 + \frac{L^*}{L} \varphi_0 = \frac{L^*}{L} \Phi. \quad (8)$$

For later use, we rewrite expression (2) for  $U$  in terms of  $\chi$  as follows:

$$U = E_J^0 \left[ -\cos[\chi(2\pi) - \chi(0) + \varphi_0] - [\chi(2\pi) - \chi(0)] \sin \varphi_0 + \pi \frac{L^*}{L} \int_0^{2\pi} d\theta \left( \frac{\partial \chi}{\partial \theta} \right)^2 + \frac{L}{2L^*} \sin^2 \varphi_0 \right]. \quad (9)$$

Here a new (“angular”) coordinate  $\theta = 2\pi x/L$  has been introduced.

The dc Josephson effect at zero temperature is described fully by the  $\Phi$  dependence of the ground-state energy,  $E_{\text{gr}}$ , of the system under consideration. Because only the term  $U$  of the energy depends explicitly on  $\Phi$ , the persistent current  $J(\Phi) \equiv (2e/\hbar) \partial E_{\text{gr}} / \partial \Phi$  can be expressed in terms of the average  $\langle U \rangle$  over the ground-state wave function:

$$J(\Phi) = \frac{2e}{\hbar} \left( 1 + \frac{L}{L^*} \cos \varphi_0 \right)^{-1} \left\langle \frac{\partial U}{\partial \varphi_0} \right\rangle. \quad (10)$$

### III. QUALITATIVE ANALYSIS

The behavior of  $J$  as a function  $\Phi$  is very different in the two limiting cases  $L \ll L^*$  and  $L \gg L^*$ , which we will discuss qualitatively below.

#### A. The case $L \ll L^*$

For a relatively short loop,  $L \ll L^*$ , the kinetic energy of the supercurrent dominates over the Josephson energy, and the phase difference  $\varphi_0$  across the junction is completely determined by the flux threading the loop,  $\varphi_0 \approx \Phi$ , as can be seen from Eq. (8). Classically, i.e., in the absence of phase fluctuations ( $\chi = 0$ ), the dependence of the persistent current on  $\Phi$  is given by  $J(\Phi) = J_c^0 \sin \Phi$ , with  $J_c^0 = 2eE_J/\hbar$ . We will see below that this classical result holds in the limit  $1/g \rightarrow 0$ .

In the quantum case  $\chi \neq 0$ , we can neglect the effect of the Josephson junction on phase fluctuations when calculating the average in Eq. (10) for  $L \ll L^*$ . In other words, for a short loop only the term with the integral in Eq. (9), which corresponds to the kinetic energy of the supercurrent along the loop, is important. Substituting Eq. (9) into Eq. (10), and neglecting terms  $\mathcal{O}(L/L^*)$ , we find

$$J(\Phi) = J_c^0 \langle \cos[\chi(2\pi) - \chi(0)] \rangle \sin \Phi. \quad (11)$$

The evaluation of the average  $\langle \cos[\chi(2\pi) - \chi(0)] \rangle$  does not differ in fact from the well-known calculation of the Debye-Waller factor.<sup>23</sup> We quantize the fluctuating field  $\chi(\theta)$  in the standard way as follows:

$$\chi(\theta) = \chi_0 + \frac{1}{\sqrt{g}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos\left(\frac{n\theta}{2}\right) [a_n^\dagger + a_n], \quad (12)$$

where operators  $a_n$  satisfy canonical commutation relations. Using Eq. (12), it is straightforward to evaluate  $\langle \cos[\chi(2\pi) - \chi(0)] \rangle$  where the average is taken with respect to the quadratic Hamiltonian

$$H = \sum_{n=1}^{\infty} \frac{\hbar v_{\text{pl}} n \pi}{L} \left[ a_n^\dagger a_n + \frac{1}{2} \right].$$

At low temperatures and  $L \gg \hbar v_{\text{pl}}/\Delta$ , we finally obtain  $J(\Phi) = J_c \sin \Phi$ , with a renormalized critical current

$$J_c = J_c^0 \left( \frac{\hbar v_{\text{pl}}}{L\Delta} \right)^{1/g} \left[ \frac{\pi L k_B T / \hbar v_{\text{pl}}}{\sinh(\pi L k_B T / \hbar v_{\text{pl}})} \right]^{1/g}. \quad (13)$$

The energy  $\Delta$  in Eq. (13) appears as a high energy cutoff for the plasmon waves; the result (13) holds for temperatures  $k_B T \ll \Delta$ . The classical result  $J_c \rightarrow J_c^0$  is recovered in the limit  $1/g \rightarrow 0$ .

At this point we would like to note that the Lagrangian defined by Eqs. (1) and (2) has been derived in the limit of small phase fluctuations,  $\langle (\nabla \chi)^2 \rangle \ll 1/\xi^2(0)$ . This poses an upper bound on the allowed values of  $1/g$ ,

$$1/g \ll \frac{e^2}{\hbar v_F} (k_F a)^2 \frac{\ln(R/a)}{\epsilon}. \quad (14)$$

The right-hand side of this inequality is proportional to the number  $(k_F a)^2$  of quantum channels in the wire, and typi-

cally is large. We will be interested in superconducting wires characterized by  $g \sim 1$ . For such wires, Eq. (14) is satisfied, and the local quantum fluctuations of the current due to the propagating plasma mode<sup>12</sup> are smaller than the critical current. We therefore neglect phase slip events<sup>24,25</sup> in the wire.

Let us estimate the exponent  $1/g$  for an Al wire. If the wire is very dirty, with a mean free path  $l \sim 1$  nm, we estimate the zero-temperature coherence length to be  $\xi(0) \sim 40$  nm and the London penetration depth to be  $\lambda_L(0) \sim 500$  nm. Present-day technology enables one to fabricate wires with a cross-sectional area  $S = a^2 \sim (50 \text{ nm})^2$ . Such wires would be characterized by an exponent  $1/g \sim 1.2$ . For cleaner wires with  $l \sim a$ , the exponent can be expressed as  $1/g = [16/a(\text{nm})]^{3/2}$ . We conclude that typical values of the exponent should be in the range  $1/g \leq 1$ .

Quantum fluctuations suppress the maximum Josephson current below its mean-field value  $J_c^0$ . This suppression depends on the loop length  $L$ ; according to Eq. (13)  $J_c \rightarrow 0$  when  $L \rightarrow \infty$ . As we will discuss below, this is an artefact of the lowest order of perturbation theory, where the effect of the junction on the fluctuations in the attached wire is disregarded completely.

#### B. The case $L \gg L^*$

As we have seen above, if  $L \ll L^*$ , the solution  $\varphi_0(\Phi)$  of Eq. (8) that provides the absolute minimum of energy varies continuously with  $\Phi$ . For a large loop with  $L \gg L^*$ , the solution  $\varphi_0(\Phi)$  has discontinuities:

$$\begin{aligned} \varphi_0 &\approx \frac{L^*}{L} \Phi & \text{if } 0 \leq \Phi < \pi, \\ \varphi_0 &\approx 2\pi + \frac{L^*}{L} (\Phi - 2\pi) & \text{if } \pi \leq \Phi < 2\pi. \end{aligned} \quad (15)$$

Because the Josephson energy dominates over the kinetic energy of the supercurrents, the phase  $\varphi_0$  remains ‘‘pinned’’ to the minima of the cosine potential. Correspondingly, in the absence of fluctuations the equilibrium persistent current  $J(\Phi)$  has cusps,

$$\begin{aligned} J(\Phi) &\approx J_c^0 \frac{L^*}{L} \Phi & \text{if } 0 \leq \Phi < \pi, \\ J(\Phi) &\approx J_c^0 \frac{L^*}{L} (\Phi - 2\pi) & \text{if } \pi \leq \Phi < 2\pi. \end{aligned} \quad (16)$$

We expect quantum fluctuations (i) to renormalize the bare Josephson energy and thus to suppress the slope of the sawtooth dependence; (ii) to smear the cusps at  $\Phi = (2n+1)\pi$ , as quantum tunneling will remove the degeneracy between pairs of states having the same values of energy but different values of  $\varphi_0$ . This is shown schematically in Fig. 1.

However, due to the fact that the Josephson energy is not a weak perturbation if  $L > L^*$ , we have to take its effect on the quantum fluctuations into account. This is very similar to the problem of pinning of a 1D crystal.<sup>15</sup> The decrease in  $\langle \cos[\chi(2\pi) - \chi(0)] \rangle$  with a growing length of the loop  $L$  should saturate when  $L$  exceeds the characteristic length  $L^*$ . The saturation occurs because the Josephson coupling pins the low-frequency modes, thus preventing the logarithmic divergence of the phase fluctuations at the junction.

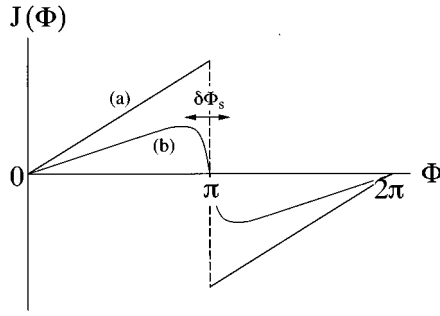


FIG. 1. Schematic dependence of the Josephson current  $J$  on  $\Phi$  for a large loop,  $L \gg L^*$ : (a) sawtooth dependence, found in the absence of fluctuations; (b) quantum fluctuations suppress the slope of the sawtooth dependence and smear the cusps over a typical width  $\delta\Phi_s$ .

#### IV. RENORMALIZATION OF THE JOSEPHSON ENERGY

In this section we will analyze the renormalization of the Josephson energy by the charge fluctuations in the framework of the renormalization group (RG) approach. This approach will enable us to treat both cases  $L < L^*$  and  $L > L^*$  in a unifying manner. We will restrict our analysis first to zero applied flux,  $\Phi = 0$ . It is convenient to perform a Wick rotation to imaginary time  $\tau$  and to consider the Euclidean action  $\mathcal{S}$  for the fluctuating field  $\tilde{\chi}(\theta, \tau) \equiv \chi(\theta, \tau) - \chi(2\pi - \theta, \tau)$ , which can be easily obtained from the Lagrangian  $\mathcal{L}$ . Next one integrates out the fluctuations in  $\tilde{\chi}(\theta, \tau)$  away from the junction, and obtains the effective action for the field  $\tilde{\chi}(\theta = 0)$ ,

$$\mathcal{S} = \frac{\hbar g}{4\pi} \int \frac{d\omega}{2\pi} |\tilde{\chi}(\omega)|^2 |\omega| - \int d\tau E_J^0 \cos \tilde{\chi}(\theta = 0, \tau). \quad (17)$$

This action can be studied by a standard perturbative RG method.<sup>13</sup> We introduce a running cutoff energy  $\mu$ , and find a flow equation<sup>26</sup> for the dimensionless Josephson coupling energy  $\bar{E}_J \equiv E_J / \mu$ :

$$\frac{d\bar{E}_J}{dl} = (1 - 1/g)\bar{E}_J + \mathcal{O}(\bar{E}_J^3), \quad dl = -d\mu/\mu. \quad (18)$$

This equation describes how the Josephson coupling  $E_J$  is renormalized when high energy degrees of freedom are integrated out.<sup>27</sup> From Eq. (18) it follows that, upon decreasing  $\mu$ , the energy  $\bar{E}_J$  flows to zero if  $g < 1$ , whereas  $\bar{E}_J$  increases if  $g > 1$ .<sup>28</sup> Note that in the latter case perturbation theory breaks down as soon as  $\bar{E}_J \sim 1$ .

In order to investigate these cases in more detail, we integrate Eq. (18) from the high energy cutoff  $\mu_h = \Delta$  at which  $E_J = E_J^0$  down to a value  $\mu_l = \hbar v_{pl}/l_0$ , characterized by some length  $l_0$ . As a result, we find

$$E_J(\mu_l) = E_J^0 \left( \frac{\hbar v_{pl}}{l_0 \Delta} \right)^{1/g}. \quad (19)$$

*The case  $g < 1$ .* In this case, the result (19) remains valid for  $\mu_l \rightarrow 0$ , i.e., for the largest possible values of  $l_0$ . Putting  $l_0 \sim L$  we thus recover our earlier result (13) at  $T = 0$ . At any finite length  $L > \hbar v_{pl}/\Delta$ , the renormalized Josephson energy  $E_J$  is smaller than the plasmon level spacing of the loop,

$\hbar v_{pl}/L$ . Therefore the perturbative analysis of Sec. III A applies, and we find for the flux-dependent Josephson current

$$J(\Phi) = \frac{2eE_J^0}{\hbar} \left( \frac{\hbar v_{pl}}{L\Delta} \right)^{1/g} \sin\Phi. \quad (20)$$

We see that quantum fluctuations will completely suppress the Josephson current as  $L \rightarrow \infty$  if  $g < 1$ .

*The case  $g > 1$ .* In this case, the situation is quite different. The RG procedure should be stopped when  $\bar{E}_J \sim 1$ , i.e., when the cutoff energy  $\mu$  reaches a value  $\mu_l$  which satisfies the condition  $\mu_l = E_J(\mu_l)$ . As a result  $E_J$  is renormalized down to a value  $E_J^{\text{eff}}$ , and should be determined from the condition of self-consistency,

$$E_J^{\text{eff}} = E_J^0 \left( \frac{E_J^{\text{eff}}}{\Delta} \right)^{1/g}, \quad (21)$$

which yields

$$E_J^{\text{eff}} = E_J^0 \left( \frac{E_J^0}{\Delta} \right)^{1/(g-1)}. \quad (22)$$

The value  $E_J^{\text{eff}}$  is reached for  $l_0 \sim (L^*/g)(\Delta/E_J^0)^{1/(g-1)}$ . We thus conclude that if  $g > 1$ , the result (13) at  $T = 0$  holds as long as  $L \leq L^*$ ; for larger values of  $L$ , the decrease of the Josephson coupling slows down, and eventually  $E_J$  saturates at the value  $E_J^{\text{eff}}$  given by Eq. (22). Further suppression of the Josephson energy is prevented by the fact that the modes  $\tilde{\chi}(\omega)$  at frequencies  $\omega < E_J^{\text{eff}}/\hbar$  are pinned by the Josephson coupling, and hence cannot participate in the renormalization.

When using  $\Delta$  as an upper energy cutoff in the derivation of Eqs. (19) – (22), we assumed condition (4) to be satisfied. In fact, the above treatment remains valid, even if Eq. (4) is violated, but the weaker condition  $L^* \gg C$ , holds; in this case,  $\Delta$  should be replaced with  $E_c$  in Eqs. (19) – (22).

Result (16) for the flux-dependent Josephson current in a loop with  $L > L^*$  remains valid for values of flux away from the cusp at  $\Phi = \pi$ ; we just should replace  $E_J^0$  by  $E_J^{\text{eff}}$ . The behavior of  $J(\Phi)$  for  $\Phi \sim \pi$  is strongly affected by quantum tunneling, which we will study in the next section.

#### V. QUANTUM TUNNELING OF PHASE

As we have seen in Sec. III B, in the classical limit the flux dependence of the persistent current has cusps at  $\Phi = (2n+1)\pi$  for a large loop,  $L \gg L^*$ . At these values of  $\Phi$ , a degeneracy occurs: Two states having a phase difference at the junction given by  $\varphi_0 = 2n\pi$  and  $\varphi_0 = 2(n+1)\pi$ , respectively, have the same energy. This degeneracy may be lifted by the quantum fluctuations of phase at  $1/g \neq 0$ . Tunneling between the two macroscopic states characterized by different values of  $\varphi_0$  induces a shift  $\delta E$  of the ground-state energy of the system. As a result, the cusps in the function  $J(\Phi)$  will be smeared. We will show below that the tunnel splitting  $\delta E \ll \hbar v_{pl}/L$ , i.e., it is smaller than the gap between the degenerate ground state and the first excited state of the loop with the junction. Thus, at zero temperature, we are dealing with an effective two-state sys-

tem, and the flux dependence of  $J(\delta\Phi \equiv \Phi - \pi)$  near  $\Phi = \pi$  will be given by<sup>29</sup>

$$J(\delta\Phi) \approx \frac{2eE_J L^*}{\hbar L} \delta\Phi \times \left\{ 1 - \frac{\pi}{\sqrt{(L^*/L)^2 \delta\Phi^2 + [\delta E / (\pi E_J)]^2}} \right\}. \quad (23)$$

In particular, we see that the smearing is characterized by a width  $\delta\Phi_s \sim (L/L^*)(\delta E/E_J)$ , see Fig. 1.

The tunnel splitting  $\delta E$  is proportional to the amplitude  $t$  for tunneling through a barrier of height  $E_J$ . This tunneling involves a varying phase field  $\varphi(x)$  along the loop, and therefore occurs in a multidimensional potential landscape. The dominant contribution to the tunneling action grows logarithmically with the system size  $L$ . For large  $L$  the amplitude  $t$  can thus be obtained within the WKB approximation. However the preexponential factor has to be retained, as the leading term of the WKB approximation yields a power-law, rather than an exponential, decay of  $t$  with  $L$ . The calculation of  $\delta E$  is therefore conveniently performed with the use of instanton techniques<sup>16</sup> which generalize the WKB method to higher dimensions and enable one to evaluate the preexponential factor directly.

We start our analysis by introducing a new phase variable  $\phi(x) \equiv \varphi(x) - \varphi(L-x) + (2x/L-1)\Phi$  [note that  $\phi(L/2) = 0$ ]. In imaginary time, the Lagrangian for this phase field reads

$$\mathcal{L} = \frac{\hbar g v_{\text{pl}}}{4\pi} \int_0^{L/2} dx \left[ \frac{1}{v_{\text{pl}}^2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 \right] + E_J \cos \phi(0, \tau). \quad (24)$$

Here, we put  $\Phi = \pi$ , i.e., we consider the degeneracy point. The Josephson coupling energy  $E_J$  appearing in Eq. (24) is assumed to be renormalized by the high energy degrees of freedom of the loop. It is a complicated problem to actually perform such a renormalization procedure, due to the strong anharmonicity of the potential  $U$  [Eq. (2)] close to the degeneracy point. For our subsequent treatment of the quantum tunneling process the following qualitative description of the renormalization scheme will be sufficient. We imagine integrating out high energy degrees of freedom in the spirit of Sec. IV, starting from  $\Delta$  down to a cutoff  $\mu_l$ , which satisfies the inequality  $E_J^{\text{eff}} \ll \mu_l \ll \Delta$ . These degrees of freedom are so fast that they “follow” the tunneling process and merely adiabatically renormalize the Josephson energy  $E_J^0$  down to the value  $E_J$ , where  $E_J^{\text{eff}} < E_J < E_J^0$ . As will be discussed below, the tunneling process itself consists of a slow part and a fast part. The former involves the remaining low energy degrees of freedom with energies up to  $E_J$ , whereas the latter involves those with energies between  $E_J$  and  $\mu_l$ .

We will be interested in tunneling between the initial phase configuration  $\phi_i(x, -T/2) = -(2\pi/L)(x-L/2)$  at time  $\tau = -T/2$  and the final configuration  $\phi_f(x, T/2) = (2\pi/L)(x-L/2)$  at time  $\tau = T/2$ . Both are classical configurations which minimize the energy  $U$ , see Eq. (2). The tunneling amplitude is characterized by the matrix element

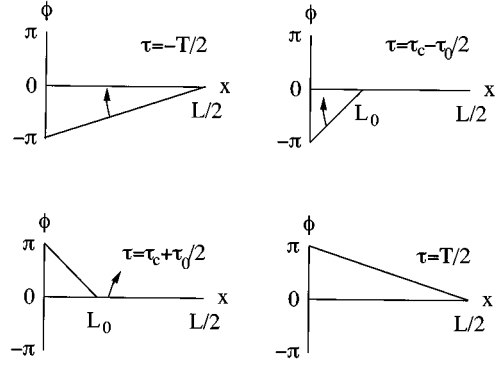


FIG. 2. Four configurations of the phase field  $\phi$  which occur during the tunneling process; these configurations are discussed in the text.

$$\langle \phi_f | e^{-H\mathcal{T}/\hbar} | \phi_i \rangle = N \int \mathcal{D}\phi e^{-S/\hbar}. \quad (25)$$

Here,  $N$  is a normalization constant,  $H$  the Hamiltonian, and  $S = \int_{-T/2}^{T/2} d\tau \mathcal{L}(\tau)$  the action. The matrix element (25) can be used to determine the ground-state energy  $E_{\text{gr}}$ , because it decays as  $e^{-E_{\text{gr}}\mathcal{T}/\hbar}$  for  $\mathcal{T} \rightarrow \infty$ .

The matrix element (25) will be evaluated in the so-called dilute instanton gas approximation.<sup>16</sup> We first will construct a single instanton (SI), i.e., a classical trajectory in the inverted potential  $-U$  between the configurations  $\phi_i$  and  $\phi_f$  that passes once, at a time  $\tau_c$ , through the minimum of  $-U$ . According to Refs. 14 and 15, the tunneling process consists of a fast and a slow part. The actual tunneling at the junction (i.e., the passage through the minimum at  $\tau = \tau_c$ ) happens within a short time  $\tau_0$ , and involves a part of the phase field with a length  $L_0 = v_{\text{pl}}\tau_0$ . The length  $L_0$  is determined by minimizing the total action of the SI; as we will see below  $L_0 \sim L^*$ , in agreement with Refs. 14 and 15. The rest of the phase field makes the transition  $\phi_i \rightarrow \phi_f$  slowly. Therefore the SI consists of three steps, see Fig. 2: (i)  $-T/2 < \tau < \tau_c - \tau_0/2$ : slow adjustment of the phase field away from the junction from the initial configuration  $\phi_i$  to the intermediate configuration  $\phi(x > L_0) = 0$ ; (ii)  $\tau_c - \tau_0/2 < \tau < \tau_c + \tau_0/2$ : fast tunneling at the junction involving a part of the phase field with length  $L_0$ ; (iii)  $\tau_c + \tau_0/2 < \tau < T/2$ : slow adjustment of the phase field away from the junction from the intermediate configuration to the final configuration  $\phi_f$ .

The matrix element (25) for a SI can be written as a product of two amplitudes, one corresponding to the slow and one corresponding to the fast contribution.

We describe the *slow adjustment* by decomposing the phase field  $\phi(x, \tau)$  into modes on the loop

$$\phi(x, \tau) = \sum_{n=1}^{n_{\text{max}}} \phi_n(\tau) \sin(2\pi n x/L). \quad (26)$$

The upper cutoff  $n_{\text{max}} \sim L/L_0$  indicates that slow adjustment involves the phase field away from the junction ( $x > L_0$ ) only. For the initial (final) state of each mode we have  $\phi_n(\tau = -(+)T/2) = -(+)2\pi/n$ ; the dynamics of the modes is determined by the Lagrangian (24) without the cos term.

This is a quadratic problem and the contribution to the matrix element (25) can be calculated exactly. We find

$$\langle \phi_f | e^{-HT\hbar} | \phi_i \rangle_{\text{slow}}^{\text{SI}} = \prod_{n=1}^{n_{\text{max}}} \sqrt{\frac{m_s \omega_n}{\pi \hbar}} e^{-\omega_n T} e^{-g/n}, \quad (27)$$

where  $m_s = 2\hbar g L / (\pi v_{\text{pl}})$  is the ‘‘mass’’ of each mode and  $\omega_n = 2\pi n v_{\text{pl}} / L$  its frequency. The SI action for slow adjustment of the phase is easily calculated to be

$$S_{\text{slow}}^{\text{SI}} \approx \hbar g \ln(L/L_0). \quad (28)$$

The *fast tunneling* of the phase field close to the junction ( $x < L_0$ ) can be described by the following ansatz:<sup>14</sup>

$$\phi(x, \tau) = \phi_0(\tau) [1 - x/L_0], \quad -\pi < \phi_0 < \pi. \quad (29)$$

Substituting this into Eq. (24), we find the Lagrangian for  $\phi_0(\tau)$ :

$$\mathcal{L}_0 = \frac{\hbar g v_{\text{pl}}}{4\pi} \left[ \frac{L_0}{3v_{\text{pl}}^2} \left( \frac{d\phi_0}{d\tau} \right)^2 + \frac{1}{L_0} \phi_0^2 \right] + E_J \cos(\phi_0). \quad (30)$$

This Lagrangian describes the ‘‘rigid’’ tunneling of a part of the phase field with length  $L_0$  in terms of the motion of a particle with ‘‘mass’’  $m_f = \hbar g L_0 / (6\pi v_{\text{pl}})$  and ‘‘coordinate’’  $\phi_0$  in an inverted double-well potential  $V(\phi_0) = -\hbar g v_{\text{pl}} \phi_0^2 / (4\pi L_0) - E_J \cos(\phi_0)$ . The action corresponding to a single passage through the minimum of  $V$  at  $\phi_0 = 0$  can be estimated to be<sup>14</sup>

$$S_{\text{fast}}^{\text{SI}} \approx E_J L_0 / v_{\text{pl}} + \alpha \hbar g, \quad (31)$$

where  $\alpha$  is a constant of order unity. Minimizing the total SI action  $S_t^{\text{SI}} = S_{\text{slow}}^{\text{SI}} + S_{\text{fast}}^{\text{SI}}$  with respect to  $L_0$  we find  $L_0 = \hbar g v_{\text{pl}} / E_J \sim L^*$ . Following the standard treatment for tunneling in a 1D double-well potential outlined in Ref. 16 one can easily estimate the fast SI contribution to Eq. (25),

$$\langle \phi_f | e^{-HT\hbar} | \phi_i \rangle_{\text{fast}}^{\text{SI}} \sim \sqrt{g} \frac{v_{\text{pl}} T}{L^*} \exp\left\{ -\frac{v_{\text{pl}} T}{2L^*} \right\} \exp\left\{ -\frac{S_{\text{fast}}^{\text{SI}}}{\hbar} \right\}. \quad (32)$$

The total matrix element (25) can now be calculated in the dilute instanton approximation.<sup>16</sup> One sums over all configurations of single instantons and anti-instantons that involve transitions  $\phi_i \rightarrow \phi_f$  and  $\phi_f \rightarrow \phi_i$ , respectively. This is done under the assumption that SI's do not overlap, which is justified as long as  $S_t^{\text{SI}} \gg \hbar$ , i.e., for  $(L/L^*)^g \gg 1$ . As a result we find

$$\begin{aligned} \langle \phi_f | e^{-HT\hbar} | \phi_i \rangle &\sim \sqrt{g} \exp\left\{ -\frac{v_{\text{pl}} T}{2L^*} \right\} \prod_{n=1}^{n_{\text{max}}} \left\{ \sqrt{\frac{m_s \omega_n}{\pi \hbar}} e^{-\omega_n T} \right\} \\ &\times \left[ \exp\left\{ \frac{v_{\text{pl}} T}{L^*} e^{-S_t^{\text{SI}}/\hbar} \right\} \right. \\ &\left. - \exp\left\{ -\frac{v_{\text{pl}} T}{L^*} e^{-S_t^{\text{SI}}/\hbar} \right\} \right]. \quad (33) \end{aligned}$$

From the behavior of Eq. (33) for  $T \rightarrow \infty$  we infer that there are two low-lying energy eigenstates with energies  $E_{\text{gr}}^{\pm} = E_{\text{gr}}^0 \pm \delta E$ , where  $E_{\text{gr}}^0$  is an irrelevant reference energy and

$$\delta E \sim \frac{\hbar v_{\text{pl}}}{L^*} e^{-S_t^{\text{SI}}} \sim \frac{\hbar v_{\text{pl}}}{L} \left( \frac{L^*}{L} \right)^{g-1}. \quad (34)$$

We see that indeed  $\delta E \ll \hbar v_{\text{pl}} / L$  for large  $L$ , consistent with the assumption leading to Eq. (23).

## VI. DISCUSSION

The interplay between disorder and Coulomb correlations strongly influences the properties of low-dimensional superconductors. This is well known for thin films:<sup>30</sup> A transition from superconducting ( $S$ ) to insulating ( $I$ ) behavior occurs upon decreasing the thickness of the film. Only recently, developments in fabrication techniques made experimental studies on *in situ* grown quasi-1D wires<sup>31,32</sup> possible. These indicate that a similar transition might occur in a superconducting wire upon decreasing its cross-sectional area  $S$ . Correspondingly, one may expect that if the parameter  $g$  is smaller than a certain threshold value  $g_c$ , the wire should behave as an insulator on length scales even shorter than the loop circumference  $L$ , and the theory presented above ceases to be valid.

Recent attempts to extend the description of  $T_c$  suppression in homogeneous thin films<sup>30</sup> to include quasi-1D homogeneous wires have met with considerable difficulties.<sup>33</sup> On the other hand, in a 1D boson system disorder is known to induce a localized-delocalized transition, which occurs for strongly attractive interactions between the bosons.<sup>34</sup> More specifically, in terms of our model, the results of Ref. 34 would correspond to a transition to insulating behavior at a value  $g_c = 3/2$ . This threshold in interaction strength is reduced in the case of two coupled chains,<sup>35</sup> and one may conjecture a reduction to  $g_c = 1$  for a multimode wire, in agreement with Ref. 25.

An interesting model system which shows a  $S$ - $I$  transition is a 1D array of Josephson junctions.<sup>36,37</sup> The behavior of such an array is determined by a competition of the Josephson coupling  $E_j$  between the islands (which favors a phase-coherent superconducting state) with the electrostatic energy  $E_0$  (which localizes Cooper pairs on the superconducting islands). If the array is superconducting,  $(E_j/8E_0)^{1/2} > 2\pi$ , its low lying excitations are ‘‘phase waves’’ with a linear dispersion,  $\omega(k) \sim \sqrt{8E_j E_0} k$ . We therefore speculate that a 1D array containing a junction which is weakly coupled to its neighbors could be used to study the renormalization of the Josephson energy discussed in this paper. An advantage of this system is that the energies  $E_j$  and  $E_0$ , which depend on properties of the array, are well known and controllable in a typical experiment;<sup>38</sup> in particular the Josephson coupling  $E_j$  can be chosen from a large range of values. This also would enable one to systematically probe the regime close to the  $S$ - $I$  transition.

In conclusion, we considered a thin superconducting loop which contains a Josephson junction. The ‘‘environment modes’’ of the loop, which consist of propagating plasmon modes with a soundlike dispersion, were found to renormal-

ize the Josephson energy of the junction. The strength of the renormalization is determined by the dimensionless zero-frequency impedance of the loop. In order to observe this renormalization we propose to phase bias the junction by threading the loop with a flux and measure the corresponding Josephson current. For a relatively short loop, the kinetic energy of the supercurrent dominates over the Josephson energy and the phase difference across the junction is completely determined by the flux threading the loop. The supercurrent depends on flux in a sinusoidal fashion. Quantum fluctuations suppress the amplitude of this dependence. In the opposite limit of a large loop, the phase difference remains more or less “pinned” to the minima of the Josephson energy. Correspondingly the persistent current shows a sawtooth dependence on flux in the classical limit. Quantum fluctuations not only suppress the amplitude of the oscillations, but also affect their shape. While the impedance of the loop increases, the cusps of the sawtooth dependence are smeared; as the impedance tends to the quantum unit value

the shape of the oscillations approaches a sinusoidal form. At any finite length of the loop, the renormalized current amplitude is finite and shows a power-law dependence on the junction conductance, with an exponent depending on the impedance of the loop.

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