Propagation of light waves in Thue-Morse dielectric multilayers

Nian-hua Liu*

Fundan–T. D. Lee Physics Laboratory, Fudan University, Shanghai, 200433, People's Republic of China and Institute of Materials Science, Nanchang University, Nanchang, 330047, People's Republic of China

(Received 4 September 1996)

We investigate the propagation of light waves in an aperiodic multilayer which is composed of dielectric slabs arranged following the Thue-Morse sequence. Both refractive index modulation and optical thickness modulation are considered. The recursion relations of the transfer matrix elements are derived to calculate the transmission. By introducing a localization index we explore the localization properties of light in the aperiodic system. Around a particular center frequency (the midgap frequency of a periodic quarter-wavelength multilayer), the transmission is more sensitive to the optical thickness modulation. [S0163-1829(97)01505-1]

In recent years, the propagation of electronic and classical waves in aperiodic systems has received considerable interest.[1–8] Two particular interesting systems are the Fibonacci sequence and Thue-Morse (TM) sequence, which are generated simply by two symbols, A and B, following the inflation rules: $A\rightarrow AB$, $B\rightarrow A$ for the Fibonacci sequence, and $A\rightarrow AB$, $B\rightarrow BA$ for the TM sequence. According to these inflation rules, the successive Fibonacci chains are $A,AB,ABA,ABAAB,\ldots$, and the successive TM chains are $AB,ABBA,ABBABAAB,\ldots$. The symbols A and B may be thought of as on-site energies, nearest-neighbor hopping integrals, heights or widths of potential barriers, dielectric constants, and so on, depending on the considered physical problem.

It was recognized that the Fibonacci lattice is a quasiperiodic system in view of the fact that its Fourier spectrum contains self-similar Bragg peaks.[9] The TM lattice, however, is not quasiperiodic but deterministically aperiodic for the singular continuous Fourier spectrum.[10,11] This means that the structure of the TM lattice is more ''disordered'' than the quasiperiodic one. In other words, the TM lattice has a degree of order intermediate between quasiperiodic and disordered systems. Contrary to the results of the structure factor, the electronic behaviors of the TM chain show that it is more similar to a periodic system.[12] This apparent contradiction brings about a number of detailed investigations of the TM lattices.

Some important physical properties, such as electronic spectra, spin excitations, light transmission, diamagnetic properties of TM superconducting wire networks and Josephson-junction arrays, were studied by Kolář *et al.* through the trace-map approach.[7] In order to compare the localization properties of the TM chain and Fibonacci chain, Huang *et al.* calculated the mean resistance of the one-dimensional wire with position modulation and scattering strength modulation.[13] The results show that for the position modulation, the TM chain is more localized than the Fibonacci chain. Lin and Tao proved analytically the existence of extended states in the TM chains and generalized TM chains.[14] The numerical evidence of existing extended states in TM chains was provided by Ryu, Oh, and Lee who considered both the tight-binding model and Kronig-Penney model.[8]

It is known that localization is essentially a wave phenomenon. Using classical waves to study localization has distinct advantages: there are no other involved interactions, such as electron-electron, electron-phonon, spin-orbit effect, which make the problem more complex. Recently, Kohmoto *et al.* have investigated the localization of light waves in Fibonacci dielectric multilayers.[1] They demonstrated that the transmission coefficient has a scaling property. In this paper, we concern ourselves with the behaviors of light waves in TM dielectric multilayers. Particularly, we investigate the effects of optical thickness modulation on the transmission, which has not been considered in the works of Kolář and Kohmoto. The optical thickness modulation means phase shift modulation because the phase shift is proportional to the optical thickness. In analogy with electronic problems, the optical thickness modulation may be considered as position modulation, while the refractive index modulation may be thought of as scattering strength modulation since the reflection of the light on interfaces depends on the contrast of the refractive indexes. We will see that the phase shift modulation strongly influences the wave propagation in the aperiodic systems.

We consider the normal propagation of light waves in a multilayer which is composed of dielectric slabs A and B stacked alternately following the TM sequence. The refractive indexes of the slabs are n_A and n_B, and their thicknesses are d_A and d_B, respectively. In order to describe the electromagnetic field, we introduce a two-component wave function

$$\chi=\begin{pmatrix} E \\ icB \end{pmatrix}, \tag{1}$$

where c is the velocity of light in vacuum. At two different positions $z+\Delta z$ and z, the wave function satisfies

$$\chi(z+\Delta z)=\mathbf{M}_\mu(\Delta z)\chi(z), \tag{2}$$

where

$$\mathbf{M}_\mu(\Delta z)=\begin{pmatrix} \cos\frac{\omega}{c}n_\mu\Delta z & -n_\mu^{-1}\sin\frac{\omega}{c}n_\mu\Delta z \\ n_\mu\sin\frac{\omega}{c}n_\mu\Delta z & \cos\frac{\omega}{c}n_\mu\Delta z \end{pmatrix},$$

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$$\mu = A,B. \tag{3}$$

Let z_0 and z_N denote the coordinates of the beginning and the end of the nth order TM multilayer, we can connect $\chi(z_N)$ and $\chi(z_0)$ through

$$\chi(z_N) = \Gamma_n \chi(z_0), \tag{4}$$

where Γ_n is a transfer matrix given by the following recursion relation

$$\Gamma_n = \tilde{\Gamma}_{n-1}\Gamma_{n-1} \tag{5}$$

with

$$\Gamma_1 = \mathbf{M}_B(d_B)\mathbf{M}_A(d_A), \tag{6}$$

where $\tilde{\Gamma}_{n-1}$ is the complement of Γ_{n-1} obtained by interchanging A and B in Γ_{n-1}.

For simplicity, we assume that the regions $z<z_0$ and $z>z_N$ are vacuum. The transmission then can be expressed as

$$T = \frac{4}{x_n^2 + y_n^2}, \tag{7}$$

where $x_n = \Gamma_{n,11} + \Gamma_{n,22}$ is the trace of Γ_n, and $y_n = \Gamma_{n,21} - \Gamma_{n,12}$ is related to the off-diagonal elements.

According to Eq. (3), $\mathbf{M}_A(d_A)$ and $\mathbf{M}_B(d_B)$ are unimodular, and the diagonal elements are identical. Using these conditions we have

$$\tilde{\Gamma}_n = \sigma \Gamma_n^{-1} \sigma \tag{8}$$

for $n=$odd, and

$$\Gamma_{n,11} = \Gamma_{n,22} = \tilde{\Gamma}_{n,11} = \tilde{\Gamma}_{n,22} = 1 + 2\Gamma_{n-1,12}\Gamma_{n-1,21}, \tag{9}$$

$$\Gamma_{n,12} = 2\Gamma_{n-1,12}\Gamma_{n-1,22}, \tag{10}$$

$$\Gamma_{n,21} = 2\Gamma_{n-1,21}\Gamma_{n-1,11}, \tag{11}$$

$$\tilde{\Gamma}_{n,12} = 2\Gamma_{n-1,12}\Gamma_{n-1,11}, \tag{12}$$

$$\tilde{\Gamma}_{n,21} = 2\Gamma_{n-1,21}\Gamma_{n-1,22} \tag{13}$$

for $n=$even, where

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

is a Pauli matrix. From the above equations we can derive the recursion relation of the matrix elements:

$$\Gamma_{n,11} = \Gamma_{n-2,11} x_{n-2}(x_{n-1} - 2) + 1, \tag{15}$$

$$\Gamma_{n,22} = \Gamma_{n-2,22} x_{n-2}(x_{n-1} - 2) + 1, \tag{16}$$

$$\Gamma_{n,12} = \Gamma_{n-2,12} x_{n-2} x_{n-1} + [1 + (-1)^n](\Gamma_{n-1,12} - \Gamma_{n-2,12} x_{n-2}), \tag{17}$$

$$\Gamma_{n,21} = \Gamma_{n-2,21} x_{n-2} x_{n-1} + [1 + (-1)^n](\Gamma_{n-1,21} - \Gamma_{n-2,21} x_{n-2}). \tag{18}$$

Obviously, Eq. (15) plus Eq. (16) yields the well-known trace map[15]

$$x_n = x_{n-2}^2 (x_{n-1} - 2) + 2, \tag{19}$$

and Eq. (18) minus Eq. (17) leads to

$$y_n = y_{n-2} x_{n-2} x_{n-1} + [1 + (-1)^n](y_{n-1} - y_{n-2} x_{n-2}). \tag{20}$$

Using Eqs. (19) and (20) with the following initial conditions,

$$x_1 = 2\cos\alpha\cos\beta - \left(\frac{n_A}{n_B} + \frac{n_B}{n_A}\right)\sin\alpha\sin\beta, \tag{21}$$

$$x_2 = 2\cos 2\alpha\cos 2\beta - \left(\frac{n_A}{n_B} + \frac{n_B}{n_A}\right)\sin 2\alpha\sin 2\beta, \tag{22}$$

$$y_1 = (n_A + n_A^{-1})\sin\alpha\cos\beta - (n_B + n_B^{-1})\cos\alpha\sin\beta, \tag{23}$$

$$y_2 = (n_A + n_A^{-1})\sin 2\alpha\cos 2\beta + \frac{1}{2}\sin 2\beta\left[(n_A + n_A^{-1})\left(\frac{n_A}{n_B} + \frac{n_B}{n_A}\right) - (n_A - n_A^{-1})\left(\frac{n_A}{n_B} - \frac{n_B}{n_A}\right)\cos 2\alpha\right], \tag{24}$$

where $\alpha = (\omega/c)n_A d_A$ and $\beta = (\omega/c)n_B d_B$, we can calculate recurrently the transmission for any generation.

Suppose the dielectric materials A and B are silicon dioxide and titanium dioxide respectively. The refractive indexes are $n_A = 1.45$ and $n_B = 2.30$ around the center wavelength $\lambda_0 = 700$ nm. These parameters are the same as in Ref. 1. For simplicity, we introduce the reduced optical thicknesses a and b such that $n_A d_A = a\lambda_0$ and $n_B d_B = b\lambda_0$, and write the phase shifts in the forms of $\alpha = 2\pi a\Omega$ and $\beta = 2\pi b\Omega$, where $\Omega = \omega/\omega_0 = \lambda_0/\lambda$ is the reduced frequency.

From Eqs. (15)–(18) we see that when $x_{l-2} = 0$, the further recursion of the off-diagonal matrix elements is cut off, namely, $\Gamma_{n,12} = \Gamma_{n,21} = 0$ for $n \geq l$, and the matrix Γ_n becomes an identical matrix. This means the system becomes completely transparent. In Fig. 1 we plot the frequencies which give rise to the identical matrix for $a = b = 0.25$ (quarter-wavelength). All of the completely transparent states are determined by $x_{n-2} = 0$ except for $\Omega = 1$. A straightforward calculation shows that the matrix $\Gamma_n (n \geq 2)$ is an identical matrix when $a = b = 0.25$ and $\Omega = 1$. The spectrum shown in Fig. 1 exhibits a self-similar pattern, and around $\Omega = 1$ is a wide quasicontinuous band. It is known that the frequency $\Omega = 1$ is the midgap frequency of a periodic quarter-wavelength multilayer. The transmission properties around this frequency are detailedly investigated in Ref. 1 for the Fibonacci dielectric multilayers. At this frequency, the quasiperiodicity is most effective, and the transfer matrixes have a six-cycle which causes the scaling property of the transmission for the Fibonacci multilayers.[1] We will also focus our attention on the frequency region around $\Omega = 1$.

For the completely transparent electromagnetic state, the wave function, namely, the electric field distribution, is similar to the structure of the TM sequence. In Fig. 2, we show

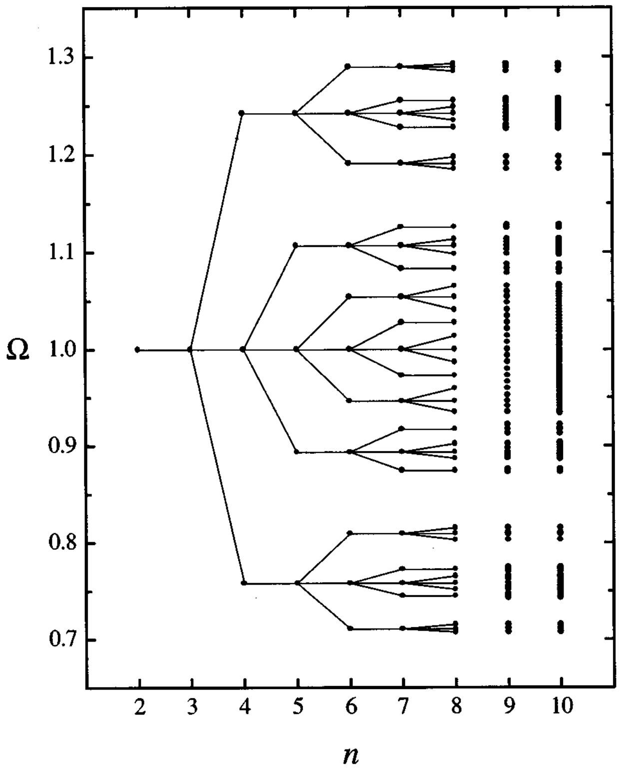

FIG. 1. The reduced frequency Ω giving rise to identical matrix vs generation n for $a=b=0.25$ (quarter-wavelength). Around $\Omega=1$ is a quasicontinuous band.

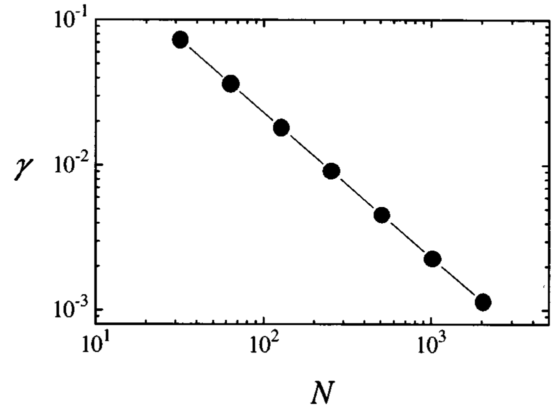

FIG. 3. Localization index γ vs the number of layers N for the latticelike electric field distributions at a completely transparent frequency $\Omega=0.758\,181$.

the latticelike electric distributions for $\Omega=0.758\,181$ and for $n=5$(a), $n=6$(b), and $n=7$(c), respectively. These electric distributions are analogous to the latticelike wave functions in the electronic problem.[8]

In a deterministic aperiodic system, the wave function is neither Bloch-type extended state as in periodic systems, nor exponentially localized state as in disordered systems. To describe the localization properties of the aperiodic systems, a notion of critical wave function has been introduced.[16] The critical wave functions are rather oscillatory, thus the localization properties cannot be characterized simply by a local-

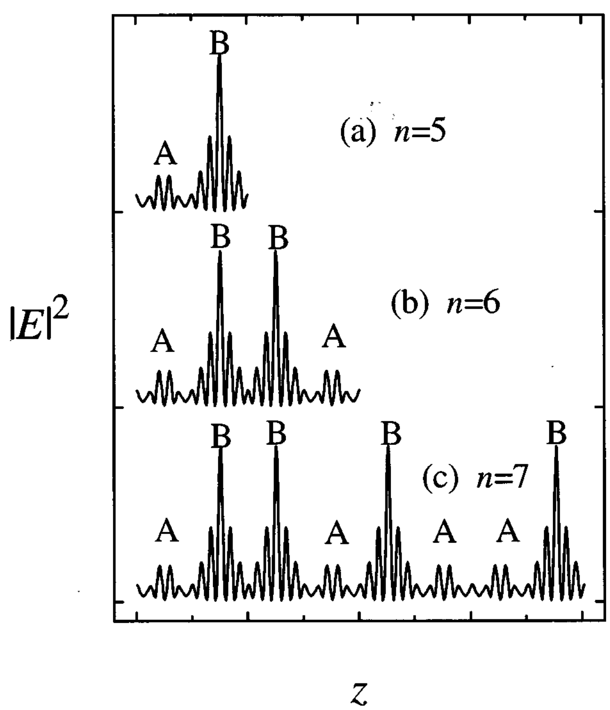

FIG. 2. The latticelike electric field distributions in the TM multilayers for a completely transparent frequency $\Omega=0.75\,818\,1$: (a) $n=5$ (32 layers), (b) $n=6$ (64 layers), and (c) $n=7$ (128 layers).

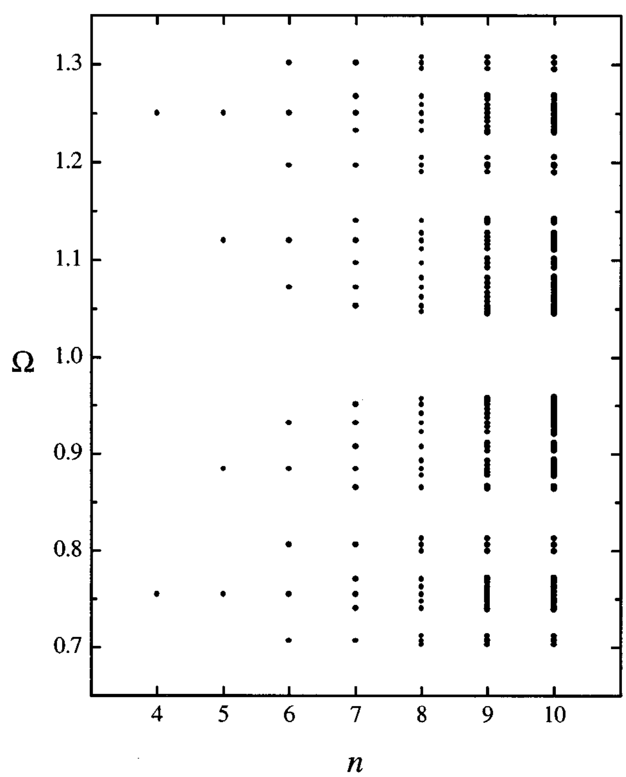

FIG. 4. The reduced frequency Ω giving rise to identical matrix vs generation n for $a=0.2$ and $b=0.3$. In contrast to Fig. 1, a wide gap is opened around $\Omega=1$.

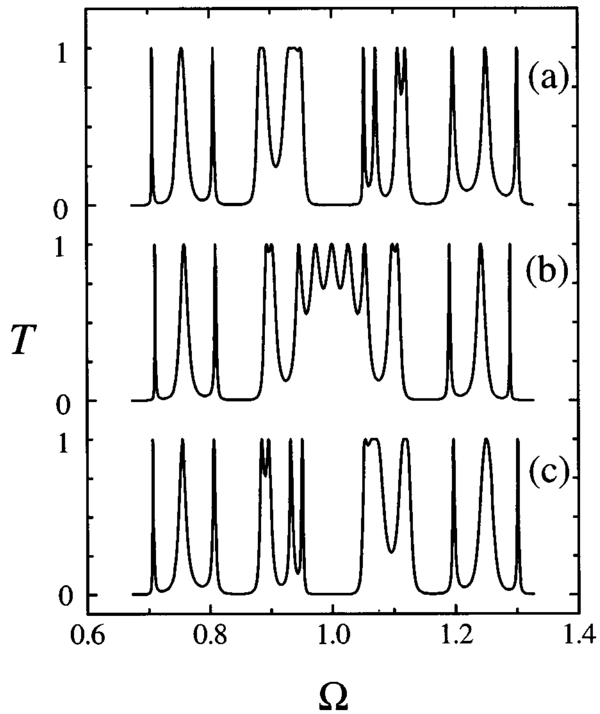

FIG. 5. The transmission through the sixth-order TM multilayers: (a) $a=0.2$, $b=0.3$, (b) $a=b=0.25$, and (c) $a=0.3$, $b=0.2$. With a (and b) deviating from 0.25, the transmission band around $\Omega=1$ becomes a gap.

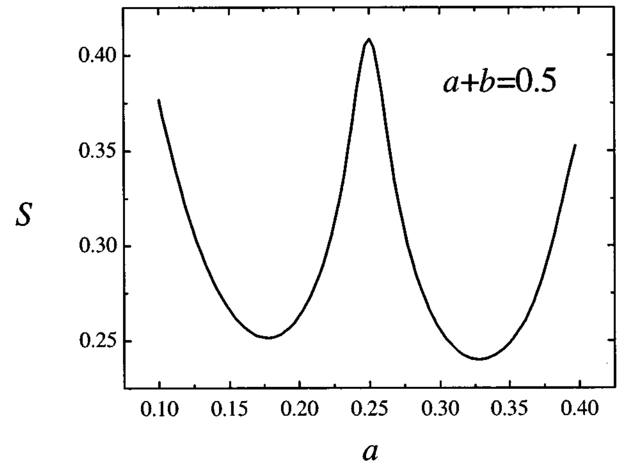

FIG. 6. The average transmission S vs the reduced optical thickness a for the sixth-order TM multilayer, we have kept $a+b=0.5$ as a constant.

ization length. To measure the degree of localization we introduce a localization index defined as

$$\gamma=\frac{\Sigma_{i=1}^{N} I_i^2}{(\Sigma_{i=1}^{N} I_i)^2}, \tag{25}$$

where N is the total number of the layers, and

$$I_i=\int_{z_{i-1}}^{z_i} |E(z)|^2 dz \tag{26}$$

is the intensity distributed in the ith layer. There are two extreme cases: the most extended state of which the electric field is distributed uniformly in the whole system, and the most localized state of which the electric field distribution is concentrated in one layer. For the most extended state, γ takes the minimum value $1/N$ because I_i is the same for each layer, but for the most localized state, $I_i=I_j\delta_{ij}$(assuming the electric field distribution is concentrated in jth layer), and γ takes the maximum value 1. Generally, $1/N<\gamma<1$. The larger of γ, the more localized of the state. Figure 3 shows the localization index γ vs the number of layers N for the latticelike wave function at $\Omega=0.758\,181$. Notice that we have used the log-log scale. It is shown that $\gamma \sim N^{-\delta}$. As N increase, the states become more and more extended.

The condition $a=b$ implies that the phase shift in each layer is the same. From a wave point of view, the layers A and B are equivalent when the phase shifts are the same. In this case, the difference of $\mathbf{M}_A(d_A)$ and $\mathbf{M}_B(d_B)$ is only reflected in the coefficients of the off-diagonal elements, and the aperiodic modulation is a refractive index modulation, namely, a scattering strength modulation. In fact, the wave propagation in aperiodic systems should be more sensitive to the phase shift modulation. Thus, we consider the situation that $a\neq b$. This means the phase shift and the scattering strength are both modulated aperiodically.

In our calculations, we vary a and b simultaneously to keep $a+b=0.5$ as a constant (half-wavelength). When $a\neq b$, the frequency $\Omega=1$ dose not give rise to an identical matrix for any n. On the contrary, a large gap is opened around this frequency, as shown in Fig. 4, where we have plotted the spectrum of the completely transparent frequencies for $a=0.2$ and $b=0.3$. The frequencies around $\Omega=1$ are altered obviously, but those far from $\Omega=1$ has only a slight shift. In Fig. 5, we plot the transmission through the sixth-order TM multilayer for different a and b. It is shown that around $\Omega=1$, there is a wide transmission band for $a=b=0.25$ [curve (b)], but the band becomes a gap when $a=0.2$, $b=0.3$ [curve (a)] and when $a=0.3$, $b=0.2$ [curve (c)]. This means the transmission around $\Omega=1$ is very sensitive to the optical thickness modulation.

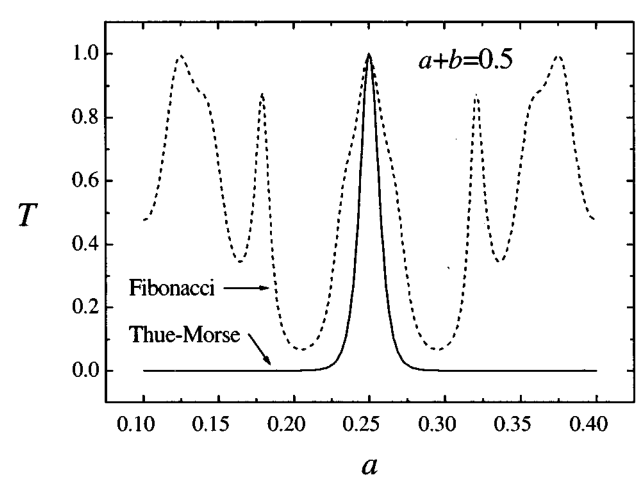

FIG. 7. The transmission at $\Omega=1$ vs the reduced optical thickness a for the sixth-order TM multilayer (64 layers) and for the ninth-order Fibonacci multilayer (55 layers), we have kept $a+b=0.5$ as a constant. The transmission shows quite different behaviors when we vary a and b.

We have to calculate the average transmission which is defined as

$$S=\frac{1}{\Omega_f-\Omega_i}\int_{\Omega_i}^{\Omega_f}T(\Omega)\,d\Omega. \tag{27}$$

The integral region is from $\Omega_i=0.85$ to $\Omega_f=1.15$, which covers the main transmission region around $\Omega=1$. In Fig. 6, we plot S as a function of a. Also, we keep $a+b=0.5$ as a constant. The figure illustrates that S rapidly decreases with a (and b) deviating from 0.25. When a is about 0.175 and 0.325, S reaches to the minima. If a is further decreased from 0.175 or increased from 0.325, the system tends to become a bulk materials and S increases.

To compare the transmission behaviors of TM multilayers with Fibonacci multilayers at $\Omega=1$, we plot transmission as a function of a for the sixth TM multilayer (64 layers) and the ninth Fibonacci mutilayer (55 layers) in Fig. 7. When a is deviated from 0.25 (still assuming $a+b=0.5$), we see two quite different behaviors: the transmission decreases monotonously and rapidly for the TM system, but it exhibits an oscillatory feature for the Fibnacci one.

In summary, we have derived the recursion relation of the transfer matrix elements in order to study the propagation of light waves in TM multilayers. The frequencies, which give rise to an identical transfer matrix, have a self-similar structure. At these frequencies the system becomes completely transparent, and the electric field distributions are lattice-like, namely, similar to the structure of the TM sequence. By introducing a localization index we have explored the localization properties of light in the aperiodic system. As the layers' number becomes larger, the electric field distributions of the completely transparent states become more and more extended. The effect of phase shift modulation on the transmission is considered. It is shown that around a center frequency (the midgap frequency of a periodic quarter-wavelength stuck), the transmission is very sensitive to the optical thickness modulation. At this frequency, the TM and Fibonacci multilayers show quite different transmission behaviors when the optical thickness is modulated aperiodiclly.

The author would like to thank Professor Ruibao Tao and Dr. Zhifang Lin for their helpful discussions. Part of this work was supported by the Youth Science Foundation of Jiangxi, China.

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[*]Permanent address: Institute of Materials Science, Nanchang University, Nanchang, Jiangxi 330047, P. R. China.

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