Edge and strip plasmons in a two-dimensional electron fluid

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We study plasma oscillations in a bounded two-dimensional electron fluid. The use of a variational method allows a simple analysis of plasma modes in a strip of finite width. For the exactly solvable model that neglects the pressure gradient in a plasma confined to a semi-infinite plane a simple one-parameter variational function gives the edge plasmon frequency reasonably close to the exact value. In the case of the semi-infinite plane we also incorporate the effects of the pressure gradient in our variational solution. In a two-dimensional strip of finite width we find the symmetric and antisymmetric plasma modes of different frequencies. The method used here can be readily extended to include the effects of applied magnetic fields and different confinement geometries. [S0163-1829(97)08107-1]

I. INTRODUCTION

Plasma excitations have been of considerable interest in recent studies of semiconductor quantum wells. The confinement provided by the doped heterostructure allows the formation of a quasi-two-dimensional electron gas. The intrasubband plasmons that are associated with a single electron subband can be studied as collective excitations in a two-dimensional electron fluid. These excitations will be affected by the presence of boundaries in the confinement plane of the electron fluid. The existence of edge modes in a bounded two-dimensional (2D) electron fluid is well known.[1–4] Their dispersion in an applied magnetic field has been studied theoretically by various methods and the localization of the magnetoplasmon near the edge is known to produce a gapless excitation.[2,3] The edge mode propagates along the edge of the bounded 2D system and decays exponentially away from the edge. An integral equation describing this mode in a semi-infinite 2D system can be obtained from the hydrodynamic model[2–4] and it allows an exact solution by the Wiener-Hopf method.[3,5] The early approximate method[1,2,4] replaces the kernel of the integral equation with a simpler exponential form similar to the one that appears in the equation for surface plasmons.[2] This approach was recognized as unsatisfactory and approximate numerical solutions of the integral equation were proposed[6,7] based on expansion in the basis of the Laguerre polynomials. Such solutions give a good approximation for the energy of the edge magnetoplasmon without reproducing the detailed behavior of the exact solution near the edge. A 2D system with two boundaries, on the other hand, does not allow an exact solution. The existing treatment of the plasmons in a strip of a finite width[5] is suitable only for strong magnetic fields, when the width of the region of localization is much smaller than other lengths in the problem.

In this work we derive an integral equation for the strip plasmons using a hydrodynamic model of the 2D electron fluid, and propose a variational solution for the density oscillations. In order to test this solution we consider at first a semi-infinite 2D system neglecting the gradient of pressure terms in the equation of motion for the electrons. The energy of the edge plasmon is shifted down from the 2D bulk value, and we find that a simple one-parameter variational function gives the edge plasmon energy shift within 15% of the exact value of the shift. Better agreement with the exact solution can be obtained if one incorporates an inverse square-root divergence of the density oscillations near the edge into the variational function. When the pressure-gradient term and electron scattering are neglected the only characteristic length in the semi-infinite system is given by the wavelength of the plasmon. When the pressure-gradient term is included, resulting in the nonzero speed of sound, the localization length of the plasma oscillations depends on the value of the plasmon wave vector along the edge.

In the case of a 2D strip an additional length is introduced, the width of the strip. Here we investigate the oscillation modes that are localized near the edges and decay toward the middle of the strip. We refer to these modes as the strip plasmons. Using a one-parameter variational function we find the lowest-frequency plasmon represented by the symmetric nodeless solution of the integral equation. We also find an antisymmetric solution of higher frequency. In the zero-width limit the lowest symmetric solution reproduces the dispersion of the 1D plasmon while the energy of the antisymmetric solution becomes infinite. The variational method we used here can be extended to the cases of different geometries of the bounded-2D system. It can be also used when the effects of an applied magnetic field are included.

II. VARIATIONAL METHOD FOR THE EDGE PLASMONS

We consider a semi-infinite plasma layer whose three-dimensional density $\rho_b(\mathbf{r},z)$ can be written as

$$\rho_b(\mathbf{r},z)=\delta(z)\rho(\mathbf{r})\theta(x), \tag{1}$$

where $\mathbf{r}$ is a 2D position vector in the (x,y) plane with the y axis chosen along the edge, $\theta(x)$ is the step function. We define a 2D gradient operator

$$\nabla\equiv\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}. \tag{2}$$

We treat the electron gas as a compressible negatively charged fluid placed in a rigid neutralizing positive background.[2] The equation of the continuity and the Euler equation are projected on the (x,y) plane to give the equations for the 2D plasma:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0,$$

$$m\rho\frac{\partial\mathbf{v}}{\partial t}+m\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=-e\rho\mathbf{E}-\nabla p, \tag{3}$$

where $\mathbf{E}$ is an electric field projection on the $z=0$ plane, $\mathbf{E}=(E_x,E_y)=-\nabla\phi$; p is the electron fluid pressure, m is the electron effective mass, and $\mathbf{v}$ is the 2D velocity field. In the equilibrium $\rho(\mathbf{r})=n_0$ and $\mathbf{E}=0$. Under a density perturbation we have

$$\rho(\mathbf{r},t)=n_0+\delta\rho(\mathbf{r},t), \tag{4}$$

where $\delta\rho(\mathbf{r},t)$ describes the time-dependent plasma oscillations and determines the electric potential $\phi(\mathbf{r},z,t)$ through the 3D Poisson equation

$$\nabla^2\phi+\frac{\partial^2\phi}{\partial z^2}=\frac{4\pi e}{\epsilon_s}\,\delta\rho(\mathbf{r})\delta(z)\theta(x), \tag{5}$$

where ϵ_s is a background dielectric constant. We define the 1D space-time Fourier transformation so that

$$\delta\rho(\mathbf{r},t)=(2\pi)^{-2}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}d\omega\; e^{i(qy-\omega t)}\delta\rho_q(x,q,\omega). \tag{6}$$

Then from Eq. (5) we obtain a relation[2] between 1D Fourier components of ρ and ϕ:

$$\phi_q(x,t)=-\frac{2e}{\epsilon_s}\int_0^{\infty}dx'\,\delta\rho_q(x',q,t)K_0(q|x-x'|), \tag{7}$$

where K_0 is the modified Bessel function.[8]

For an adiabatic process the pressure $p(\mathbf{r},t)$ in Eq. (3) is a function of the density $\rho(\mathbf{r},t)$, and we can write

$$\nabla p=\left(\frac{dp}{d\rho}\right)_0\nabla\rho=ms^2\nabla\rho, \tag{8}$$

where s is the speed of sound. When this equation is applied to the degenerate electron gas one obtains $s^2=v_F^2/2$ where v_F is the Fermi velocity. The hydrodynamic model uses average quantities ρ and $\mathbf{v}$ and does not take into account correlation effects. If one uses the Boltzman-Vlasov equation for the velocity distribution $f(\mathbf{v})$ the long-wavelength expansion gives s^2 in terms of the second moment of $f(\mathbf{v})$.[9] When the resulting expression is applied to a degenerate electron gas one obtains

$$s^2=\tfrac{3}{4}v_F^2. \tag{9}$$

This is identical to the result obtained in a self-consistent quantum model[10] and is different from the value obtained from the inverse compressibility. We will use the form in Eq. (9) for s^2. For small oscillations, $\delta\rho\ll n_0$, we linearize Eq. (3) in $\delta\rho$ and $\mathbf{v}$. The consideration of nonlinear effects in a bounded fluid is beyond the scope of this work.[11]

Let us define an axillary 3D density n_3 and a dimensionless 2D density N as

$$n_3\equiv n_0/a_3,\quad N\equiv n_0a_2^2 \tag{10}$$

where a_3 and a_2 are, respectively, the 3D and 2D electron Bohr radii, $a_3=2a_2=\hbar^2\epsilon_s/me^2$. We define an axillary frequency ω_p to be the bulk plasmon frequency of the 3D electron gas with the density n_3 and also define a dimensionless frequency Ω:

$$\omega_p^2\equiv\frac{4\pi e^2n_3}{\epsilon_s m},\quad \Omega^2\equiv\frac{\omega^2}{\omega_p^2}. \tag{11}$$

and a dimensionless coordinate $\xi\equiv xq$. From the Fourier transformation (6) of Eqs. (3) and using Eq. (7) we obtain the following integro-differential equation for the density oscillations:

$$\Omega^2\delta\rho(\xi)+\frac{a_2q}{\pi}\left(\frac{\partial^2}{\partial\xi^2}-1\right)\int_0^{\infty}d\xi' K_0(|\xi-\xi'|)\delta\rho(\xi')+\frac{s^2q^2}{\omega_p^2}\left(\frac{\partial^2}{\partial\xi^2}-1\right)\delta\rho(\xi)=0. \tag{12}$$

We impose the boundary condition of a sharp edge in the density distribution by requiring the normal component of the electron velocity to be zero at $x=0$:

$$v_x(x=0)=0. \tag{13}$$

From Eqs. (3) and (7) this can be expressed in terms of $\delta\rho(\xi)$. Let us define a function $f(\xi)$ as

$$f(\xi)\equiv\frac{a_2q}{\pi}\int_0^{\infty}d\xi' K_0(|\xi-\xi'|)\delta\rho(\xi')+\frac{s^2q^2}{\omega_p^2}\,\delta\rho(\xi). \tag{14}$$

Then the boundary condition (13) leads to the condition on f:

$$\frac{\partial f}{\partial\xi}(\xi=0^+)=0. \tag{15}$$

The integro-differential Eq. (12) with boundary condition (15) can be transformed into an integral equation[6] using the Green's function $G(\xi,\xi')$ defined by the following equations:

$$\left(\frac{\partial^2}{\partial\xi^2}-1\right)G(\xi,\xi')=\delta(\xi-\xi'),$$

$$\frac{\partial G}{\partial\xi}(\xi=0^+)=0. \tag{16}$$

Its explicit form is

$$G(\xi,\xi')=-\tfrac{1}{2}[e^{-(\xi+\xi')}+e^{-|\xi-\xi'|}], \quad \xi,\xi'>0. \tag{17}$$

The system of Eqs. (12) and (15) is equivalent to one homogeneous integral equation:

$$f(\xi)+\Omega^2\int_0^\infty d\xi' G(\xi,\xi')\delta\rho(\xi')=0. \tag{18}$$

Let us rewrite this equation in an operator form:

$$\hat{A}\psi=\Omega^2\hat{B}\psi, \tag{19}$$

where $\hat{B}$ and $\hat{A}$ are integral operators with kernels $B(\xi,\xi')$ and $A(\xi,\xi')$ given by

$$B(\xi,\xi')=-G(\xi,\xi')\theta(\xi)\theta(\xi'), \tag{20}$$

$$A(\xi,\xi')=\left[\frac{a_2 q}{\pi}K_0(|\xi-\xi'|)+\frac{s^2q^2}{\omega_p^2}\delta(\xi-\xi')\right]\theta(\xi)\theta(\xi'). \tag{21}$$

Applying the variational principle[12] to Eq. (19) we minimize the functional

$$J\{\psi\}=\frac{\langle\psi\hat{A}\psi\rangle}{\langle\psi\hat{B}\psi\rangle}, \tag{22}$$

where the scalar products are defined as

$$\langle\psi\hat{A}\psi\rangle=\int d\xi\, d\xi'\,\psi(\xi)A(\xi,\xi')\psi(\xi'). \tag{23}$$

As a trial function we choose a simple exponential form

$$\psi(\xi)\equiv\delta\rho(\xi)=ce^{-b\xi}\theta(\xi), \tag{24}$$

where c is a constant. Transforming to coordinates $u_\pm=\xi\pm\xi'$ and using a particular integral representation[8] for K_0 we can evaluate the integrals in Eq. (22) analytically. We define a dimensionless momentum,

$$k\equiv qa_2 \tag{25}$$

and minimizing the functional (22) with ψ given by Eq. (24) we obtain

$$\Omega^2=J_1(b)k+\tfrac{3}{2}J_2(b)k^2, \tag{26}$$

where

$$J_1(b)=\frac{2}{\pi}\frac{(1+b)^2}{1+2b}\left[\frac{\theta(1-b)}{\sqrt{1-b^2}}\arccos b+\frac{\theta(b-1)}{2\sqrt{b^2-1}}\ln\left(\frac{b+\sqrt{b^2-1}}{b-\sqrt{b^2-1}}\right)\right] \tag{27}$$

$$J_2(b)=\frac{(1+b)^2}{1+2b}. \tag{28}$$

If we set $s^2=0$, the minimum is reached at $b\to1$. This value corresponds to $\omega/\omega_{2D}=(8/3\pi)^{1/2}\approx0.921$ where $\omega_{2D}(q)$ is the frequency of the 2D plasmon in an infinite system, $\omega_{2D}(q)=(a_2q)^{1/2}\omega_p$. When the s^2 term is retained, the dispersion is given by Eq. (26). We find that $b(k)$ is a monotonically decreasing function of k as shown in Fig. 1. The frequency of the edge plasmon approaches the 2D plasmon frequency at large k where both modes tend to become a sound wave. We can compare our trial function in Eq. (24) to the density oscillations obtained in the exact solution[5] for a semi-infinite plane with $s^2=0$. It follows then that better values of ω will be obtained with a two-parameter form for the trial function, namely,

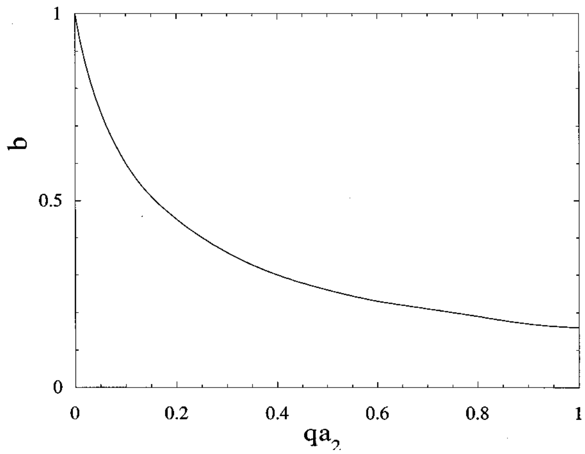

FIG. 1. Variational parameter b for the edge plasmon in a semi-infinite 2D electron fluid. It gives the spatial rate of decay of the density oscillations away from the edge and is shown here as a function of the wave vector along the edge. a_2 is the 2D electron Bohr radius. Parameter b decreases with q because at large q the plasmon becomes a sound wave.

$$\delta\rho(\xi)=c\xi^{-\alpha}e^{-b\xi}\theta(\xi), \tag{29}$$

where $\alpha\to\frac{1}{2}$ for $s^2\to0$, and $\alpha(p)<\frac{1}{2}$ for $s^2\neq0$. With the trial function given by Eq. (29) the integrals in Eq. (22) would have to be evaluated numerically.

III. STRIP PLASMONS

We consider a strip of width d and choose the y axis in the middle so the edges are at $x=\pm d/2$. The boundary conditions are

$$v_x(x=\pm d/2)=0. \tag{30}$$

The integral equation for the plasma oscillations is derived in the same way as in the previous section:

$$\frac{a_2q}{\pi}\int_{-\xi_0/2}^{\xi_0/2}d\xi' K_0(|\xi-\xi'|)\delta\rho(\xi')+\frac{s^2q^2}{\omega_p^2}\delta\rho(\xi)+\Omega^2\int_{-\xi_0/2}^{\xi_0/2}d\xi' G(\xi,\xi')\delta\rho(\xi')=0, \tag{31}$$

where $\xi=qx$, $\xi_0=qd$. The Green's function $G(\xi,\xi')$ is defined by Eq. (16) and the boundary condition of zero derivatives at $\xi=\pm\xi_0/2$. Its explicit form in the interval $[-\xi_0/2,\xi_0/2]$ is

$$G(\xi,\xi')=-\frac{1}{\sinh\xi_0}[\cosh(\xi+\xi_0/2)\cosh(\xi_0/2-\xi')\times\theta(\xi'-\xi)+\cosh(\xi_0/2-\xi)\cosh(\xi'+\xi_0/2)\times\theta(\xi-\xi')]. \tag{32}$$

Equation (31) and boundary conditions (30) are invariant under the reflection $x\to-x$. Therefore, the nondegenerate solutions have a definite parity, i.e., they are either even or odd. We can rewrite Eq. (31) in the operator form as in Eq. (19) and look for the variational solution. To simplify the discussion for the strip plasmons we set $s^2=0$, corresponding to the long-wavelength limit $qs \ll \omega_p$. The definition of operators $\hat{A}$ and $\hat{B}$ that appear in the eigenvalue equation (19) follow from Eq. (31). The operators are real and symmetric and the solutions form an orthogonal set. Variational solutions are found by minimizing the functional defined in Eq. (22). For the two lowest-frequency solutions we take the following trial functions to describe the x dependence of the plasma oscillations localized near the edges:

$$\delta\rho_1(\xi)=c_1\cosh(b\xi)\theta(\xi_0/2-|\xi|), \tag{33}$$

$$\delta\rho_2(\xi)=c_2\sinh(b\xi)\theta(\xi_0/2-|\xi|). \tag{34}$$

In the y direction the density variation oscillates as $\exp(iqy)$. Transforming the coordinates to $u_\pm=\xi\pm\xi'$ we can evaluate $\langle\hat{B}\rangle$ in terms of elementary functions and $\langle\hat{A}\rangle$ in terms of a simple integral. The frequencies of the even and odd plasmons are shown in Fig. 2 as functions of qd in the units of $\omega_0=(a_2/d)^{1/2}\omega_p$. In the limit $qd\to\infty$ the solutions become degenerate and reproduce the dispersion of the edge plasmon. In the opposite limit $qd\to 0$ we expand $\langle\hat{A}\rangle$ and $\langle\hat{B}\rangle$ in powers of qd and find that the even solution $\delta\rho_1(\xi)$ reproduces the dispersion of the 1D plasmon:

$$\omega_1^2\to\frac{2e^2}{m\epsilon_s}n_0dq^2\ln(1/qd), \tag{35}$$

where n_0d gives the linear density of electrons in one dimension. For the odd solution $\delta\rho_2(\xi)$ in the limit $qd\to 0$ the frequency approaches a constant value:

$$\omega_2^2\to\frac{15e^2n_0}{\epsilon_s md}. \tag{36}$$

Such behavior of the odd mode can be understood by considering this mode as a 2D plasma wave whose wave-vector

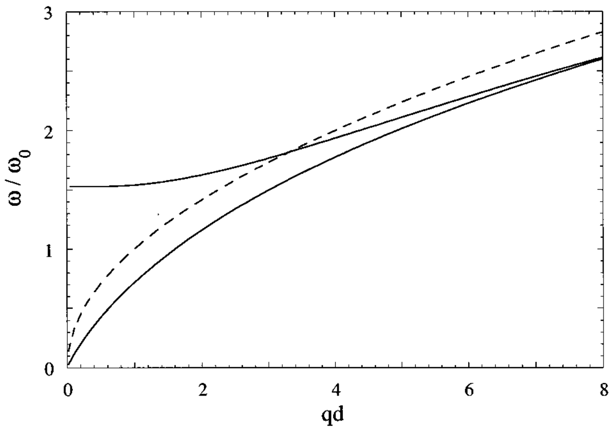

FIG. 2. The frequency of the strip plasmon in units of $\omega_0=(a_2/d)^{1/2}\omega_p$. The lower solid curve shows the dispersion of the lowest-frequency even mode, the upper solid curve shows the dispersion of the odd mode. They are shown as functions of the wave vector q along the strip of width d. At small q the even solution becomes a 1D plasmon. The frequency of the odd solution approaches a constant value $\propto(n_0/d)^{1/2}$. At large q both frequencies approach the edge plasmon frequency that is shifted downward from the 2D plasmon dispersion, shown here by the broken curve.

component in the x direction is π/d. Then the frequency is $C\omega_{2D}(q=\pi/d)$ and the variational solution gives $C=(15/2\pi^2)^{1/2}$ and Eq. (36). If the linear electron density n_0d is fixed as $d\to 0$, the frequency ω_2 diverges as $1/d$. Therefore in the narrow strip limit only the lowest-frequency mode will exist, with dispersion given in Eq. (35).

We want to mention possible extensions of the variational calculation presented above. It is simple to incorporate the effect of the pressure gradient in the variational analysis of the strip plasmons just as we did for the semi-infinite plane in the previous section. We can also apply the variational method for different geometries and boundary conditions, for example, assuming metal electrodes on both sides of the strip. Boundary conditions will then be imposed on the electrostatic potential. The variational method can also be applied to the strip magnetoplasmons and the results will be given elsewhere.

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