Higher-order-magnetization-cumulant universality of the two-dimensional Ising model

K. K. Mon
Department of Physics and Astronomy, and Center for Simulational Physics, University of Georgia, Athens, Georgia 30602
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We study the higher-order cumulant universality of the order-parameter distribution at criticality for the two-dimensional nearest-neighbor ferromagnetic Ising model with Monte Carlo methods. These cumulants are interesting because they are a quantitative measure of the shape of the critical order parameter distribution. This shape has been predicted to be universal. Up to now, only the fourth-order cumulant has been studied in any detail. The set of higher-order cumulants as a whole is more sensitive to the details of the distribution and hence is a stronger test. It cannot be sampled efficiently with standard Monte Carlo sampling. To the best of our knowledge, no definite numerical results have been reported in the literature. Using the umbrella-sampling technique, we are able to obtain accurate estimates for the fourth-, sixth-, eighth-, and tenth-order cumulants for both the square and triangular lattices of different lattice sizes. These results support universality. [S0163-1829(97)00702-9]

I. INTRODUCTION

At the critical point, the scaled order parameter distribution function is predicted to be universal.[1–7] Different models and physical systems in the same universality class will have the same normalized distribution function at criticality. A quantitative measure of this universality is that all the cumulants of the distribution have the same universal values at criticality. For dimensions below the upper critical dimension, the relevant length scale near the critical point is the correlation length. That serves as the scaling length in finite-size scaling.[8,2,9] It becomes infinite at criticality. Thus, all sufficiently large finite systems at bulk criticality are equivalent and will have the same values for the cumulant. For this reason, Monte Carlo simulations of finite systems have been used widely to estimate these constants and probe universality.

The fourth-, sixth-, eighth-, and tenth-order cumulants of the order parameter (m) distribution are defined in terms of the moments as

$$g_4=(\langle m^4\rangle-3\langle m^2\rangle^2)/\langle m^2\rangle^2, \tag{1}$$

$$g_6=(\langle m^6\rangle-15\langle m^4\rangle\langle m^2\rangle+30\langle m^2\rangle^3)/\langle m^2\rangle^3, \tag{2}$$

$$g_8=(\langle m^8\rangle-28\langle m^6\rangle\langle m^2\rangle-35\langle m^4\rangle^2+420\langle m^4\rangle\langle m^2\rangle^2 -630\langle m^2\rangle^4)/\langle m^2\rangle^4, \tag{3}$$

$$g_{10}=(\langle m^{10}\rangle-45\langle m^8\rangle\langle m^2\rangle-210\langle m^6\rangle\langle m^4\rangle +1260\langle m^6\rangle\langle m^2\rangle^2+3150\langle m^4\rangle^2\langle m^2\rangle -18\,900\langle m^4\rangle\langle m^2\rangle^3+22\,680\langle m^2\rangle^5)/\langle m^2\rangle^5. \tag{4}$$

These are for the nearest-neighbor ferromagnetic Ising model, where all the odd-order moments and cumulants vanish by symmetry.

Prior to the present study, only the fourth- and lower-order cumulants of the magnetization distributions have been studied in any detail.[7,1,5,10] The higher-order cumulants have not been probed extensively because they are very sensitive to the tails of the distribution and cannot be sampled efficiently with standard Monte Carlo sampling. A few early estimates[7,1] for these higher-order cumulants exist, but no definite numerical results have been reported in the literature. There are also some simulation studies which consider directly the magnetization distribution function.[5,6] Since the set of higher-order cumulants as a whole is more sensitive to the details of the distribution, it is a stronger test for universality.

Using the umbrella-sampling technique,[11] we are able to obtain accurate estimates for the fourth-, sixth-, eighth-, and tenth-order cumulants. The possibility of universality for these higher-order cumulants is studied by considering the Ising model with the square and the triangular lattices. A range of lattice sizes is used at bulk criticality and with these results, we demonstrate universality for all the higher-order cumulant considered.

In the next section, a brief description of the umbrella-sampling technique is given. In Sec. III, Monte Carlo results are presented and the paper concludes with some remarks in Sec. IV.

II. UMBRELLA-SAMPLING TECHNIQUE

The umbrella-sampling technique[11] was proposed many years ago to overcome the difficulty of sampling configurations in regions of phase space which have a very small probability in the Boltzmann distribution. Exponentially long Markov chains are needed to sample these states in the standard Metropolis method. This problem is especially important in the sampling of the higher-order moments of the distribution. For nth moments, the configurations are weighted by m^n. Since the tails of the distribution have a larger m, the tail configurations are given larger weight in the sampling. However, the tails of the distribution have exponentially small probability of being sampled. In the umbrella-sampling technique, the total distribution is sampled in a number of separate samplings. The total range of the order parameter is divided into a nonoverlapping, but contiguous sequence of

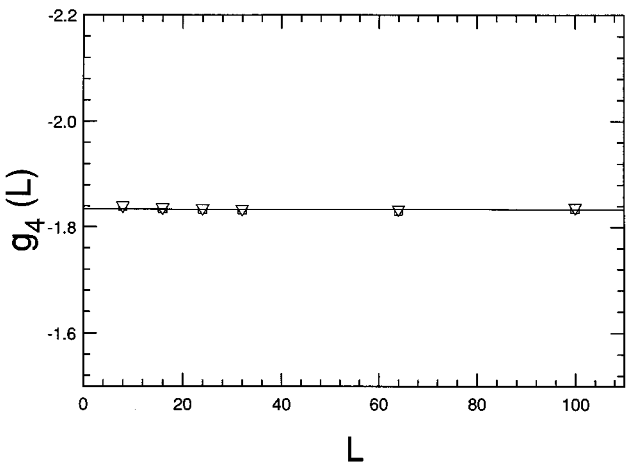

FIG. 1. Finite-size dependence of the fourth-order cumulant of the magnetization distribution at bulk criticality for $L\times L$ lattices. The estimated errors are less than or about the size of the symbols. The solid line has the value -1.834. Diamonds and squares are for the triangular and square lattices, respectively. See text.

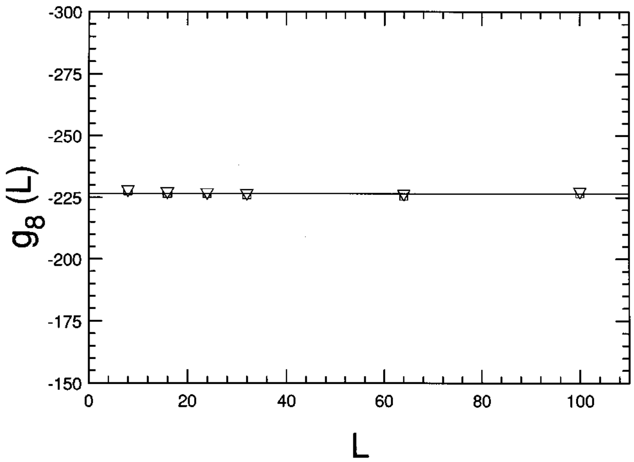

FIG. 3. Finite-size dependence of the eighth-order cumulant of the magnetization distribution at bulk criticality for $L\times L$ lattices. The estimated errors are less than or about the size of the symbols. The solid line has the value -226.6. Diamonds and squares are for the triangular and square lattices, respectively. See text.

bins. (See Ref. 11, for a clear introduction.) Each one samples over only a single bin that spans a sufficiently small range of the order parameter to ensure good statistics. To combine these separate samplings to form a single normalized distribution, one considers an additional set of ''umbrella samplings'' with a range of sampling that overlaps adjacent bins. This overlap can be described as a sort of ''umbrella'' covering regions of phase space common to two different bins. By imposing the condition that distributions from different bins with the same order parameter have the same numerical value, one can combine the separate bin distributions to form a single normalized distribution.

For our application of the Ising model, the order parameter is the magnetization per spin m. m ranges from -1 to $+1$ and is divided into a number of contiguous bins with bin width of δm for $-0.8\leqslant m\leqslant 0.8$ and two tail bins of (0.8 to 1.0) and (-0.8 to -1.0). Typical δm ranges from 0.02 to 0.08. We use a second set of bins with the same bin width δm, but ranges from $(-[0.8+\delta m/2])$ to $(+[0.8+\delta m/2])$. The two sets of bins represent a sequence of bins with maximum overlaps between neighboring bins. Within each bin, standard Monte Carlo sampling[12] is used with the additional restriction that moves into m values outside of the bin are forbidden.

We consider $L\times L$ square and triangular lattices with an Ising spin degree of freedom (± 1) on each site, coupled to the nearest-neighbor site spin by ferromagnetic coupling J. The magnetic field at each site is zero. We employ lattice sizes of $L=8$–100 and sample each bin with up to about 10^5 Monte Carlo steps per spin after equilibrium. By varying the bin widths, we have checked that within the estimated errors, the results are not sensitive to the widths of the bins

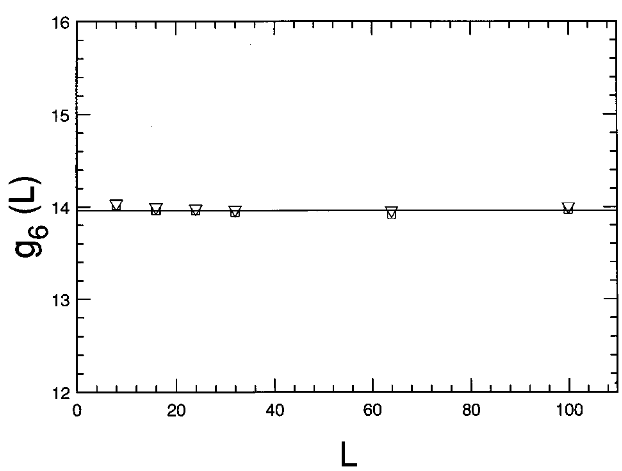

FIG. 2. Finite-size dependence of the sixth-order cumulant of the magnetization distribution at bulk criticality for $L\times L$ lattices. The estimated errors are less than or about the size of the symbols. The solid line has the value 13.96. Diamonds and squares are for the triangular and square lattices, respectively. See text.

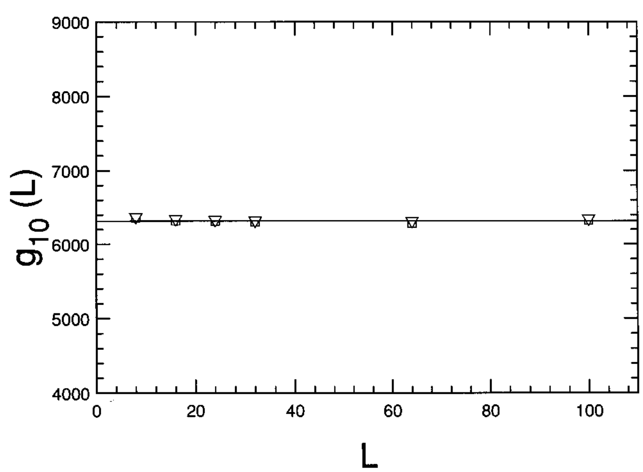

FIG. 4. Finite-size dependence of the tenth-order cumulant of the magnetization distribution at bulk criticality for $L\times L$ lattices. The estimated errors are less than or about the size of the symbols. The solid line has the value 6314.0. Diamonds and squares are for the triangular and square lattices, respectively. See text.

for sufficiently small bin widths. Standard periodic boundary conditions are implemented. The order-disorder transition in the bulk limit is known exactly for both lattices[13,14] to be at $J/k_B T_c = \frac{1}{2}\ln(\sqrt{2}+1)$ and 0.274 653 1 . . . ,[14] for the square and triangular lattice, respectively.

III. MONTE CARLO RESULTS

In Fig. 1, we present the finite-size dependence of the fourth-order cumulant of the magnetization distribution at bulk criticality for $L\times L$ lattices. A wide range of lattice size for both the square and triangular lattices are used to test the universality predictions. The statistical errors are estimated from different runs and standard block averaging.[12] They are less than or about the size of the symbols. The convergence to the large lattice limit is very rapid. This limit is denoted by the solid line with the value $-1.83(4)$. That is in very good agreement with previous estimates[1,5,10] of ~ -1.83. Diamonds and squares are for the triangular and square lattices, respectively.

The finite-size dependence of the sixth-order cumulant of the magnetization distribution at bulk criticality is given in Fig. 2. Here the bulk limit is given by the solid line which has the value 13.9(6), and the convergence in size is also rapid. Figure 3 contains the finite-size dependence of the eighth-order cumulant of the magnetization distribution at bulk criticality. The solid line represents the estimated bulk-limit value of $-226(6)$. The size-dependence convergence continues to be very rapid. Finally, Fig. 4 has the finite-size dependence of the tenth-order cumulant of the magnetization distribution at bulk criticality. The solid line has the bulk-limit value of 631(4). The universality of these critical cumulants is evident and is indeed consistent with theoretical predictions that they are the same for all members of the same universality class.

IV. REMARKS

In this paper, we present the higher-order cumulants of the order parameter distribution at criticality for the two-dimensional nearest-neighbor ferromagnetic Ising model universality class. The new results are obtained with umbrella sampling Monte Carlo simulations. These cumulants are significant because they represent a quantitative test of the shape universality of the critical-order parameter distribution. This universality has been predicted and is the same for all members of the same universality class. Up to now, to the best of our knowledge, only the fourth-order cumulant of the magnetization distributions has been studied in any detail. The set of higher-order cumulants represents an even more demanding test for universality, but has been difficult to obtain accurately. This is because they are very sensitive to the tails of the distribution, which cannot be sampled efficiently with standard Monte Carlo sampling. Using the umbrella-sampling technique, we are able to obtain accurate estimates for the fourth-, sixth-, eighth-, and tenth-order cumulants for both the square and triangular lattices. A range of lattice sizes is simulated at bulk criticality and with these results, we demonstrate universality for all the higher-order cumulants considered.

From our experiences with the present study, even higher-order cumulants can be estimated accurately with the sampling technique used here. With a set of accurate estimates for the cumulants, an accurate analytic representation for the distribution function can be obtained using various moment-expansion methods.[15]

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