Magnus force in superfluids and superconductors

E. B. Sonin
Low Temperature Laboratory, Helsinki University of Technology, 02150 Espoo, Finland and Ioffe Physical Technical Institute, St. Petersburg 194021, Russia
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The forces on the vortex, transverse to its velocity, are considered. In addition to the superfluid Magnus force from the condensate (superfluid component), there are transverse forces from thermal quasiparticles and external fields violating the Galilean invariance. The forces between quasiparticles and the vortex originate from interference of quasiparticles with trajectories on the left and on the right from the vortex like similar forces for electrons interacting with the thin magnetic-flux tube (the Aharonov-Bohm effect). These forces are derived for phonons from the equations of superfluid hydrodynamics, and for BCS quasiparticles from the Bogolyubov–de Gennes equations. The effect of external fields breaking Galilean invariance is analyzed for vortices in the two-dimensional Josephson junction array. The symmetry analysis of the classical equations for the array shows that the total transverse force on the vortex vanishes. Therefore the Hall effect which is linear in the transverse force is absent also. This means that the Magnus force from the superfluid component *exactly* cancels with the transverse force from the external fields. The results of other approaches are also brought together for discussion. [S0163-1829(97)04502-5]

I. INTRODUCTION

The Magnus force on a vortex has long been known in classical hydrodynamics.[1] This force appears if the vortex moves with respect to a liquid. The force is normal to the relative vortex velocity and therefore is reactive and does not produce a work. In general, such a force arises always when a body with a flow circulation around it moves through a liquid or a gas (the Kutta-Joukowski theorem). The most important example is the lift force on a wing of an aeroplane which keeps the aeroplane in the air.[2]

The key role of the Magnus force in vortex dynamics has become clear from the very beginning of studying superfluid hydrodynamics.[3,4] The superfluid Magnus force was defined as a force between a vortex and a superfluid and therefore was proportional to the superfluid density ρ_s. But in the two-fluid hydrodynamics the superfluid Magnus force is not the only force on the vortex transverse to its velocity: there was also a transverse force between the vortex and quasiparticles moving with respect to the vortex. The transverse force from rotons was found by Lifshitz and Pitaevskii[5] from the quasiclassical scattering theory. Later Iordanskii[6] revealed the transverse force from phonons which was equal in magnitude and opposite in sign with the quasiclassical force of Lifshitz and Pitaevskii. From the very beginning the Iordanskii force was a controversial matter. Iordanskii suggested that his force and the Lifshitz-Pitaevskii force were of different origins and for rotons they should be summed. As a result, he concluded that the transverse force from rotons vanished. But the analysis done in Ref. 7 demonstrated that the Iordanskii force for rotons is identical to the Lifshitz-Pitaevskii force and they must not be added. In addition, the Lifshitz-Pitaevskii force from rotons was calculated in the original paper[5] with a wrong sign. After its correction the transverse force on the vortex had a same sign and a value both for rotons (the Lifshitz-Pitaevskii force) and for phonons (the Iordanskii force). In the same paper[7] it was pointed out that the Iordanskii force for phonons and rotons results from interference between quasiparticles which move past the vortex on the left and on the right sides with different phase shifts, like in the Aharanov-Bohm effect.[8]

In the theory of superconductivity the Magnus force appeared first in the paper by Nozières and Vinen.[9] In clean superconductors the BCS quasiparticles produce an additional transverse force on the vortex[10,11] analogous to the Iordanskii force in superfluids. The total transverse force is responsible for the Hall effect in the mixed state of a superconductor. But the Hall effect was rather weak in classical superconductors. An explanation of it was suggested by Kopnin and Kravtsov[12] (see also Ref. 13): impurities interact with quasiparticles bound in the vortex core and this interaction produces an additional transverse force on the vortex. In contrast with the quasiparticle transverse force which increases the total transverse force, the impurity force decreases it and in a dirty superconductor the Magnus force is very small. As a result, the strong Hall effect is possible only in superclean superconductors. The transverse force from the bound states in the core has been recently rephrased in terms of the spectral flow through the quasiparticle bound states.[14]

A new wave of interest to the Magnus force came with discovery of high-T_c superconductivity. A few reasons of this interest might be mentioned. (i) The so-called Hall anomaly was observed:[15] near T_c the sign of the Hall voltage is opposite to that expected from the standard vortex dynamics. (ii) It has become possible to obtain superclean single crystals with large Hall angle[16] as predicted in the superclean limit of the theory of Kopnin and Kravtsov. This made possible to observe magnetoresonances in the ac response of the superconducting single crystals connected with waves propagating along vortices.[17] (iii) The effective Magnus force governs quantum vortex nucleation in clean high-T_c superconductors intensively discussed now.[18,19]

Despite a lot of work done to understand and calculate the

Magnus force, it still remains to be a matter of controversy with a number of conflicting points of view on it. A new discussion has been launched by the paper of Ao and Thouless.[20] The main claim of Ao and Thouless is that there is an universal *exact* expression for the total transverse force on the vortex (the effective Magnus force) which does not depend on the presence of quasiparticles or impurities. This force derived from the concept of the geometrical phase (the Berry phase) coincides with the superfluid Magnus force and therefore is proportional to the superfluid density. According to Ao and Thouless, there is no transverse force on the vortex from quasiparticles and impurities, though they might influence the value of the superfluid density and thereby influence the amplitude of the Magnus force.

The Ao-Thouless theory is in an evident disagreement with the previous calculations of the transverse force on the vortex (the effective Magnus force) in superfluids and superconductors reviewed above. It attracted a great attention and has been supported in a number of recent publications of other theorists (see, e.g., Refs. 21–23). If the Ao-Thouless theory were true, it would be necessary to revise the whole basis of the vortex dynamics. For example, on the basis of the Ao-Thouless theory Šimanek[22] suggested that quantum vortex tunnelling is governed by the Magnus force obtained from the Berry phase approach, i.e., proportional to the superfluid density, in contradiction to the previous theory.[18,19] Therefore it is important to understand what the Magnus force is and whether the Ao-Thouless theory is true or not.

The present paper is to analyze this controversy. Among the sources of controversy there is semantics. Therefore it is important to define force terminology from the very beginning. The word *force* itself is only a label to describe a transfer of momentum between two objects. Before using these labels one must analyze the momentum balance for any object and only then to label various contributions to these balances as forces. Keeping this in mind, the forces under discussion may be defined as the following:

(i) There is a momentum transfer between a vortex and the rest part of a superfluid. One reveals it analyzing the momentum balance for the superfluid moving with the velocity $\vec{v}_s$ whereas the vortex moves through the superfluid with a different velocity $\vec{v}_L$. This momentum transfer is the *superfluid Magnus force*. It is proportional to the superfluid density ρ_s and transverse to the relative velocity $\vec{v}_L-\vec{v}_s$.

(ii) Analyzing the momentum balance for the *whole* Galilean-invariant liquid (including the superfluid and normal parts of it) around the vortex one may reveal a contribution presenting the momentum transfer between the vortex moving with the velocity $\vec{v}_L$ and the normal fluid (the gas of quasiparticles) moving with the velocity $\vec{v}_n$. This force is proportional to the relative velocity $\vec{v}_L-\vec{v}_n$ and has the components longitudinal and transverse to $\vec{v}_L-\vec{v}_n$. The transverse component of this force includes the *Iordanskii force*.

(iii) If there is no Galilean invariance, as in a dirty superconductor, the momentum balance for the whole liquid must include also forces external for the liquid, namely, the momentum transfer to the impurities rigidly connected with the crystal lattice. When the latter is at rest, this momentum transfer, or the force from impurities, is proportional to the vortex velocity $\vec{v}_L$ in the laboratory reference frame (the frame connected with the crystal at rest). Its component transverse to $\vec{v}_L$ is the *Kopnin-Kravtsov force*.

(iv) The momentum balance for the whole liquid around the vortex is at the same time an equation from which one must find the vortex velocity $\vec{v}_L$. Therefore it is useful to collect all the terms proportional to $\vec{v}_L$ together. After it the term uniting all contributions transverse to $\vec{v}_L$ is the total transverse force on the vortex, or the *effective Magnus force*.

Thus in general three forces contribute to the effective Magnus force: the superfluid Magnus force, the force from quasiparticles (the Iordanskii force), and the force from impurities, or other external fields breaking the Galilean invariance (the Kopnin-Kravtsov force). Usually in experiment they can determine the effective Magnus force, but not the ''bare'' superfluid Magnus force. In rotating superfluids the effective Magnus force determines the mutual friction. One can find the latest reviews of the experiment and the theory on mutual friction in ^{3}He in Refs. 24 and 25. In superconductors the Hall conductivity[15,26] and the acoustic Faraday effect for the transverse ultrasound wave propagating along vortices[27] are linear in the effective Magnus force. The process of vortex quantum tunnelling is also influenced by the effective, but not the superfluid Magnus force. So the final outcome of the theory must be the amplitude of the effective Magnus force. Its presentation as a combination of three forces is an intermediate stage of the theory. In fact, this presentation is valid only if (i) the number of quasiparticles is not too large and their mutual interaction is weak; (ii) external fields breaking Galilean invariance are not too strong. The first condition is violated close to T_c, where other approaches based on the Ginzburg-Landau theory (or its analogue for superfluids, the Ginzburg-Pitaevskii theory) must be used.[28,29] The second condition does not hold in the Josephson junction array considered in the present paper (see below). In these cases the theory deals directly with the effective Magnus force in the equation of vortex motion: its decomposition on the ''bare'' superfluid Magnus force and the forces from quasiparticles or impurities becomes conventional and of a little physical sense.

Whereas there is a consensus among theorists on the superfluid Magnus force, the Ao-Thoules theory rejects the Iordanskii force from quasiparticles and the Kopnin-Kravtsov force from impurities claiming that amplitudes of the effective and the superfluid Magnus forces are exactly equal. Therefore the present paper considers the effect on the force balance (i) from quasiparticles, and (ii) from external fields breaking the Galilean invariance of the superfluid.

In my analysis of the quasiparticle effect I chose the phonon-vortex interaction which may be described by the nonlinear Schrödinger equation long ago suggested for a weakly nonideal Bose gas (the Gross-Pitaevskii theory[30]). The nonlinear Schrödinger equation yields the usual superfluid hydrodynamics. It is a good starting point for further discussion, which one may expect a consensus of all parties on (see discussion in Ref. 31). The next step is to analyze scattering of the sound wave (phonon) by the vortex in hydrodynamics. Just at this stage a disagreement appears. Ao and Thouless believe that this scattering can produce only a dissipative force on the vortex, but not a transverse one.[32] Recently Demircan, Ao, and Niu[33] tried to prove it using the Born approximation. But they ignored peculiarities of the phonon Born scattering at small angles which resulted in the

Iordanskii force. It is important to note that the controversy arises not from a difference in ideology; anyone is free to choose a language to derive the Magnus force at $T=0$: either the standard hydrodynamics, or newest topological concepts of the geometrical phase. But there is a disagreement in calculation of integrals describing the phonon scattering in the first order of the perturbation theory. We hope to show in this paper at which point Demircan, Ao, and Niu[33] missed to take into account the Aharonov-Bohm interference of phonons which was ignored by the Ao-Thouless theory.

Now discussions around the transverse force on the vortex in the presence of impurities concentrate mostly on the contribution of the core bound states in Fermi superfluids. This requires a rather sophisticated analysis (see Ref. 14 and references therein). In the present paper I chose another example when the Galilean invariance is absent: the two-dimensional Josephson junction array (JJA). This is a regular lattice of nodes with the Josephson coupling between them. Experimentally, any node corresponds to a superconducting island in an artificially prepared JJA, or to a grain in a granular superconducting film. The behavior of the JJA in an external magnetic field is usually described in the picture of moving vortices similar to the mixed state of type II superconductors. The dynamics of vortices in JJA attracts a great interest of experimentalists[34,35] and theorists.[36–45] There is an intrinsic pinning of vortices at the JJA cells, and vortices can move only if the driving supercurrent is more than the critical value. But when they start to move, in many cases a good approximation is to replace the lattice by a continuous superconducting film. However, the hydrodynamic derivation of the Magnus force is not valid since it assumes the momentum conservation law and the Galilean invariance. In the present paper it will be shown that the Hall effect is *exactly* absent in the classical theory of JJA which neglects the charge quantization. Since the Hall effect is linear in the amplitude of the effective Magnus force, the latter also vanishes in the classical JJA. This statement directly follows from the symmetry of the dynamic equations. At the same time the superfluid density is finite in the continuum limit of JJA and therefore the superfluid Magnus force does not vanish. Therefore the theory based on the Berry phase approach[23] predicted a finite effective Magnus force and the Hall effect for JJA in disagreement with our symmetry analysis. Our result might be interpreted as that the superfluid Magnus force is compensated by some force external for the liquid, like the Kopnin-Kravtsov force in a dirty superconductor. But as mentioned above, JJA is a system with a strong violation of Galilean invariance, for which this interpretation is purely formal. Only the resultant effective Magnus force has a physical meaning.

We start from Sec. II which shows how the Magnus force appears in the phenomenological theory of neutral and charged superfluids. The force terminology is also introduced explaining to which term and in which equation any force under discussion corresponds. Section III is devoted to the transverse force between quasiparticles and the vortex (the Iordanskii force) and its connection with the Aharonov-Bohm effect. Section III A recalls connection between the nonlinear Schrödinger equation for the condensate (the Gross-Pitaevskii theory) and superfluid hydrodynamics and phonons. Scattering of the sound wave (phonon) by the vortex in hydrodynamics is analyzed in Sec. III B. It is shown that the standard scattering-theory approach fails to reveal the transverse Iordanskii force because of the divergence of the scattering amplitude at small angles of scattering. The analysis of the small-angle scattering is presented in Sec. III C. It reveals the interference between quasiparticles with trajectories on the left and on the right from the vortex. In Sec. III D the same results is rederived using the partial-wave expansion, and the analogy with the Aharonov-Bohm effect is shown. In the end of this subsection I show how an oscillatory motion of a free vortex line induced by the sound wave must be taken into account by the partial-wave method. A more general quasiclassical derivation of the transverse force from the quasiparticles with an arbitrary spectrum is presented in Sec. III E. In Sec. IV the transverse force between the BCS quasiparticles and the vortex is derived using the Bogolyubov–de Gennes equations. Section V presents the symmetry analysis of the classical dynamical equations for JJA which shows that the total transverse force on the vortex in JJA vanishes and as a result of it the Hall effect is possible only in the quantum theory of JJA which takes into account charge quantization. The last section VI contains the summary and the discussion of other approaches to the problem.

II. WHERE AND HOW THE MAGNUS FORCE APPEARS

A. The Magnus force in classical hydrodynamics

For a better understanding of the origin of the Magnus force it is worth recalling how the Magnus force arises in classical hydrodynamics.

Let us consider an isolated straight vortex line in an incompressible inviscid liquid. The line along the axis z induces the velocity field

$$\vec{v}_v(\vec{r})=\frac{\vec{\kappa}\times\vec{r}}{2\pi r^2}. \qquad (1)$$

Here $\vec{r}$ is the position vector in the plane xy, and $\vec{\kappa}$ is the circulation vector directed along the axis z. The circulation, given by

$$\kappa=\oint \vec{v}_v\cdot d\vec{l}, \qquad (2)$$

may have arbitrary values in classical hydrodynamics. In addition, there is a fluid current past the vortex line with a transport velocity $\vec{v}_{\rm tr}$. Then the net velocity field around the line is

$$\vec{v}(\vec{r})=\vec{v}_v(\vec{r})+\vec{v}_{\rm tr}. \qquad (3)$$

The Euler equation for the liquid is

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho}\vec{\nabla}P+\frac{\vec{F}}{\rho}\delta_2(\vec{r}). \qquad (4)$$

Here ρ is the liquid density and P is the pressure. This equation suggests that an external δ-function force $\vec{F}$ is applied to the liquid at the vortex line.

Assuming that the vortex line moves with the constant velocity $\vec{v}_L$, i.e., replacing the position vector $\vec{r}$ by $\vec{r}-\vec{v}_L t$, one obtains that

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v}_L \cdot \vec{\nabla})\vec{v}. \tag{5}$$

Then the Euler equation (4) yields the Bernoulli law for the pressure:

$$P = P_0 - \frac{1}{2}\rho[\vec{v}(\vec{r}) - \vec{v}_L]^2$$
$$= P_0' - \frac{1}{2}\rho \vec{v}_v(\vec{r})^2 - \rho \vec{v}_v(\vec{r}) \cdot (\vec{v}_{\rm tr} - \vec{v}_L). \tag{6}$$

Here P_0 and $P_0' = P_0 - \frac{1}{2}\rho(\vec{v}_{\rm tr} - \vec{v}_L)^2$ are constants which are of no importance for the following derivation.

Next one should consider the momentum balance for a cylindrical region of a radius r_0 around the vortex line. The momentum-flux tensor is given by[2]

$$\Pi_{ij} = P\delta_{ij} + \rho v_i(\vec{r}) v_j(\vec{r}), \tag{7}$$

or in the reference frame moving with the vortex velocity $\vec{v}_L$:

$$\Pi'_{ij} = P\delta_{ij} + \rho(v_i - v_{Li})(v_j - v_{Lj}). \tag{8}$$

The momentum conservation law requires that the external force $\vec{F}$ on the vortex line must be equal to the momentum flux through the entire cylindrical boundary in the reference frame moving with the vortex velocity $\vec{v}_L$. The latter is given by the integral $\int dS_j \Pi'_{ij}$ where dS_j are the components of the vector $d\vec{S}$ directed along the outer normal to the boundary of the cylindrical region and equal to the elementary area of the boundary in magnitude. Then using Eqs. (1), (6), and (8), the momentum balance yields the following relation:

$$\rho[(\vec{v}_L - \vec{v}_{\rm tr}) \times \vec{\kappa}] = \vec{F}. \tag{9}$$

On the left-hand side of this equation one can see the Magnus force as it comes in the classical hydrodynamics. A half of this force is due to the Bernoulli contribution to the pressure, Eq. (6); another half is due to the convection term $\propto v_i v_j$ in the momentum flux. The Magnus force balances the resultant of all external forces applied to the liquid at the vortex line (the force $\vec{F}$). In the absence of external forces the vortex moves with the transport velocity of the liquid: $\vec{v}_L = \vec{v}_{\rm tr}$ (the Helmholtz theorem).

This derivation demonstrates the classical origin of the Magnus force: quantization of circulation is not necessary for its existence. During the derivation we referred to the hydrodynamic equations only at large distance from the vortex line. It might seem as if the fluid in the vortex core did not matter at all. However, the derivation is based on the assumption that the momentum is a well-defined *conserved* quantity everywhere even inside the vortex core where the hydrodynamic theory does not hold.

B. The superfluid Magnus force

In the superfluid hydrodynamics one can refer this derivation to the superfluid component with density ρ_s. The Euler equation for the superfluid component[2] after adding the external δ-function force $\vec{F}$ applied at the vortex line is

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\vec{\nabla}\mu + \frac{\vec{F}}{\rho_s}\delta_2(\vec{r}), \tag{10}$$

where μ is the chemical potential.

For charged superfluids (superconductors) the Euler equation should include also the electromagnetic forces. In particular, the chemical potential must be replaced by the electrochemical potential. But outside of the vortex core one may use the quasineutrality condition that the total electron charge is approximately equal to the background ion charge. Then one may neglect the chemical potential gradient. Finally the Euler equation may be written as

$$\frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = \frac{e}{m}\left(\vec{E} + \frac{1}{c}[\vec{v}_s \times \vec{H}]\right) + \frac{\vec{F}}{\rho_s}\delta_2(\vec{r}), \tag{11}$$

where $\vec{E}$ and $\vec{H}$ are the electric and the magnetic fields.

Let us consider a vortex line in a neutral superfluid with the velocity field Eq. (1). Now circulation is quantized, and the circulation quantum is $\kappa = h/m$ in a Bose superfluid and $\kappa = h/2m$ in a Fermi superfluid. In the two-fluid theory the momentum-flux tensor is

$$\Pi_{ij} = P\delta_{ij} + \rho_s v_{si} v_{sj} + \rho_n v_{ni} v_{nj}, \tag{12}$$

and the chemical-potential variation is determined from the Gibbs-Duhem relation:[28]

$$\delta P = \rho\,\delta\mu + S\,\delta T + \rho_n(\vec{v}_n - \vec{v}_s) \cdot \delta(\vec{v}_n - \vec{v}_s)$$
$$= \rho\,\delta\mu + S\,\delta T + \frac{1}{2}\rho_n \delta(\vec{v}_n - \vec{v}_s)^2. \tag{13}$$

According to the Euler equation (10) $\delta\mu = -\frac{1}{2}v_s^2$, then the Bernoulli law for the pressure is

$$P = P_0 - \frac{1}{2}\rho v_s(\vec{r})^2 + \frac{1}{2}\rho_n[\vec{v}_s(\vec{r}) - \vec{v}_n]^2$$
$$= P_0 - \frac{1}{2}\rho_s v_s(\vec{r})^2 - \rho_n \vec{v}_s \cdot \vec{v}_n + \frac{1}{2}\rho_n v_n^2. \tag{14}$$

In these expressions only contributions which depend on the superfluid velocity $\vec{v}_s$ are of importance; those which depend on the normal velocity $\vec{v}_n$ are assumed to be incorporated by the force $\vec{F}$ external with respect to the superfluid component. Comparing Eqs. (14) and (12) with Eqs. (6) and (7) one can see that in the case of the superfluid ρ_s and $\vec{v}_s$ replace ρ and $\vec{v}$. We repeat the analysis of the momentum balance for a superfluid component in a cylindrical region around the vortex line assuming that $\vec{v}_s(\vec{r}) = \vec{v}_v(\vec{r}) + \vec{v}_{s\rm tr}$. Then instead of Eq. (9) one has

$$\rho_s[(\vec{v}_L-\vec{v}_s)\times\vec{\kappa}]=\vec{F}. \tag{15}$$

Here and later on we omit the subscript ''tr'' replacing $\vec{v}_{\rm str}$ by $\vec{v}_s$. But one should remember that the superfluid velocity $\vec{v}_s$ in the expression for the Magnus force is in fact the superfluid velocity far from the vortex line.

The force $\vec{F}$, which enters the theory as a δ-function force, is distributed over some vicinity of the vortex line in reality. The dimension of this vicinity may exceed the vortex-core size, but must be smaller than all relevant hydrodynamic scales (e.g., the intervortex distance, or the vortex line curvature radius) in order to justify the assumption of a force localized at the vortex line. We suppose that the force $\vec{F}$ incorporates all interactions with quasiparticles and impurities.

A similar derivation can be done for a charged superfluid with the electric potential instead of the chemical potential. Bearing in mind Eq. (15), one can rewrite the Euler equations for the neutral and the charged superfluids:

$$\frac{\partial\vec{v}_s}{\partial t}+(\vec{v}_s\cdot\vec{\nabla})\vec{v}_s=-\vec{\nabla}\mu+[(\vec{v}_L-\vec{v}_s)\times\vec{\kappa}]\delta_2(\vec{r}), \tag{16}$$

$$\frac{\partial\vec{v}_s}{\partial t}+(\vec{v}_s\cdot\vec{\nabla})\vec{v}_s=\frac{e}{m}\left(\vec{E}+\frac{1}{c}[\vec{v}_s\times\vec{B}]\right)+[(\vec{v}_L-\vec{v}_s)\times\vec{\kappa}]\delta_2(\vec{r}). \tag{17}$$

Further transformation of the Euler equations uses the vector identity:

$$(\vec{v}_s\cdot\vec{\nabla})\vec{v}_s=\vec{\nabla}\frac{v_s^2}{2}-\vec{v}_s\times[\vec{\nabla}\times\vec{v}_s]. \tag{18}$$

In a neutral superfluid vorticity is concentrated on the vortex line: $[\vec{\nabla}\times\vec{v}_s]=\vec{\kappa}\delta_2(\vec{r})$. But in a superconductor $\vec{\nabla}\times\vec{v}_s=\vec{\kappa}\delta_2(\vec{r})-(e/mc)\vec{H}$.

Then the Euler equations are

$$\frac{\partial\vec{v}_s}{\partial t}=-\vec{\nabla}\left(\mu+\frac{v_s^2}{2}\right)+[\vec{v}_L\times\vec{\kappa}]\delta_2(\vec{r}), \tag{19}$$

$$\frac{\partial\vec{v}_s}{\partial t}=\frac{e}{m}\vec{E}-\vec{\nabla}\left(\frac{v_s^2}{2}\right)+[\vec{v}_L\times\vec{\kappa}]\delta_2(\vec{r}). \tag{20}$$

This analysis demonstrates that the total external force on the superfluid in the vicinity of the vortex line is *exactly* balanced by the superfluid Magnus force $\rho_s[(\vec{v}_L-\vec{v}_s)\times\vec{\kappa}]$. In fact, the term $\propto\vec{v}_L$ in the Euler equation may be received from a pure kinematics: it presents the flow of the vortex lines across the line between two points which changes the phase difference between them (the phase slip). After replacing the external force by the Magnus force in the Euler equation, the latter does not contain any information on the nature and the magnitude of the external force. But the Euler equation is not sufficient for description of superfluid motion: an additional equation for the vortex velocity $\vec{v}_L$ is necessary. In order to derive it, one should specify the force $\vec{F}$. This may be done by considering the momentum balance of the whole liquid, but not only its superfluid component.

C. Equation of vortex motion and effective Magnus force

The momentum balance for the whole liquid in a cylindrical region around the vortex line yields the equation of vortex motion which is a linear relation imposed on three velocities $\vec{v}_s$, $\vec{v}_n$, and $\vec{v}_L$. In a Galilean invariant liquid this equation must depend only on relative velocities. However, one can also include into this balance some interactions with external fields, e.g., with impurities in superconductors. Then for a medium with axial symmetry in the plane normal to the vortex line the most general form of the equation of vortex motion is

$$\rho_s[(\vec{v}_L-\vec{v}_s)\times\kappa]=-D(\vec{v}_L-\vec{v}_n)-D'[\hat{z}\times(\vec{v}_L-\vec{v}_n)]-d\vec{v}_L-d'[\hat{z}\times\vec{v}_L]. \tag{21}$$

Comparing it with Eq. (15), one obtains the expression for $\vec{F}$:

$$\vec{F}=-D(\vec{v}_L-\vec{v}_n)-D'[\hat{z}\times(\vec{v}_L-\vec{v}_n)]-d\vec{v}_L-d'[\hat{z}\times\vec{v}_L]. \tag{22}$$

The forces proportional to D and D' are due to scattering of free quasiparticles by the vortex, therefore they are proportional to the difference between the drift velocity of quasiparticles (the normal velocity $\vec{v}_n$) and the vortex velocity $\vec{v}_L$. The forces proportional to d and d' are due to interaction between the vortex line and impurities which are frozen into the crystal and therefore do not move if the crystal is at rest. Therefore they are determined by the vortex velocity $\vec{v}_L$ in the laboratory reference frame connected with the crystal. The case when the crystal is not at rest is discussed in Ref. 27.

According to Kopnin and Kravtsov[12] the force from impurities $\propto d,d'$ originates from interaction of impurities with the quasiparticles bound in the vortex core, and therefore moving with $\vec{v}_L$, but not with $\vec{v}_n$. However, in a Fermi superfluid, like ^{3}He, quasiparticles localized at the bound states in the vortex core interact also with free quasiparticles drifting with the normal velocity $\vec{v}_n$. This interaction contributes to the forces linear in $\vec{v}_L-\vec{v}_n$ ($\propto D$ and $\propto D'$).[25] But this contribution, however important it is, is not considered in the present paper. This means that the force $\propto D'$ in Eq. (22) includes only the Iordanskii force from quasiparticles scattered by the velocity field around the vortex line.

One can rewrite the equation (21) of vortex motion collecting together the terms proportional to the velocity $\vec{v}_L$:

$$\rho_M[\vec{v}_L\times\vec{\kappa}]+\eta\vec{v}_L=\rho_s[\vec{v}_s\times\vec{\kappa}]+D\vec{v}_n+D'[\hat{z}\times\vec{v}_n]. \tag{23}$$

The forces on the left-hand side of the equation are the *effective Magnus force* $\propto\rho_M=\rho_s-(D'+d')/\kappa$ and the friction force $\propto\eta=D+d$. The forces on the right-hand side are driving forces produced by the superfluid and normal flows. In the theory of superconductivity the force $\vec{F}_L=\rho_s[\vec{v}_s\times\vec{\kappa}]=(1/c)[\vec{j}_s\times\vec{\Phi}_0]$, proportional to the superfluid velocity $\vec{v}_s$ (or to the supercurrent $\vec{j}_s=en_s\vec{v}_s$), is called the Lorentz force. Here $\Phi_0=hc/2e$ is the magnetic-flux quantum and the vector $\vec{\Phi}_0$ is parallel to $\vec{\kappa}$. There are also forces on the vortex produced by the normal current $\vec{j}_n=en_n\vec{v}_n$. One can find discussion of the effect of the normal-current force on electrodynamics of a type II superconductors in Ref. 46.

The left-hand side of Eq. (23) presents the response of the vortex to these driving forces. The factor ρ_M, which determines the amplitude of the effective Magnus force on the vortex, is not equal to the superfluid density ρ_s in general: it may be more or less than ρ_s. Note that the Hall conductivity is governed by ρ_M, but not by ρ_s. At low magnetic fields the normal current is small compared to the supercurrent, i.e., the total current $\vec{j}\approx\vec{j}_s=en_s\vec{v}_s$ and one may neglect the terms $\propto\vec{v}_n$ on the right-hand side of Eq. (23). On the other hand, the electric field is connected with the vortex velocity by the Josephson relation $\vec{E}=1/c[\vec{H}\times\vec{v}_L]$. Then the equation of vortex motion is equivalent to the Ohm law connecting the current and the electric field. One can easily check that the Hall component of the conductivity is linear in ρ_M.

In the superfluidity theory they usually present the equation of vortex motion using the mutual friction parameters B and B' introduced by Hall and Vinen.[3] Because of Galilean invariance $d=d'=0$ for superfluids, and neglecting the normal motion ($\vec{v}_n=0$) the equation is

$$\vec{v}_L=\left(1-\frac{\rho_n}{2\rho}B'\right)\vec{v}_s+\frac{\rho_n}{2\rho}B[\hat{z}\times\vec{v}_s]$$
$$=\frac{\rho_s\rho_M\kappa^2}{D^2+(\rho_M\kappa)^2}\vec{v}_s+\frac{\rho_s\kappa D}{D^2+(\rho_M\kappa)^2}[\hat{z}\times\vec{v}_s]. \tag{24}$$

In the next section we shall calculate the amplitude D' of the Iordanskii force analyzing the interaction of the vortex with phonons in the long-wavelength limit.

III. IORDANSKII FORCE AND AHARONOV-BOHM EFFECT

A. Nonlinear Schrödinger equation and two-fluid hydrodynamics

The Gross-Pitaevskii theory[30] has used the nonlinear Schrödinger equation to describe a weakly nonideal Bose gas:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V|\psi|^2\psi. \tag{25}$$

Here $\psi=a\exp(i\phi)$ is the condensate wave function and V is the amplitude of two-particle interaction. Using the Madelung transformation,[47] this equation for a complex function may be transformed into two real equations for the liquid density $\rho=ma^2$ and the liquid velocity $\vec{v}=(\kappa/2\pi)\vec{\nabla}\phi$ where $\kappa=h/m$ is the circulation quantum. Far from the vortex line these equations are hydrodynamic equations for an ideal inviscid liquid:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}(\rho\vec{v})=0, \tag{26}$$

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}\mu. \tag{27}$$

Here $\mu=Va^2/m$ is the chemical potential.

Suppose that a plane sound wave propagates in the liquid generating the phase variation $\phi(\vec{r},t)=\phi_0\exp(i\vec{k}\cdot\vec{r}-i\omega t)$. Then the liquid density and velocity are functions of the time t and the position vector $\vec{r}$ in the plane xy and can be written in the form

$$\rho(\vec{r},t)=\rho_0+\rho_{(1)}(\vec{r},t),\quad \vec{v}(\vec{r},t)=\vec{v}_0+\vec{v}_{(1)}(\vec{r},t), \tag{28}$$

where ρ_0 and $\vec{v}_0$ are the average density and the average velocity in the liquid, $\rho_{(1)}(\vec{r},t)$ and $\vec{v}_{(1)}(\vec{r},t)=(\kappa/2\pi)\vec{\nabla}\phi$ are periodical variations of the density and the velocity due to the sound wave ($\langle\rho_{(1)}\rangle=0$, $\langle\vec{v}_{(1)}\rangle=0$). They should be determined from the hydrodynamic equations (26) and (27) after their linearization. In particular, Eq. (27) gives the relation between the density variation and the phase ϕ:

$$\rho_{(1)}=\frac{\rho_0}{c^2}\mu_{(1)}=-\frac{\rho_0}{c^2}\frac{\kappa}{2\pi}\left\{\frac{\partial\phi}{\partial t}+\vec{v}_0\cdot\vec{\nabla}\phi(\vec{r})\right\}, \tag{29}$$

where c is the sound velocity. Substitution of this expression into Eq. (26) yields the wave equation for a moving liquid. The sound wave has the spectrum $\omega=ck+\vec{k}\cdot\vec{v}_0$. The sound propagation is accompanied with the transport of mass. This is an effect of the second order with respect to the wave amplitude. In the reference frame moving with the average velocity $\vec{v}_0$ of the liquid the average mass current $\vec{j}^{\rm ph}$ is

$$\vec{j}^{\rm ph}(\vec{p})=\langle\rho_{(1)}\vec{v}_{(1)}\rangle=\rho_0\phi_0^2\frac{\kappa^2k}{8\pi^2c}\vec{k}=n(\vec{p})\vec{p}. \tag{30}$$

This expression supposes that the plane sound wave corresponds to a number $n(\vec{p})$ of phonons with the momentum $\vec{p}=\hbar\vec{k}$ and the energy $E=\varepsilon(\vec{p})+\vec{p}\cdot\vec{v}_0$ where $\varepsilon(\vec{p})=cp$ is the energy in the reference frame moving with the liquid velocity $\vec{v}_0$. Then the total mass current in the laboratory reference frame is

$$\vec{g}=\rho_0\vec{v}_0+\frac{1}{h^3}\int d_3\vec{j}^{\rm ph}(\vec{p})=\rho_0\vec{v}_0+\frac{1}{h^3}\int d_3\vec{p}\,n(\vec{p})\vec{p}. \tag{31}$$

In the thermal equilibrium at $T>0$, the phonon numbers are given by the Planck distribution $n(\vec{p})=n_0(E,\vec{v}_n)$ with the drift velocity $\vec{v}_n$ of quasiparticles:

$$n_0(E,\vec{v}_n)=\frac{1}{\exp\dfrac{E(\vec{p})-\vec{p}\cdot\vec{v}_n}{T}-1}$$
$$=\frac{1}{\exp\dfrac{\varepsilon(\vec{p})+\vec{p}\cdot(\vec{v}_0-\vec{v}_n)}{T}-1}. \tag{32}$$

Linearizing Eq. (32) with respect to the relative velocity $\vec{v}_0-\vec{v}_n$, one obtains from Eq. (31) that

$$\vec{g}=\rho_0\vec{v}_0+\rho_n(\vec{v}_n-\vec{v}_0). \quad (33)$$

This expression is equivalent to the two-fluid expression $\vec{g}=\rho\vec{v}_s+\rho_n(\vec{v}_n-\vec{v}_s)=\rho_s\vec{v}_s+\rho_n\vec{v}_n$ assuming that $\rho=\rho_0=\rho_s+\rho_n$, $\vec{v}_0=\vec{v}_s$, and the normal density is given by the usual two-fluid-theory expression:

$$\rho_n=-\frac{1}{3h^3}\int\frac{\partial n_0(\varepsilon,0)}{\partial E}p^2 d_3\vec{p}. \quad (34)$$

It is important to emphasize a difference between sound waves in a liquid and in an elastic solid. The sound wave in the elastic solid is not accompanied by real mass transport in the laboratory reference frame: all atoms oscillate near their equilibrium positions in the crystal lattice. Within our present formalism this means that the second-order contribution $\langle\rho_{(1)}\vec{v}_{(1)}\rangle$ to the mass flow is compensated by $\rho_0\langle\vec{v}_{(2)}\rangle$ where $\langle\vec{v}_{(2)}\rangle$ the second-order contribution to the average velocity $\vec{v}_0$. But in the case of the liquid the latter contribution is not essential since one assumes that the fixed final average velocity $\vec{v}_0$ incorporates *all* contributions to it. Thus when we say about the mass transport by phonons we mean the transport in the reference frame moving with the average velocity $\vec{v}_0$. In the presence of phonons the latter is different from the center-of-mass velocity $\vec{g}/\rho_0$. This difference is possible because of using Euler variables: the average velocity $\vec{v}_0$ relates to a *given* point in the space, whereas the average velocity related to a *given* particle (a Lagrange variable) coincides with the center-of-mass velocity.

In the same manner one may derive the two-fluid expression Eq. (12) for the momentum flux tensor. Expanding the momentum-tensor up to the terms of the second-order with respect to the sound wave amplitude one obtains

$$\Pi_{ij}=P_0\delta_{ij}+\rho_0 v_{0i}v_{0j}+\Pi_{ij}^{\text{ph}}, \quad (35)$$

where the second-order phonon contribution is

$$\Pi_{ij}^{\text{ph}}=\langle P_{(2)}\rangle\delta_{ij}+\langle\rho_{(1)}(v_{(1)})_i\rangle v_{0j}+\langle\rho_{(1)}(v_{(1)})_j\rangle v_{0i}+\rho_0\langle(v_{(1)})_i(v_{(1)})_j\rangle. \quad (36)$$

The second-order contribution $P_{(2)}$ to the pressure can be obtained from the Gibbs-Duhem relation $\delta P=\rho\delta\mu$ for an ideal fluid at $T=0$ using expansions $\rho=\rho_0+\rho_{(1)}$ and $\mu=\mu_0+\mu_{(1)}+\mu_{(2)}$, where μ_0 is the chemical potential without the sound wave. This yields the second-order phonon contribution $P_{(2)}=\rho_0\mu_{(2)}+(\partial\mu/\partial\rho)(\rho_{(1)}^2/2)$ to the pressure, where $\partial\rho/\partial\mu=\rho_0/c^2$. According to the Euler equation (4) the second-order contribution to the chemical potential is $\mu_{(2)}=-(v_{(1)}^2/2)$. Then

$$\langle P_{(2)}\rangle=\frac{c^2}{\rho_0}\frac{\langle\rho_{(1)}^2\rangle}{2}-\rho_0\frac{\langle v_{(1)}^2\rangle}{2}. \quad (37)$$

The last term in Eq. (36) may be transformed as follows:

$$\rho_0\langle(v_{(1)})_i(v_{(1)})_j\rangle=\frac{1}{h^3}\int d_3\vec{p}\,n_0(E,\vec{v}_n)\frac{c}{p}p_ip_j\approx\frac{1}{h^3}\int d_3\vec{p}\,n_0(\varepsilon,0)\frac{c}{p}p_ip_j+\frac{1}{2h^3}\int d_3\vec{p}\frac{\partial^2 n_0(\varepsilon,0)}{\partial\varepsilon^2}[\vec{p}\cdot(\vec{v}_n-\vec{v}_0)]^2\frac{\partial\varepsilon}{\partial p_i}p_j$$

$$=\frac{1}{3h^3}\int d_3\vec{p}\,n_0(\varepsilon,0)cp\,\delta_{ij}-\frac{1}{2h^3}\int d_3\vec{p}\frac{\partial n_0(\varepsilon,0)}{\partial\varepsilon}[\vec{p}\cdot(\vec{v}_n-\vec{v}_0)]^2\delta_{ij}$$

$$-\frac{1}{h^3}\int d_3\vec{p}\frac{\partial n_0(\varepsilon,0)}{\partial\varepsilon}[\vec{p}\cdot(\vec{v}_n-\vec{v}_0)](v_{ni}-v_{0i})p_j$$

$$=\left[\frac{1}{3h^3}\int d_3\vec{p}\,n_0(\varepsilon,0)cp+\frac{1}{2}\rho_n(\vec{v}_n-\vec{v}_0)^2\right]\delta_{ij}+\rho_n(v_{ni}-v_{0i})(v_{nj}-v_{0j}) \quad (38)$$

Bearing in mind that $\langle\rho_{(1)}\vec{v}_{(1)}\rangle=\rho_n(\vec{v}_n-\vec{v}_0)$, Eq. (35) at $\rho_0=\rho$, $\vec{v}_0=\vec{v}_s$ coincides with the two-fluid momentum-flux tensor Eq. (12) where the pressure incorporates a phonon contribution for a liquid at rest and the Bernoulli terms given by Eq. (14).

This analysis demonstrates that two-fluid hydrodynamics with phonon quasiparticles is identical to the nonlinear Schrödinger equation with thermally excited sound waves. A next step is to analyze scattering of phonons by the vortex in hydrodynamics of an ideal liquid.

B. Scattering of phonons by the vortex in hydrodynamics

The phonon scattering by a vortex line was studied beginning from the works by Pitaevskii[48] and Fetter.[49] Let us consider a sound wave $\phi(\vec{r},t)=\phi_0\exp(i\vec{k}\cdot\vec{r}-i\omega t)$ propagating in the plane xy normal to a vortex line (the axis z). Then in linearized hydrodynamic equations of the previous section the fluid velocity $\vec{v}_0$ should be replaced by the velocity $\vec{v}_v(\vec{r})$ around the vortex line. The hydrodynamic equations linearized with respect to the wave amplitude are

$$\frac{\partial\rho_{(1)}}{\partial t}+\rho_0\vec{\nabla}\cdot\vec{v}_{(1)}=-\vec{v}_v\cdot\vec{\nabla}\rho_{(1)}, \quad (39)$$

$$\frac{\partial\vec{v}_{(1)}}{\partial t}+\frac{c^2}{\rho_0}\vec{\nabla}\rho_{(1)}=-\vec{\nabla}\cdot[(\vec{v}_v\cdot\vec{\nabla})\vec{v}_{(1)}+(\vec{v}_{(1)}\cdot\vec{\nabla})\vec{v}_v]. \quad (40)$$

Using the vector identity Eq. (18) for the velocity $\vec{v}=\vec{v}_v+\vec{v}_{(1)}$, Eq. (40) can be rewritten as

$$\frac{\partial \vec{v}_{(1)}}{\partial t}+\frac{c^2}{\rho_0}\vec{\nabla}\rho_{(1)}=-\vec{\nabla}(\vec{v}_v\cdot\vec{v}_{(1)})+[\vec{v}_{(1)}\times\vec{\kappa}]\delta_2(\vec{r}). \tag{41}$$

One can see from this equation that the perturbation from the vortex (the right-hand side) contains strong singularities related to the fact that the vortex line is not at rest when the sound wave propagates past the vortex. In order to weaken these singularities in Ref. 7 the time-dependent vortex velocity $\vec{v}_v(\vec{r},t)$ was introduced as a zero-order approximation for the velocity field. This means that $\vec{r}$ in Eq. (1) must be replaced by $\vec{r}-\vec{v}_L t$ and $\partial\vec{v}_v/\partial t=-(\vec{v}_L\cdot\vec{\nabla})\vec{v}_v=-\vec{\nabla}(\vec{v}_L\cdot\vec{v}_v)+[\vec{v}_L\times\vec{\kappa}]\delta_2(\vec{r})$. Here $\vec{v}_L$ is the velocity of the singular vortex line. Since there is no external force on the liquid, the vortex moves with the velocity in the sound wave: $\vec{v}_L=\vec{v}_{(1)}(0,t)$. Now in the linearization procedure the fluid acceleration in Eq. (27) must be presented as $\partial\vec{v}/\partial t=\partial\vec{v}_v/\partial t+\partial\vec{v}_{(1)}/\partial t$. As a result Eq. (41) is replaced by

$$\frac{\partial \vec{v}_{(1)}}{\partial t}+\frac{c^2}{\rho_0}\vec{\nabla}\rho_{(1)}=\vec{\nabla}[\vec{v}_v\cdot\vec{v}_{(1)}(r)]-\vec{\nabla}[\vec{v}_v\cdot\vec{v}_{(1)}(0)]. \tag{42}$$

Now the perturbation from the vortex line on the right-hand side is free from singularities of Eq. (41). Equation (42) yields

$$\rho_{(1)}=-\frac{\rho_0}{c^2}\frac{\kappa}{2\pi}\left\{\frac{\partial\phi}{\partial t}+\vec{v}_v\cdot[\vec{\nabla}\phi(\vec{r})-\vec{\nabla}\phi(0)]\right\}. \tag{43}$$

Substitution of $\rho_{(1)}$ in Eq. (39) yields the linear equation for the phonon-induced phase:

$$\frac{\partial^2\phi}{\partial t^2}-c^2\vec{\nabla}^2\phi=-2\vec{v}_v(\vec{r})\cdot\vec{\nabla}\frac{\partial}{\partial t}\left[\phi(\vec{r})-\frac{1}{2}\phi(0)\right]. \tag{44}$$

In the long-wavelength limit $k\to 0$ one may treat interaction with the vortex velocity field [the right-hand side of Eq. (44)] as a small perturbation. It is equivalent to the Born approximation. The small perturbation parameter $\kappa k/c$ is on the order of the ratio of the wavelength $2\pi/k$ to the vortex core radius $r_c\sim\kappa/c$. Then after substituting the plane wave into the right-hand side of Eq. (44) the solution of this equation is

$$\phi=\phi_0\exp(-i\omega t)\left\{\exp(i\vec{k}\cdot\vec{r})+\frac{ik}{4c}\int d_2\vec{r}_1 \times H_0^{(1)}(k|\vec{r}-\vec{r}_1|)\vec{k}\cdot\vec{v}_v(\vec{r}_1)[2\exp(i\vec{k}\cdot\vec{r}_1)-1]\right\}. \tag{45}$$

Here $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind, and $(i/4)H_0^{(1)}(k|\vec{r}-\vec{r}_1|)$ is the Green function for the two-dimensional wave equation, i.e., satisfies to the equation

$$(k^2-\vec{\nabla}^2)\phi(\vec{r})=\delta_2(\vec{r}-\vec{r}_1). \tag{46}$$

The standard procedure in the scattering theory is the following. One uses the asymptotic expression for the Hankel function at large values of the argument:

$$\lim_{z\to\infty}H_0^{(1)}(z)=\sqrt{2/\pi z}\,e^{i(z-\pi/4)}. \tag{47}$$

Then it is assumed that the perturbation is confined to a finite vicinity of the line, where $r_1\ll r$, and

$$|\vec{r}-\vec{r}_1|\approx r-\frac{(\vec{r}_1\cdot\vec{r})}{r}. \tag{48}$$

Finally integration in Eq. (45) yields the phase field at large values of kr in a form of a superposition of the plane wave $\propto\exp(i\vec{k}\cdot\vec{r})$ and the scattered wave $\propto\exp(ikr)$:

$$\phi=\phi_0\exp(-i\omega t)\left[\exp(i\vec{k}\cdot\vec{r})+\frac{ia(\varphi)}{\sqrt{r}}\exp(ikr)\right]. \tag{49}$$

Here $a(\varphi)$ is the scattering amplitude which is a function of the angle φ between the initial wave vector $\vec{k}$ and the wave vector $\vec{k}'=k\vec{r}/r$ after scattering:

$$a(\varphi)=-\sqrt{k/2\pi}\frac{1}{c}e^{i(\pi/4)}[\vec{\kappa}\times\vec{k}']\cdot\vec{k}\frac{1}{q^2}\left(1-\frac{q^2}{2k^2}\right)=\frac{1}{2}\sqrt{k/2\pi}\frac{\kappa}{c}e^{i(\pi/4)}\frac{\sin\varphi\,\cos\varphi}{1-\cos\varphi}, \tag{50}$$

where $\vec{q}=\vec{k}-\vec{k}'$ is the momentum transferred by the scattered phonon to the vortex, and $q=2k\sin(\varphi/2)$.

This scattering amplitude is the same as obtained by Pitaevskii.[48] In the expressions for the scattering amplitude by Fetter[49] and Demircan *et al.*[33] the factor $(1-q^2/2k^2)=\cos\varphi$ is absent. This disagreement was explained either by using the Born approximation in the theory of Pitaevskii,[49] or by algebra mistakes in his paper.[33] Indeed, there is some confusion with numerical factors and signs in the original paper by Pitaevskii.[50] But the factor $\cos\varphi$ is not a result of wrong algebra. As was shown in Ref. 7, the factor arises from vortex oscillatory motion ignored by Fetter[49] and by Demircan *et al.*[33] Therefore their result is correct only for a *fixed* vortex line which is kept at rest by some external forces, e.g., by strong pinning. If the vortex-line singularity were at rest, in Eq. (50) the second term $q^2/2k^2$ in parentheses would be absent and Eq. (50) would agree with Fetter.[49] Fetter used the method of partial waves. In order to take into account vortex motion within this method, one should modify Fetter's analysis for the partial waves with $l=\pm 1$ as shown in the end of Sec. III D.[51] Pitaevskii[48] did not discuss the effect of vortex line motion in his paper. He started from Eqs. (39) and (40) which correspond to a fixed vortex line as a zero-order approximation. But transformation of Eq. (40) to Eq. (41) shows that the δ-function perturbation from the vortex motion is present in these equations. Due to this term the scattering amplitude by Pitaevskii coincides with Eq. (50) and with results of the partial-wave analysis for a free vortex line in the long-wavelength limit.

Thus the vortex plays a role of a line defect which scatters the sound wave. Scattering results in the momentum transfer from the sound wave to the line defect, i.e., the sound wave produces a force on the defect. In order to find this force, one must determine the phonon contribution $F_i^{ph}=\int dS_j \Pi_{ij}^{ph}$ to the total momentum flux through the cylindrical surface around the line defect. If the perturbation by the line defect is confined to a finite vicinity of the line, the phonon contribution to the momentum-flux tensor is

$$\Pi_{ij}^{ph}=\frac{1}{2}\left(\frac{c^2}{\rho_0}\langle\rho_{(1)}^2\rangle-\rho_0\langle v_{(1)}^2\rangle\right)\delta_{ij}+\rho_0\langle(v_{(1)})_i(v_{(1)})_j\rangle. \tag{51}$$

In Appendix A it is shown that in this case the force on the line from the sound wave,

$$\vec{F}^{ph}=\sigma_\parallel c\vec{j}^{ph}-\sigma_\perp c[\hat{z}\times\vec{j}^{ph}], \tag{52}$$

is determined by two effective cross sections: the transport cross section for the dissipative force component,

$$\sigma_\parallel=\int\sigma(\varphi)(1-\cos\varphi)d\varphi, \tag{53}$$

and the transverse cross section for the transverse force component,

$$\sigma_\perp=\int\sigma(\varphi)\sin\varphi d\varphi. \tag{54}$$

The differential cross section $\sigma(\varphi)=|a(\varphi)|^2$ in these expressions is known due to Eq. (50) for the scattering amplitude $a(\varphi)$. It is quite natural that in the Born approximation the transverse cross section vanishes since the differential cross section is quadratic in the circulation κ.

However, the standard scattering-theory approach fails to describe the phonon scattering at small angles φ. Indeed, the velocity v_v induced around the vortex is decreasing very slowly, as $1/r$. Therefore the terms $\propto v_v$ in the phonon momentum flux are important in the momentum balance. The total momentum-flux tensor can be obtained from Eqs. (35) and (36) assuming $\vec{v}_0(\vec{r})=\vec{v}_v(\vec{r})+\vec{v}_s$ neglecting some unimportant terms:

$$\Pi_{ij}=-\rho_0(\vec{v}_s-\vec{v}_L)\cdot\vec{v}_v\delta_{ij}+\rho_0 v_{0i}v_{0j}+\langle\rho_{(1)}(v_{(1)})_i\rangle v_{vj}+\langle\rho_{(1)}(v_{(1)})_j\rangle v_{vi}+\rho_0\langle(v_{(1)})_i(v_{(1)})_j\rangle. \tag{55}$$

In addition, the scattering amplitude is divergent at $\varphi\to 0$:

$$\lim_{\varphi\to 0}a(\varphi)=\sqrt{k/2\pi}\frac{\kappa}{c}e^{i(\pi/4)}\frac{1}{\varphi}. \tag{56}$$

This divergence is integrable in the integral for the transport cross section, Eq. (53). So the calculation of the transport cross section is reliable. Contrary to it, the integrand in Eq. (54) for the transverse cross-section has a pole at $\varphi=0$, and the contribution of this pole requires an additional analysis. A proper analysis of the phonon small-angle scattering was fulfilled in Refs. 6 and 7. However, in a recent publication Demircan, Ao, and Niu[33] considered the phonon scattering by the vortex ignoring special features of the small-angle scattering. This is the reason why they could not find the transverse force from phonons on the vortex.

C. Small-angle phonon scattering and the Iordanskii force

At small scattering angles $\varphi\lesssim 1/\sqrt{kr}$ the asymptotic expansion given by Eq. (49) does not hold. The accurate calculation of the integral in Eq. (45) for small angles was done in Ref. 7. A simplified version of this calculation is presented in Appendix B. It yields that at $\varphi\ll 1$

$$\phi=\phi_0\exp(-i\omega t+i\vec{k}\cdot\vec{r})\left[1+\frac{i\kappa k}{2c}\Phi(\varphi\sqrt{kr/2i})\right]. \tag{57}$$

Using an asymptotic expression for the error integral

$$\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt\to\frac{z}{|z|}+\sqrt{2/\pi z}\exp(-z^2) \tag{58}$$

at $|z|\to\infty$, one obtains for angles $1\gg\varphi\gg 1/\sqrt{kr}$:

$$\phi=\phi_0\exp(-i\omega t)\left[\exp(i\vec{k}\cdot\vec{r})\left(1+\frac{i\kappa k}{2c}\frac{\varphi}{|\varphi|}\right)+\frac{i\kappa}{c}\sqrt{k/2\pi r}\frac{1}{\varphi}\exp\left(ikr+i\frac{\pi}{4}\right)\right]. \tag{59}$$

The second term in square brackets coincides with scattering wave at small angles $\varphi\ll 1$ when the scattering amplitude is given by Eq. (56). But now one can see that the standard scattering theory misses to reveal a very important nonanalytical correction to the incidental plane wave. We shall see in Sec. III E that the factor $\pm\kappa k/c$ which determines this correction is exactly the phase shift of the sound wave along the quasiclassical trajectories past the vortex on the right and left sides. This is a manifestation of the Aharonov-Bohm effect:[8] the sound wave after its interaction with the vortex velocity field has different phases on the left and on the right of the vortex line, and this phase difference results in an interference.

In the interference region the velocity induced by the sound wave is obtained by taking the gradient of the phase given by Eq. (57). The velocity component normal to the wave vector $\vec{k}$,

$$v_{(1)\perp}=\frac{\kappa}{2\pi r}\frac{\partial\phi}{\partial\varphi}=\phi_0\exp(-i\omega t+ikr)\frac{i\kappa^2 k}{2\pi c}\sqrt{k/2\pi r}\times\exp\left[i\left(\frac{1}{2}kr\varphi^2-\frac{\pi}{4}\right)\right], \tag{60}$$

determines the interference contribution to the transverse force:

$$\int dS_j \rho_0 \langle (v_{(1)})_\perp (v_{(1)})_j \rangle$$
$$= \int \rho_0 \langle (v_{(1)})_\perp (v_{(1)})_r \rangle r d\varphi$$
$$= \frac{1}{16\pi^2\sqrt{\pi}} \rho_0 \phi_0^2 \frac{\kappa^3 k^2}{c} \sqrt{kr} \int d\varphi \cos\left(\frac{1}{2} kr \varphi^2\right)$$
$$= \frac{1}{16\pi^2} \rho_0 \phi_0^2 \frac{\kappa^3 k^2}{c} = \frac{\kappa}{2} j^{\mathrm{ph}}. \tag{61}$$

Note, that this force contribution arises from the interference region with the transverse dimension $d_{\mathrm{int}} \sim \sqrt{r_0/k}$. Here r_0 is the large distance from the vortex line where the momentum balance is considered. But the interference region corresponds to very small scattering angles $\sim d_{\mathrm{int}}/r_0 = 1/\sqrt{kr_0}$. Thus an infinitesimally small angle interval yields a finite contribution to the transverse force. One could not reveal such a contribution from the standard scattering theory using the differential cross section.

Exactly the same contribution to the transverse force, as in Eq. (61), arises from the term $v_{v_i} \langle \rho_{(1)} (v_{(1)})_j \rangle$ in the momentum-flux tensor, Eq. (55). In this term the mass flow $\langle \rho_{(1)} (v_{(1)})_j \rangle$ for the plane wave may be used [see Eq. (30)], since we take into account only terms which are of the first order in the small parameter $\kappa k/c$. Finally the momentum balance $\int dS_j \Pi_{ij} = 0$ yields the relation

$$\rho_0 [(\vec{v}_L - \vec{v}_s) \times \vec{\kappa}] - [\vec{j}^{\mathrm{ph}}(\vec{p}) \times \vec{\kappa}] = 0. \tag{62}$$

The second vector product on the left-hand side is a transverse force given by the transverse cross section $\sigma_\perp = \kappa/c$. It is linear in the circulation quantum κ and therefore cannot be obtained from the differential cross section quadratic in κ. If there is the Planck distribution of phonons, $\vec{j}^{\mathrm{ph}}$ must be replaced by $(1/h^3)\int d_3\vec{p}\, n_0(\varepsilon, \vec{v}_n - \vec{v}_L)\vec{p} \approx -(1/h^3)\int d_3\vec{p}\, \partial n_0(\varepsilon,0)/\partial\varepsilon [\vec{p}\cdot(\vec{v}_n - \vec{v}_L)]\vec{p} = \rho_n(\vec{v}_n - \vec{v}_L)$. Then the momentum balance equation is

$$\rho_s [(\vec{v}_L - \vec{v}_s) \times \vec{\kappa}] + \rho_n [(\vec{v}_L - \vec{v}_n) \times \vec{\kappa}] = 0. \tag{63}$$

The force $\propto (\vec{v}_L - \vec{v}_n)$ is the Iordanskii force which corresponds to $D' = -\kappa\rho_n$ in Eq. (21). The longitudinal force $\propto D$ is not present in Eq. (63) since we ignored terms of the second order in κ in order to simplify our derivation. According to Eq. (63) the vortex moves with the center-of-mass velocity $\vec{v} = (\rho_s/\rho)\vec{v}_s + (\rho_n/\rho)\vec{v}_n$. But one should remember that the normal velocity $\vec{v}_n$ in this expression may differ from the normal velocity far from the vortex line because of the effect of *viscous drag* known from the first works on superfluid vortex dynamics[3] (see also Ref. 28).

D. Partial-wave analysis and the Aharonov-Bohm effect

In this subsection we shall rederive the Iordanskii force using expansion in partial waves in order to demonstrate the analogy with the Aharonov-Bohm effect.

Let us consider the equation which describes interaction of an electron with the magnetic flux Φ confined to a thin tube (the Aharonov-Bohm effect):[8]

$$E\psi(\vec{r}) = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \psi(\vec{r}). \tag{64}$$

Here ψ is the electron wave function with energy E and the electromagnetic vector is connected with the magnetic flux Φ by the relation similar to that for the velocity $\vec{v}_v$ around the vortex line [Eq. (1)]:

$$\vec{A} = \Phi \frac{[\hat{z} \times \vec{r}]}{2\pi r^2}. \tag{65}$$

An analogous equation for the sound wave with frequency $\omega = ck$,

$$k^2\phi - \left(-i\vec{\nabla} + \frac{k}{c}\vec{v}_v\right)^2 \phi = 0, \tag{66}$$

differs from the equation (44) for phonon-vortex interaction by the term of the second order in $v_v \propto \kappa$ and by absence of the contribution from the vortex-line motion [the term $\propto \phi(0)$ on the right-hand side of Eq. (44)]. This difference is unimportant for the calculation of the transverse force which is linear in κ.

Let us look for a solution of Eq. (64) as a superposition of the partial cylindrical waves using the cylindrical system of coordinates (r, φ):

$$\psi = \sum_l \psi_l(r) \exp(il\varphi). \tag{67}$$

The partial-wave amplitudes should satisfy equations

$$\frac{d^2\psi_l}{dr^2} + \frac{1}{r}\frac{d\psi_l}{dr} - \frac{(l-\gamma)^2}{r^2}\psi_l + k^2\psi_l = 0. \tag{68}$$

Here k is the wave number of the electron far from the vortex so that $E = \hbar^2 k^2/2m$ and $\gamma = \Phi/\Phi_1$ where $\Phi_1 = hc/e$ is the magnetic-flux quantum for one electron (two times larger than the magnetic-flux quantum $\Phi_0 = hc/2e$ for a Cooper pair). A solution of this equation, which has no singularity at small $r \to 0$, is the Bessel function $J_{|l-\gamma|}(kr)$ with the following asymptotics at large arguments:

$$J_{|l-\gamma|}(kr) \to \sqrt{2/\pi kr} \cos\left(kr - \frac{\pi}{2}|l-\gamma| - \frac{\pi}{4}\right). \tag{69}$$

On the other hand, the expansion of the plane wave in the partial cylindrical waves is

$$\exp(i\vec{k}\cdot\vec{r}) = \exp(ikr\cos\varphi) = \sum_l J_l(kr) \exp[il(\varphi + \pi/2)], \tag{70}$$

or at large kr:

$$\exp(i\vec{k}\cdot\vec{r})=\sqrt{2/\pi kr}\sum_l \cos\left(kr-\frac{\pi}{2}l-\frac{\pi}{4}\right)\times\exp[il(\varphi+\pi/2)]. \quad (71)$$

In order to obtain the solution ''the incoming plane wave + the scattered cylindrical wave'' like Eq. (49), one should determine the partial waves ψ_l from the condition that the incoming components$\propto\exp(-ikr)$ in the plane wave and in the solution Eq. (67) coincide. This yields that

$$\psi_l=\sqrt{2/\pi kr}\exp\left[i\frac{\pi}{2}(l-|l-\gamma|)\right]\cos\left(kr-\frac{\pi}{2}|l-\gamma|-\frac{\pi}{4}\right). \quad (72)$$

Then the solution of the Aharonov-Bohm problem takes the form of Eq. (49) with the scattering amplitude

$$a(\varphi)=\sqrt{1/2\pi k}\exp\left(i\frac{\pi}{4}\right)\sum_l [1-\exp(2i\delta_l)]\exp(il\varphi), \quad (73)$$

where

$$\delta_l=(l-|l-\gamma|)\pi/2 \quad (74)$$

is the partial-wave phase shift at $l\neq 0$. The S-wave phase shift is $\delta_0=\pm\gamma\pi/2$, but no physical effect depends on the choice of the sign in this expression.

Equation (54) for the transverse cross section may be rewritten as an expansion in partial waves:[52]

$$\sigma_\perp=\int |a(\varphi)|^2\sin\varphi d\varphi=\frac{1}{k}\sum_l \sin 2(\delta_l-\delta_{l+1}). \quad (75)$$

Using the phase shift values for the Aharonov-Bohm effect, Eq. (74), the transverse cross section is

$$\sigma_\perp=-\frac{1}{k}\sin 2\pi\gamma. \quad (76)$$

One can obtain the cross section for phonon scattering from this expression assuming that $\gamma=-\kappa k/2\pi c$ and expanding the sine function in small γ.

The cross section for the Aharonov-Bohm effect is periodical in the magnetic flux with the period equal to the one-electron flux quantum Φ_1. If the electron is scattered by the Cooper-pair flux quantum $\Phi_0=\Phi_1/2$ the transverse cross section vanishes. But presented analysis of the Aharonov-Bohm effect is based on the assumption that the total magnetic flux is concentrated in a very thin tube. Namely, the radius of the tube must be much less than the electron wavelength. This condition certainly does not hold in superconductors, where the electron wavelength is on the order of the interatomic distance and the effective radius of the magnetic-flux tube is the London penetration depth. Therefore one should consider scattering of electrons by a thick magnetic-flux tube.[52] Recently scattering of electrons by magnetic fluxons was analyzed by Nielsen and Hedegård.[53] They also confirmed the existence of the transverse force similar to the Iordanskii force. Scattering of the BCS quasiparticles by the magnetic field of the vortex also contributes to the transverse force on the vortex, but this contribution cancels with the bulk electromagnetic force.[10]

In their original paper[8] Aharonov and Bohm considered only the effect of the magnetic-flux tube on the electron wave. The force from the electrons on the fluxon was considered later by Aharonov and Casher[54] and therefore is called the Aharonov-Casher effect. So for our problem of interaction between quasiparticles and the vortex both effects, Aharonov-Bohm and Aharonov-Casher, are important. But these effects are so close one to another that throughout the paper we use only the name ''Aharonov-Bohm effect.''

In conclusion of this section let us consider how motion of a free vortex line must be incorporated into the partial-wave analysis of phonon scattering performed by Fetter.[49] He assumed that all partial waves were regular at small r, i.e., proportional to the Bessel function in Eq. (69) where $\gamma=-\kappa k/2\pi c$ for phonons is small and $|l-\gamma|\approx|l|-\gamma l/|l|$. It yielded for $l\neq 0$ the phase shifts given by Eq. (74). But if the vortex line is free to move, the velocity field is singular at $r\to 0$ where it is given by $\vec{v}(\vec{r})=\vec{v}_L+\vec{v}_{\rm sing}$. The vortex velocity $\vec{v}_L=d\vec{u}/dt$ is equal to the average liquid velocity (the Helmholtz theorem). Here $\vec{u}$ is the vortex-line displacement. The second contribution $\vec{v}_{\rm sing}=-(\vec{u}\cdot\vec{\nabla})\vec{v}_v(\vec{r})\approx-\vec{\nabla}[\vec{u}\cdot\vec{v}_v(\vec{r})]=(1/i\omega)\vec{\nabla}[(\vec{v}_L\cdot\vec{v}_v(\vec{r})]$ arises from motion of the vortex line. This contribution is of the first order in κ, but strongly singular ($\propto 1/r^2$). The phase which determines the velocity $\vec{v}(\vec{r})=(\kappa/2\pi)\vec{\nabla}\phi(\vec{r})$ at small r in cylindrical coordinates is

$$\phi(r,\varphi)\approx\frac{2\pi}{\kappa}v_L r\cos\varphi-\frac{1}{i\omega}\frac{v_L}{r}\sin\varphi=\phi_1 e^{i\varphi}+\phi_{-1}e^{-i\varphi}, \quad (77)$$

where

$$\phi_{\pm 1}=\frac{\pi}{\kappa}v_L r\pm\frac{1}{2\omega}\frac{v_L}{r}. \quad (78)$$

This assumes that the direction of $\vec{v}_L$ corresponds to $\varphi=0$. So the phase field (77) is a superposition of the partial waves $l=\pm 1$ which are singular at $r\to 0$. One must look for them in a form, more general than in Eq. (69):

$$\phi_{\pm 1}\sim J_{1\pm\gamma}(kr)+\alpha_{\pm 1}N_{1\pm\gamma}(kr). \quad (79)$$

Determining $\alpha_\pm$ one may neglect small γ in the orders of the Bessel functions. Then $J_{1\pm\gamma}(kr)\approx J_1(kr)\approx kr/2$, $N_{1\pm\gamma}(kr)\approx N_1(kr)\approx 2/\pi kr$, and Eq. (79) agrees with Eq. (78) if $\alpha_\pm=\pm(\kappa k/8c)$.

The parameters $\alpha_\pm$ are additional phase shifts due to vortex motion which must be added to the shifts $\delta_{\pm 1}$ given by Eq. (74). Performing summation in Eq. (73) Fetter obtained

$$\sum_l [1-\exp(2i\delta_l)]\exp(il\varphi)\approx 2i\sum_l \delta_l\exp(il\varphi)=\frac{\kappa k}{2c}\frac{\sin\varphi}{1-\cos\varphi}. \quad (80)$$

The shifts $\alpha_\pm$ contribute the term $-(\kappa k/2c)\sin\varphi$ to this expression, and this yields an additional factor $\cos\varphi$. After it the scattering amplitude obtained from partial waves does not differ from that in Refs. 48 and 7.

E. The transverse force in the quasiclassical theory

Now we shall show that the Iordanskii force follows also from the quasiclassical theory of scattering by the vortex, despite one cannot use the quasiclassical theory for phonons. But the quasiclassical theory is valid to describe the roton contribution to the transverse force.

Let us consider a quasiparticle with an arbitrary spectrum $\varepsilon(p)$. If the quasiparticle moves in the velocity field induced by the vortex, its energy is $E(\vec{p})=\varepsilon(p)+\vec{p}\cdot\vec{v}_v$ which may be treated as a Hamiltonian for the classic equations of motion:

$$\frac{d\vec{r}}{dt}=\frac{\partial E}{\partial\vec{p}}=v_G\frac{\vec{p}}{p}+\vec{v}_v\,,$$

$$\frac{d\vec{p}}{dt}=-\frac{\partial E}{\partial\vec{r}}=-\frac{\partial}{\partial\vec{r}}(\vec{p}\cdot\vec{v}_v), \tag{81}$$

where $v_G(p)=d\varepsilon/dp$ is the quasiparticle group velocity. As before, we assume that the quasiparticle moves in the plane xy normal to the vortex. From this equations one can find the classical trajectory of the quasiparticle moving past the vortex line. Usually it is close to a straight line, and a distance of the straight trajectory from the vortex line is the impact parameter b. We are looking for the transverse force, then we need only the variation $\delta p_\perp$ of the momentum component normal to the initial momentum $\vec{p}$ which results from quasiparticle motion past the vortex. The momentum of the quasiparticle is connected to the classical action: $\vec{p}=\partial S/\partial\vec{r}$. Then $\delta p_\perp(b)=\partial\,\delta S(b)/\partial b$ where the total variation of the classical action $\delta S(b)$ along the trajectory is a function of the impact parameter b. Solution of Eqs. (81) in the first order with respect to $\vec{v}_v$ yields that

$$\delta S(b)=-\int_{-\infty}^{\infty}dl\frac{p}{2\pi v_G(l)}\,\frac{\kappa b}{b^2+l^2}, \tag{82}$$

where l is the coordinate along the trajectory. The scattering angle of the quasiparticle is $\varphi\approx-\delta p_\perp(b)/p$ and the transverse cross section is

$$\begin{aligned}\sigma_\perp&=\int_{-\infty}^{\infty}db\ \sin\varphi\\&\approx\int_{-\infty}^{\infty}db\,\varphi(b)\\&=\frac{\delta S(-\infty)-\delta S(+\infty)}{p}\\&=\frac{1}{2\pi v_G}\int_{-\infty}^{\infty}db\frac{\partial}{\partial b}\int_{-\infty}^{\infty}dl\frac{\kappa b}{b^2+l^2}=\frac{\kappa}{v_G}.\end{aligned} \tag{83}$$

For phonons the group velocity v_G is equal to the sound velocity c and does not depend on l. Therefore the action variation δS given by Eq. (82) does not depend on the impact parameter b at $b\neq 0$. This means that there is no phonon scattering in the quasiclassical approximation: $\varphi\propto\partial\,\delta S(b)/db=0$. A nonzero transverse cross section $\sigma_\perp$ for phonons arises from a finite jump of δS at $b=0$. In fact, one may not use the quasiclassical theory to calculate this contribution. Nevertheless, Eq. (83) yields the correct value of $\sigma_\perp=\kappa/c$ for phonons.

In the case of rotons $\delta S(b)$ is a continuous function of b and the quasiclassical theory is reliable for the calculation of the Iordanskii (Lifshitz-Pitaevskii) force. But it is important to note that the double integral of Eq. (83) is improper: its value depends on what integration is done first. The correct procedure which was justified in Ref. 7 is to integrate along the trajectory first, and to integrate over the impact parameters afterwards. A way to check it is the following. We choose some finite limits in the double integral of Eq. (83) which means that the integration is restricted by some area around the vortex line. The integral depends on the shape of this area. For example, a circular border of the area yields $\sigma_\perp$ by a factor 2 less than that of Eq. (83). But the full solution of the collisionless kinetic equation for the quasiparticles made in Ref. 7 showed that the slow decrease of the velocity field produces other contributions to the momentum balance. Taking into account all of them, we arrive again to the expression for the force via the cross section given by Eq. (83). The order of integrations in Eq. (83) assumes that the integration area has a shape of a rectangular elongated along the quasiparticle trajectory. For such a shape all other contributions to the transverse force exactly cancel.

A rather simple and universal expression $D'=-\kappa\rho_n$ for the Iordanskii force amplitude tempts to claim its universal topological origin, since κ in this expression is a topological charge. However, in the next section we shall see that the expression is not universal, in fact. For quasiparticles in a BCS superconductor with energy much exceeding the gap an additional small factor should be put into this expression.

IV. IORDANSKII FORCE FOR QUASIPARTICLES IN BCS SUPERCONDUCTORS

The wave function of quasiparticles in the BCS theory has two components,

$$\psi(\vec{r})=\begin{pmatrix}u(\vec{r})\\ v(\vec{r})\end{pmatrix}, \tag{84}$$

which are determined from the Bogolyubov–de Gennes equations:

$$-\frac{\hbar^2}{2m}(\vec{\nabla}^2+k_F^2)u(\vec{r})+\Delta\ \exp[i\theta(\vec{r})]v(\vec{r})=Eu(\vec{r}), \tag{85}$$

$$\frac{\hbar^2}{2m}(\vec{\nabla}^2+k_F^2)v(\vec{r})+\Delta\ \exp[-i\theta(\vec{r})]u(\vec{r})=Ev(\vec{r}). \tag{86}$$

Here k_F is the Fermi wave number. We neglect the magnetic field effect which is not essential for the transfer of momentum from quasiparticles to the vortex if the London penetration depth is large compared to other relevant scales. Therefore only the canonical phase θ of the order parameter, but not the electromagnetic vector $\vec{A}$, is present in the phase factor of the gap Δ. Without the vortex the order parameter phase θ is a constant and the equations yield the well-known BCS quasiparticle spectrum $E=\sqrt{\xi^2+\Delta^2}$, where $\xi=(\hbar^2/2m)(k^2-k_F^2)$ is the quasiparticle energy in the normal Fermi liquid.

In Refs. 10 and 11 the Bogolyubov–de Gennes equations for a quasiparticle passing a vortex were solved with help of the partial-wave expansion, earlier used also in Ref. 52. Quasiparticles with the energy close to the gap ($\xi \ll \Delta$) behave as rotons and the transverse cross section for them is given by Eq. (83) in which the group velocity for the BCS quasiparticles is $v_G=v_F\xi/E$. Here $v_F=\hbar k_F/m$ is the Fermi velocity. In the case when the quasiparticle energy is much more than the superconducting gap, the theory yielded that the transverse cross section different from the quasiclassical result of Eq. (83) by the factor $\Delta^2/2\xi^2$. Now we rederive this result using the Born approximation similar to that for phonons in Secs. III B and III C.

We use the perturbation theory with respect to the gap Δ and the gradient of the order parameter phase $\vec{\nabla}\theta$. Then in the zero-order approximation $u=u_0\exp(i\vec{k}\cdot\vec{r})$ and $v=0$. In the first-order approximation the second Bogolyubov–de Gennes equation (86) yields

$$v=\left\{\frac{\Delta\ \exp(-i\theta)}{\xi(k)+E(k)}+\frac{\Delta\ \exp(-i\theta)}{[\xi(k)+E(k)]^2}\frac{\hbar^2}{m}(\vec{k}\cdot\vec{\nabla}\theta)\right\} \times u_0\exp(i\vec{k}\cdot\vec{r}). \tag{87}$$

The first term in curly braces yields a correction to the quasiparticle energy $\propto\Delta^2$, but does not contribute to scattering which is determined by the order-parameter phase gradients. So we keep only the second term proportional to $\vec{\nabla}\theta$. Inserting it to the first Bogolyubov–de Gennes equation (85) one obtains the following equation for the first-order correction to the plane wave:

$$(\nabla^2-k^2)u_{(1)}=(\vec{k}\cdot\vec{\nabla}\theta)\frac{\Delta^2}{2\xi^2}u_0\exp(i\vec{k}\cdot\vec{r}). \tag{88}$$

This equation is similar to equation (44) for the sound wave and using this analogy one easily obtains the expression for the transverse cross section:

$$\sigma_\perp=\frac{\Delta^2}{2\xi^2}\frac{\pi}{k_F}=\frac{\Delta^2}{2\xi^2}\frac{\kappa}{v_F}. \tag{89}$$

Since the group velocity of quasiparticles with $E\gg\Delta$ is about the Fermi velocity v_F, this expression differs from Eq. (83) by the factor $\Delta^2/2\xi^2$. After integration over the quasiparticle distribution one obtains that $D'\approx-(\Delta/2T_c)\kappa\rho_n \approx-(\Delta/4T_c)hn$ close to T_c (see Refs. 10 and 11). Here n is the total electron density. One should remember, however, that the coefficient D' may be modified by the effect of the bound states in the core[13,14] which is beyond the scope of the present paper.

V. MAGNUS FORCE IN THE JOSEPHSON JUNCTION ARRAY

In the continuum limit, the equation of motion for a vortex has been derived by Eckern and Schmid.[37] This derivation has not revealed any force normal to the vortex velocity. Absence of the Magnus force suggests that the vortices move parallel to the driving force, i.e., normal to the current, and there is no Hall effect. Then in the limit of weak dissipation the ballistic vortex motion is possible, which is a free vortex motion without friction and the driving force. The Hall effect and the ballistic motion are incompatible, since the latter is a regime with a finite electrical field and no external current which is impossible for a finite Hall resistance. Though there have been experimental evidences of the ballistic vortex motion,[35] one may suspect that a more sophisticated theory would reveal the Hall resistance, however small. In the present section it will be shown that the Hall effect is *exactly* absent in the classical limit for the JJA. It directly follows from the symmetry of the dynamic equations.

Let us consider a conductor in a magnetic field $\vec{H}$. When its symmetry is not less than the threefold (which includes a triangular and square lattices), the Ohm law is

$$\vec{E}=\rho_L\vec{I}+\rho_H\vec{n}\times\vec{I}, \tag{90}$$

where $\vec{n}=\vec{H}/H$, $\vec{E}$ is the electrical field, ρ_L is the longitudinal resistance, and ρ_H is the Hall resistance in the magnetic field $\vec{H}$. Now let us consider the transformation in which the directions of the fields $\vec{E}$ and $\vec{H}$ and the current $\vec{I}$ are reversed:

$$\vec{E}\to-\vec{E},\quad \vec{n}\to-\vec{n}(\vec{H}\to-\vec{H}),\quad \vec{I}\to-\vec{I}. \tag{91}$$

The Ohm law Eq. (90) is invariant with respect to this *field-current inversion* only for a system without the Hall effect ($\rho_H=0$). On the microscopical level the field-current-inversion invariance is a direct result of the *particle-hole symmetry* which was shown to forbid the Hall effect in the Ginzburg-Landau theory (see Ref. 29 and the references therein).

Next we consider the JJA with the energy

$$\mathcal{E}=\frac{1}{2}\sum_{\vec{l},\vec{k}}\ [Q_{\vec{l}}C^{-1}_{\vec{l},\vec{k}}\ Q_{\vec{k}}-E_J\sin(\phi_{\vec{l}}-\phi_{\vec{k}})], \tag{92}$$

and the equations of motion

$$V_{\vec{l}}=\frac{\hbar}{2e}\frac{d\phi_{\vec{l}}}{dt}, \tag{93}$$

$$\sum_{\vec{\mu}}\ C_{\vec{l},\vec{l}+\vec{\mu}}\frac{dV_{\vec{l}+\vec{\mu}}}{dt}-I_C\sum_{\vec{\mu}}\ \sin(\phi_{\vec{l}}-\phi_{\vec{l}+\vec{\mu}})+\sum_{\vec{\mu}}\ \sigma_{\vec{l},\vec{l}+\vec{\mu}}V_{\vec{l}+\vec{\mu}} =0. \tag{94}$$

Here $V_{\vec{l}}$ is the electric potential, $Q_{\vec{l}}=\Sigma_{\vec{\mu}}C_{\vec{l},\vec{l}+\vec{\mu}}V_{\vec{l}+\vec{\mu}}$ is the electric charge, $\phi_{\vec{l}}(t)$ is the gauge invariant phase at the node specified by the discrete two-dimensional position vector $\vec{l}$, E_J is the Josephson coupling energy, $I_C=2eE_J/\hbar$ is the critical current, and $C_{\vec{l},\vec{n}}$ and $\sigma_{\vec{l},\vec{n}}$ are the capacity and the conductance matrices, respectively. In the external magnetic field $\vec{H}=\vec{\nabla}\times\vec{A}$, the gauge invariant phase $\phi_{\vec{l}}$ is not single valued; in fact only its difference between neighboring nodes is well defined:

$$\phi_{\vec{l}+\vec{\mu}}-\phi_{\vec{l}}=\varphi_{\vec{l}+\vec{\mu}}-\varphi_{\vec{l}}-\frac{2\pi}{\Phi_0}\int_{(\vec{l})}^{(\vec{l}+\vec{\mu})}+\vec{A}\cdot d\vec{l}. \quad (95)$$

Here $\varphi_{\vec{l}}$ is the canonical phase at the node $\vec{l}$, and the integral over $\vec{A}$ is taken between the centers of the two neighboring nodes. The canonical phase is not single valued too, but its circulation along any closed path through the nodes of JJA is always an integer number of 2π, while the circulation of the gauge invariant phase may be any number depending on the magnetic field.

When both the external current $\vec{I}$ and the magnetic field $\vec{H}$ are applied to the JJA, the gauge-invariant phase can be presented as $\phi_{\vec{l}}=\phi_{\vec{l}}^{H}+\phi_{\vec{l}}^{I}$. Here $\phi_{\vec{l}}^{H}$ is the time-independent phase in the equilibrium state without an external current, and $\phi_{\vec{l}}^{I}$ is the time-dependent contribution to the phase from the external current $\vec{I}$. The multivaluedness of the phase related to the magnetic field is present only in the static phase $\phi_{\vec{l}}^{H}$: the time-dependent dynamical contribution $\phi_{\vec{l}}^{I}$ to the phase is single valued. One sees then that the field-current inversion [Eq. (91)] simply corresponds to the change of signs of all phases, and the equations of motion, Eqs. (93) and (94), are invariant with respect to this transformation. It proves that the Hall effect does not exist in the JJA, i.e., the effective Magnus force vanishes.

The crucial point of this simple derivation is that we used the *static* vortex solution $\phi_{\vec{l}}^{H}$ for the dynamical problem. This assumes that singularities of the phase distribution related to the presence of vortices are kept at rest despite the vortices themselves are driven by the Lorentz force. For continuous superconductors this approach is invalid and our derivation does not work (cf. the effect of vortex-line motion in Sec. III B). So there is a fundamental difference between vortices in a lattice and vortices in a continuous superconductor. Indeed, in the lattice there are no singular vortex lines. They appear only in the continuum limit. At best, one can define the lattice cell containing the vortex center. This definition has been borrowed from the continuous theory: it is the cell, around which the circulation of the phase φ is equal to 2π. However, in the lattice the circulation around a closed path is not well defined. Let us consider some closed path through a discrete number of nodes with the phase circulation 2π. One may change the phase difference by -2π between any two neighboring nodes on the path without any effect on observed physical parameters (currents, voltages and so on). Then the circulation vanishes along the path considered, but must appear along a path over other nodes. Thus one cannot locate the position of the phase singularity. In order to avoid this ambiguity in the JJA model, a special rule has been formulated: the phase difference between two neighboring nodes must not exceed π. When for some bond the phase difference achieves the value π, one must redefine the phases; as a result, the vortex center is put into another cell. This procedure is usual for numerical studies of the vortex motion in JJA.[40] However, this rule is not obligatory for the dynamic theory of JJA. Instead, one may keep 2π circulations of the phase φ at fixed cells during the dynamic process without worrying where the vortex center (defined according to the aforementioned rule) is really located.

Because of this difference between vortices in a lattice and vortices in a continuous medium any derivation of the effective Magnus force in JJA using the continuous approach is not reliable. Let us discuss this in more details. In the continuum limit the set of discrete vectors $\vec{l}$ is replaced with the continuum space of $\vec{l}$. One can define the field of the canonical (but not gauge invariant) phase $\varphi(\vec{l})$ in this space everywhere except for the singular points which are the centers of the vortices with the phase circulation 2π around them. It is assumed that the spatial variation of the phases is small and the phase differences can be replaced by the phase gradients according to

$$\varphi_{\vec{l}+\vec{\mu}}-\varphi_{\vec{l}}\approx(\vec{\mu}\cdot\vec{\nabla})\varphi(\vec{l})\ll 1. \quad (96)$$

Then one can derive the partial differential equations for a continuous field of the canonical phase $\varphi(\vec{l},t)$ which correspond to some Lagrangian $L\{\varphi(\vec{l},t)\}$ (the dissipation is neglected now). The Lagrangian may include the term proportional to the time derivative of $\varphi(\vec{l},t)$ which is called the Wess-Zumino term:[21]

$$L\{\varphi(\vec{l},t)\}=\frac{1}{2}q\frac{\partial\varphi}{\partial t}+L_0\{\varphi(\vec{l},t)\}. \quad (97)$$

It is possible to derive the equation of vortex motion from this Lagrangian following Refs. 37 and 38. One must use the phase field for a slowly moving vortex:

$$\varphi^V(\vec{l},t)=\arctan\frac{l_y-y(t)}{l_x-x(t)}, \quad (98)$$

where $\vec{r}(t)=[x(t),y(t)]$ is the two-dimensional position vector of the vortex center. Substituting the vortex solution into the Lagrangian density given by Eq. (97) and integrating over the xy plane, one obtains the effective Lagrangian which is now a functional of the trajectory for a moving vortex:

$$\mathcal{L}^V\{\vec{r}(t)\}=-\pi q\dot{\vec{r}}[\hat{z}\times\vec{r}]+\mathcal{L}_0^V\{\vec{r}(t)\}. \quad (99)$$

Varying this Lagrangian with respect to $\vec{r}(t)$, one obtains the equation of vortex motion with the effective Magnus force $\propto q$:

$$2\pi q[\dot{\vec{r}}\times\hat{z}=\vec{F}_\Sigma. \quad (100)$$

Here $\vec{F}_\Sigma$ includes all other forces (the Lorentz force and the inertia force) obtained from the Lagrangian $\mathcal{L}_0^V\{\vec{r}(t)\}$ without the Wess-Zumino term. However, the factor q is not defined. If q is constant, it has no effect on the field equation for $\varphi(\vec{l},t)$ since the Wess-Zumino term is a full time derivative in the field Lagrangian, Eq. (97). The unknown factor q should be proportional to some electric charge, since the charge is a variable conjugate to the canonical phase φ. But it remains unclear what is this charge: either the background charge determined by the whole Fermi sea of the superconducting island, or an external charge induced outside as suggested in Ref. 44. Thus in the continuum limit the problem of the effective Magnus force and the Hall effect in the JJA remains unresolved. It must not be a surprise since in the continuum limit the JJA model becomes Galilean invariant and ''forgets'' that originally it had been a lattice model without translational invariance. Meanwhile, the latter is crucial for the amplitude of the effective Magnus force. An additional physical principle beyond the continuum theory should be involved to obtain the equation for the vortex velocity. This principle is provided by our symmetry analysis. According to it $q=0$ and one should not include the Wess-Zumino term into the field Lagrangian.

The presence of the external charge has no effect on our symmetry analysis. In order to take into account the external charge, one should use the Gibbs potential $G=\mathcal{E}-V^{\mathrm{ex}}\Sigma_{\vec{l}}Q_{\vec{l}}$, where $\mathcal{E}$ is given by Eq. (92) and V^{ex} is the electric potential which creates the external charge $Q^{\mathrm{ex}}=V^{\mathrm{ex}}\Sigma_{\vec{\mu}}C_{\vec{l}+\vec{\mu}}$. Then introducing the charge deviation $Q'_{\vec{l}}=Q_{\vec{l}}-Q^{\mathrm{ex}}$, one returns back to the energy $\mathcal{E}$ with $Q'_{\vec{l}}$ instead of $Q_{\vec{l}}$. These arguments show that the external charge cannot lead to the Hall effect: its effect is restricted with the shift of the Fermi level, but the particle-hole symmetry is restored with respect to the new Fermi level. However, the external electric charge may produce the Magnus force in the quantum theory of JJA which takes into account the electron charge quantization.[45] Then the Magnus force and the Hall conductivity are periodic in the electron charge.

VI. SUMMARY AND DISCUSSION

We have shown how the Magnus force appears in the equation of motion for a superfluid component (the superfluid Magnus force) and the equation of motion for a vortex (the effective Magnus force). Whereas the superfluid Magnus force proportional to the superfluid density is known exactly (from classical hydrodynamics, or from the Berry phase approach), there is no general expression for the effective Magnus force: it depends on interaction of the vortex with quasiparticles and with the external fields, like those from impurities in a dirty superconductor. Meanwhile, it is mostly the effective Magnus force which determines the observable effects: the mutual friction in superfluids, the Hall effect and the acoustic Faraday effect in superconductors, vortex quantum tunnelling.

We have presented the contribution of quasiparticles to the effective Magnus force for phonons in a superfluid and for BCS quasiparticles in a superconductor using the Born approximation. The transverse force from quasiparticles on the vortex (the Iordanskii force) originates from interference between quasiparticles passing on different sides of the vortex (the Aharonov-Bohm effect).

Our symmetry analysis of the Josephson junction array has demonstrated that the effective Magnus force exactly vanishes in the classical limit which means that there is no Hall effect despite the finite superfluid density. One may formally interpret this result that the force from external fields breaking Galilean invariance exactly compensates the superfluid Magnus force, though the analysis is not able to reveal these two forces separately.

The Ao-Thouless approach yields only the superfluid Magnus force which appears in the momentum balance of the superfluid component (the condensate). Indeed, Gaitan[21] derived the Ao-Thouless result for a charged superfluid, analyzing the momentum balance for the condensate. In order to derive the effective Magnus force (the total transverse force on the vortex), one must consider the momentum balance for the whole system.

Ao, Niu, and Thouless[55] stated that the Iordanskii force did not appear in their Berry phase approach. Demircan, Ao, and Niu[33] tried to justify it by the analysis of the Born quasiparticle scattering. They concluded that scattering did not produce a transverse force on a vortex. This conclusion was based on a wrong analysis of the Born phonon scattering missing the contribution from the Aharonov-Bohm interference. Thus the source of controversy is not in a difference of approaches, but in the problem how to calculate integrals for the Born phonon scattering.

The Ao-Thouless theory rejects also any force on the vortex from the external fields, like the Kopnin-Kravtsov force in a dirty superconductor. Using a similar approach based on the Berry phase Gaitan and Shenoy[23] predicted the finite effective Magnus force and the Hall effect for the Josephson-junction array. This prediction contradicts to our symmetry analysis and to the experiment. Gaitan and Shenoy[23] used in their analysis the Wess-Zumino term in the Lagrangian for the continuum limit of JJA. We have shown in Sec. V why this approach is not reliable.

Makhlin and Volovik[56] suggested that the superfluid Magnus force in JJA is nearly compensated by the force from the bound states in the junctions (the spectral flow of bound states). But they did not conclude that the compensation is complete, and assumed the Fermi superfluid in islands and the superconductor–normal-metal–superconductor Josephson junctions. Our analysis shows that the Magnus force *exactly* vanishes in the classical limit of the usual JJA model independently on microscopic nature of the superconducting islands and the junctions. This shows that the bound states and the spectral flow are not the only explanation for compensation of the Magnus force in the systems without Galilean invariance.

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APPENDIX A: THE FORCE ON THE LINE SCATTERING THE SOUND WAVE

First we derive the analogue of the optical theorem for the sound wave. For the latter we use the asymptotic representation, Eq. (49), in which the scattering amplitude $a(\varphi)$ is not necessarily obtained in the Born approximation. But in general $a(\varphi)$ should satisfy the condition that the total mass flow through the cylindrical surface surrounding the scattering line vanishes.

An asymptotic expression for the average mass flow from the sound wave is

$$\vec{j}^{\mathrm{ph}}=\langle\rho_{(1)}\vec{v}_{(1)}\rangle=\rho_0\phi_0^2\frac{\kappa^2k}{8\pi^2c}\left\{\vec{k}+\frac{|a|^2}{r}\vec{k}'-(\vec{k}+\vec{k}')\frac{1}{\sqrt{r}}[\mathrm{Im}\{a\}\cos(kr-\vec{k}\cdot\vec{r})+\mathrm{Re}\{a\}\sin(kr-\vec{k}\cdot\vec{r})]\right\}. \tag{A1}$$

The condition that the total flow through the cylindrical surface around the scattering line vanishes is

$$\int j_i^{\mathrm{ph}}dS_i=\int\langle\rho_{(1)}\vec{v}_{(1)i}\rangle dS_i=\rho_0\phi_0^2\frac{\kappa^2k^2}{8\pi^2c}\int\left\{\cos\varphi+\frac{|a(\varphi)|^2}{r}-\frac{\mathrm{Im}\{a(\varphi)\}}{\sqrt{r}}(1+\cos\varphi)\cos[kr(1-\cos\varphi)]\right\}d\varphi. \tag{A2}$$

The integral over the term $\propto\mathrm{Im}\{a(\varphi)\}$ expands only over the region of small angles since $kr\gg1$. Finally this condition yields

$$-2\sqrt{\pi/k}\,\mathrm{Im}\{a(0)\}+\int|a(\varphi)|^2d\varphi=0. \tag{A3}$$

Next let us consider the momentum balance which determines the force on the scattering line from the sound wave $F_i^{\mathrm{ph}}=-\int dS_j\Pi_{ij}^{\mathrm{ph}}$ where

$$\Pi_{ij}^{\mathrm{ph}}=\left(\frac{c^2}{\rho_0}\frac{\langle\rho_{(1)}^2\rangle}{2}-\rho_0\frac{\langle v_{(1)}^2\rangle}{2}\right)\delta_{ij}+\rho_0\langle(v_{(1)})_i(v_{(1)})_j\rangle. \tag{A4}$$

The pressure term vanishes after averaging, but the convection term is essential and yields for the force on the vortex

$$\begin{aligned}\vec{F}^{\mathrm{ph}}&=-\rho_0\phi_0^2\frac{\kappa^2k}{8\pi^2}\int\left\{\vec{k}\cos\varphi+\frac{|a(\varphi)|^2}{r^2}k\vec{r}-\frac{\mathrm{Im}\{a(\varphi)\}}{\sqrt{r}}\cos[kr(1-\cos\varphi)]\left(\vec{k}+k\frac{\vec{r}}{r}\right)\right\}rd\varphi\\&\approx-\rho_0\phi_0^2\frac{\kappa^2k}{8\pi^2}\int\left[\vec{k}\cos\varphi+\frac{|a(\varphi)|^2}{r}k\vec{r}-\frac{\mathrm{Im}\{a(0)\}}{\sqrt{r^2}}\cos\left(\frac{1}{2}kr\varphi^2\right)2\vec{k}\right]rd\varphi\\&=-\rho_0\phi_0^2\frac{\kappa^2k}{8\pi^2}\left[\int\frac{|a(\varphi)|^2}{r}k\vec{r}d\varphi-2\sqrt{\pi/k}\,\mathrm{Im}\{a(0)\}\vec{k}\right].\end{aligned} \tag{A5}$$

With help of the optical theorem Eq. (A3) one obtains the expression Eq. (52) with the effective cross sections determined by Eqs. (53) and (54).

APPENDIX B: SMALL-ANGLE SCATTERED SOUND WAVE

Using the asymptotics of the Hankel function, Eq. (45) can be rewritten as

$$\phi=\phi_0\exp(-i\omega t)\left\{\exp(i\vec{k}\cdot\vec{r})+\frac{\kappa k}{c}\sqrt{i/2\pi r}\int d_2\vec{r}_1\exp(i\vec{k}\cdot\vec{r}_1+ik|\vec{r}-\vec{r}_1|)\frac{\vec{k}\cdot[\hat{z}\times\vec{r}_1]}{r_1^2}\right\}. \tag{B1}$$

Here the effect of the vortex-line motion was neglected as irrelevant for small-angle scattering. Expansion Eq. (48) is not accurate enough and next terms of the expansion must be kept:

$$|\vec{r}-\vec{r}_1|\approx r-\frac{(\vec{r}_1\cdot\vec{r})}{r}+\frac{r_1^2}{2r}-\frac{(\vec{r}_1\cdot\vec{r})^2}{2r^3}. \tag{B2}$$

The terms of the second order in r_1 are important since the perturbation is not well localized near the vortex line, but decreasing slowly when r_1 is increasing. Using the Cartesian coordinates of the position vector $\vec{r}_1(x,y)$ and the inequality $\varphi\ll1$, one obtains

$$\phi=\phi_0\exp(-i\omega t)\left\{\exp(i\vec{k}\cdot\vec{r})+\frac{\kappa k^2}{c}\sqrt{i/2\pi r}\int\int dx\,dy\,\exp\left[ik\left(r-y\varphi+\frac{y^2}{2r}\right)\right]\frac{y}{x^2+y^2}\right\}. \tag{B3}$$

The double integral in this expression may be transformed into the error integral:

$$\begin{aligned}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\exp\left[ik\left(r-y\varphi+\frac{y^2}{2r}\right)\right]\frac{y}{x^2+y^2}&=\exp\left[ikr\left(1-\frac{\varphi^2}{2}\right)\right]\int_{-\infty}^{\infty}dy\,\pi\frac{y}{|y|}\exp\left[\frac{ik}{2r}(r\varphi-y)^2\right]\\&=-\pi\sqrt{2\pi ir/k}\exp\left[ikr\left(1-\frac{\varphi^2}{2}\right)\right]\Phi(\varphi\sqrt{kr/2i}).\end{aligned}$$

Then Eq. (B3) coincides with Eq. (57).

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