

Fokker-Planck and Landau-Lifshitz-Bloch equations for classical ferromagnets

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A macroscopic equation of motion for the magnetization of a ferromagnet at elevated temperatures should contain both transverse and longitudinal relaxation terms and interpolate between the Landau-Lifshitz equation at low temperatures and the Bloch equation at high temperatures. It is shown that for the classical model where spin-bath interactions are described by stochastic Langevin fields and spin-spin interactions are treated within the mean-field approximation (MFA), such a “Landau-Lifshitz-Bloch” (LLB) equation can be derived exactly from the Fokker-Planck equation, if the external conditions change slowly enough. For weakly anisotropic ferromagnets within the MFA the LLB equation can be written in a macroscopic form based on the free-energy functional interpolating between the Landau free energy near T_C and the “micromagnetic” free energy, which neglects changes of the magnetization magnitude $|\mathbf{M}|$, at low temperatures. [S0163-1829(97)03905-2]

I. INTRODUCTION

The famous Landau-Lifshitz equation,¹ which is the basis of innumerable investigations of magnetically ordered materials, considers magnetization as a vector of fixed length and ignores its longitudinal relaxation. Such an approach is obviously unsatisfactory at elevated temperatures since magnetization is an average over some distribution function and its magnitude can change. Alternatively, semiphenomenological “soft-spin” equations of motion for the spin density allowing for the longitudinal relaxation and for the influence of the bath described by stochastic Langevin terms are known in the theory of dynamic critical phenomena.²⁻⁴ A phenomenological deterministic equation of motion for the magnetization of magnetically ordered materials with the longitudinal relaxation terms, which is a direct generalization of the Landau-Lifshitz equation, was formulated by Bar'yakhtar^{5,6} and applied to the domain wall dynamics at elevated temperatures.⁷ The Bar'yakhtar equation was conceived for the temperature range below the Curie point T_C ; the theory does not answer what happens with phenomenological relaxation terms above T_C and whether the Bloch equation used in the theory of EPR and NMR can be recovered in this region.

The simplest nontrivial model, for which the problem of finding an equation of motion for magnetization in the whole range of temperatures can be formulated, is a semiphenomenological model considering an isolated classical spin interacting with the bath modeled by stochastic Langevin fields. The spin-spin interactions in this model, which lead to the ferromagnetism, can be taken into account on the next stage on the mean-field level. Dynamics of such a spin is described by the Fokker-Planck equation (FPE), which can be solved analytically only in limiting cases, in particular, of low and high temperatures. Reduction of the FPE using the modeling of the distribution function⁸ (the accuracy of this procedure was shown to be about 7% in most situations) has led to the closed equation of motion for magnetization interpolating between the Landau-Lifshitz and Bloch equations at low and high temperatures — the so-called “Landau-Lifshitz-Bloch” (LLB) equation. The LLB equation was also derived for a *quantum* spin system interacting with a bath⁹ by the reduc-

tion of the density-matrix equation with the method similar to that used in the classical case. A kind of LLB equation taking into account the spin-spin relaxation was obtained by Plefka^{10,11} for a quantum model with long-range “spin-block” interactions.

The coefficients in the relaxation terms of such a general LLB equation are nonlinear functions of magnetization itself; the only application of this equation up to now is that to the calculation of the nonlinear mobility of domain walls (DW) in rare-earth (RE) ferrites garnets,¹² where the strongly thermally disordered spins of the RE sublattice do not interact with each other and are subject to only the combined influence of the external field and the molecular field acting from the iron sublattice. For the simplest one-sublattice weakly anisotropic ferromagnetic model below T_C the dominant term in the molecular field is the homogeneous exchange, so that the directions of the molecular field and magnetization nearly coincide. In this case the general LLB equation simplifies to its particular form similar to the Bar'yakhtar equation. The latter was applied in Refs. 14, 9, 15 to calculate the domain-wall mobility in uniaxial ferromagnets in the whole temperature range and, in particular, near the phase transition from Bloch to linear (Ising-like) walls at some $T_B < T_C$ predicted by Bulaevskii and Ginzburg.¹³ As this second-order phase transition is accompanied by changing the roles of transverse and longitudinal relaxation processes in the DW dynamics, the DW mobility has a deep minimum at T_B .^{14,9,15} This minimum, and thus the DW phase transition, was recently observed in dynamic susceptibility experiments on Ba and Sr hexaferrites.¹⁶⁻¹⁸

An important dynamical scenario is that when the rate of changing of magnetization (or of its spatial distribution), which can be controlled by an external influence, is slow in comparison to the spin-relaxation rate. This small parameter makes it possible to solve the Fokker-Planck equation exactly without making assumptions about the form of the distribution function. For example, calculation of the low-frequency imaginary part of the longitudinal susceptibility leads to the exact analytical expression for the integral relaxation time τ_{int} , which is defined as the area under the magnetization relaxation curve after an abrupt infinitesimal

change of the magnetic field.^{8,19–22} The quantity τ_{int} describes, in particular, the thermoactivation escape rate of fine ferromagnetic particles with a uniaxial anisotropy over the potential barrier, which is valid, in contrast to the well-known Brown's solution,²³ in the whole temperature range. Such a situation is also characteristic for the dynamics of domain walls, whose velocity depends on the amplitude of the driving field and can be kept whatever small. In this case the Fokker-Planck equation can be solved exactly, which leads to the exact form of the LLB equation, if the spin-spin interactions are considered within the mean-field approximation (MFA). Derivation of this exact "slow" form of the LLB equation is the main purpose of this article.

The main part of the paper is organized as follows. In Sec. II the Fokker-Planck equation for a classical spin, its low- and high-temperature solutions, and the approximate reduction of the FPE to the Landau-Lifshitz-Bloch equation is outlined. In Sec. III the FPE is exactly solved in the slow-motion case and the slow LLB equation is derived. In Sec. IV the simplified form of the latter for ferromagnets below and near T_C is worked out. In Sec. V further possible applications of the method and some unsolved problems are discussed.

II. THE FOKKER-PLANCK AND LLB EQUATIONS

We shall describe a magnetic atom as a classical spin vector \mathbf{s} of a unit length. The magnetic and mechanical moments of the atom are given by $\boldsymbol{\mu} = \mu_0 \mathbf{s}$ and $\mathbf{L} = \gamma \mu_0 \mathbf{s}$, where γ is the gyromagnetic ratio. In the case of a weak coupling with the bath the dynamics of the vector \mathbf{s} can be described with the help of the stochastic Landau-Lifshitz equation

$$\dot{\mathbf{s}} = \gamma[\mathbf{s} \times (\mathbf{H} + \boldsymbol{\zeta})] - \gamma\lambda[\mathbf{s} \times [\mathbf{s} \times \mathbf{H}]] \quad (2.1)$$

with $\lambda \ll 1$, where correlators of the $\alpha, \beta = x, y, z$ components of the Langevin field $\boldsymbol{\zeta}(t)$ are given by

$$\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = \frac{2\lambda T}{\gamma \mu_0} \delta_{\alpha\beta} \delta(t - t'). \quad (2.2)$$

The Fokker-Planck equation corresponding to Eq. (2.1) is formulated for the distribution function $f(\mathbf{N}, t) = \langle \delta(\mathbf{N} - \mathbf{s}(t)) \rangle$ on the sphere $|\mathbf{N}| = 1$, where the average is taken over the realizations of $\boldsymbol{\zeta}$. Differentiating f over t with the use of Eq. (2.1) and calculating the right part with the methods of stochastic theory (see the Appendix), one comes to the Fokker-Planck equation⁸

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{N}} \left\{ \gamma[\mathbf{N} \times \mathbf{H}] - \gamma\lambda[\mathbf{N} \times [\mathbf{N} \times \mathbf{H}]] \right. \\ \left. + \frac{\gamma\lambda T}{\mu_0} \left[\mathbf{N} \times \left[\mathbf{N} \times \frac{\partial}{\partial \mathbf{N}} \right] \right] \right\} f = 0. \end{aligned} \quad (2.3)$$

One can easily see that the distribution function

$$f_0(\mathbf{N}) \propto \exp[-\mathcal{H}(\mathbf{N})/T], \quad \mathcal{H}(\mathbf{s}) = -\mu_0 \mathbf{H} \mathbf{s} \quad (2.4)$$

satisfies Eq. (2.3) at an equilibrium. For the small coupling to the bath, $\lambda \ll 1$, Eq. (2.3) coincides with the Fokker-Planck equation derived by Brown.²³

The equation of motion for the spin polarization (the first moment of the distribution function)

$$\mathbf{m} \equiv \langle \mathbf{s} \rangle = \int d^3 \mathbf{N} \mathbf{N} f(\mathbf{N}, t) \quad (2.5)$$

of an assembly of magnetic atoms can be derived from Eq. (2.3) and has the form

$$\dot{\mathbf{m}} = \gamma[\mathbf{m} \times \mathbf{H}] - \Lambda_N \mathbf{m} - \gamma\lambda \langle [\mathbf{s} \times [\mathbf{s} \times \mathbf{H}]] \rangle \quad (2.6)$$

[cf. Eq. (2.1)], where Λ_N is the characteristic diffusional relaxation rate or, for the thermoactivation escape problem, the Néel attempt frequency given by

$$\Lambda_N \equiv \tau_N^{-1} \equiv 2\gamma\lambda T / \mu_0. \quad (2.7)$$

It can be seen that Eq. (2.6) is not closed but coupled to the second moments of the distribution function, $\langle s_i s_j \rangle$, in its last term. The behavior of Eq. (2.6) is determined by the reduced field ξ_0 given by

$$\xi_0 \equiv |\boldsymbol{\xi}_0|, \quad \boldsymbol{\xi}_0 \equiv \mu_0 \mathbf{H} / T. \quad (2.8)$$

For $\xi_0 \gg 1$ (low temperatures) the second term in Eq. (2.6) can be neglected and the last term decouples for distribution functions localized about some direction: $\langle s_i s_j \rangle \equiv m_i m_j$. In this case the Landau-Lifshitz equation of the type (2.1) for \mathbf{m} without the stochastic field $\boldsymbol{\zeta}$ is recovered. In the high-temperature case, $\xi_0 \ll 1$, the second term of Eq. (2.6) dominates over the last one, which can be neglected. Here one gets the equation of motion for \mathbf{m} with the Bloch relaxation term.

In the intermediate region, $\xi_0 \sim 1$, where the first-moment equation (2.6) is not closed, the resonance and relaxational behavior of the FPE (2.3) is *not* described by Lorentz and Debye curves, and the deviations from the latter reach 7% at $\xi_0 \sim 3$.⁸ Neglecting these features, one can obtain an *isolated* equation of motion for the spin polarization of an assembly of magnetic atoms choosing the distribution function in a form⁸

$$f(\mathbf{N}, t) = \frac{\exp[\boldsymbol{\xi}(t) \mathbf{N}]}{\mathcal{Z}(\boldsymbol{\xi})}, \quad \mathcal{Z} = 4\pi \frac{\sinh \xi}{\xi} \quad (2.9)$$

[cf. Eqs. (2.4) and (2.8)], where $\boldsymbol{\xi}(t)$ is chosen so that the first moment equation (2.6) is satisfied. Calculating the terms of Eq. (2.6) with the help of Eq. (2.9), one arrives⁸ at the LLB equation for the nonequilibrium reduced field $\boldsymbol{\xi}(t)$

$$\dot{\boldsymbol{\xi}} = \gamma[\boldsymbol{\xi} \times \mathbf{H}] - \Gamma_1 \left(1 - \frac{\boldsymbol{\xi} \boldsymbol{\xi}_0}{\xi^2} \right) \boldsymbol{\xi} - \Gamma_2 \frac{[\boldsymbol{\xi} \times [\boldsymbol{\xi} \times \boldsymbol{\xi}_0]]}{\xi^2} \quad (2.10)$$

with the longitudinal and transverse relaxation rates

$$\Gamma_1 = \Lambda_N \frac{B(\xi)}{\xi B'(\xi)}, \quad \Gamma_2 = \frac{\Lambda_N}{2} \left(\frac{\xi}{B(\xi)} - 1 \right), \quad (2.11)$$

where Λ_N is given by Eq. (2.7), $B(\xi) = \coth \xi - 1/\xi$ is the Langevin function and $B'(\xi) \equiv dB(\xi)/d\xi$. The asymptotic forms of Γ_1 and Γ_2 are given by

$$\Gamma_1 \equiv \begin{cases} \Lambda_N \left(1 + \frac{2}{15} \xi^2 \right), & \xi \ll 1 \\ \Lambda_N \xi \left(1 - \frac{1}{\xi} \right), & \xi \gg 1, \end{cases} \quad (2.12)$$

and

$$\Gamma_2 \equiv \begin{cases} \Lambda_N \left(1 + \frac{1}{10} \xi^2 \right), & \xi \ll 1 \\ \frac{1}{2} \Lambda_N \xi \left(1 + \frac{1}{\xi^2} \right), & \xi \gg 1. \end{cases} \quad (2.13)$$

The relaxation rates of such a type also appear as a result of calculation of the high-frequency longitudinal susceptibility and the far-from-resonance transverse one.⁸ The quantity Γ_1 is also proportional to the ‘‘effective eigenvalue’’ λ_{eff} of Ref. 19. One can see that the equilibrium solution of Eq. (2.10) is $\xi = \xi_0$. The nonequilibrium spin polarization \mathbf{m} is given by

$$\mathbf{m} = m \xi / \xi, \quad m = B(\xi). \quad (2.14)$$

The LLB equation for ξ , Eq. (2.10), can be written in the alternative equivalent form

$$\dot{\xi} = \gamma [\xi \times \mathbf{H}] - \Gamma_1 (\xi - \xi_0) - (\Gamma_2 - \Gamma_1) \frac{[\xi \times [\xi \times \xi_0]]}{\xi^2}. \quad (2.15)$$

Here it can be seen that in the high-temperature region, $\xi, \xi_0 \ll 1$, where $B(\xi) \equiv \xi/3$ and $\Gamma_1 \equiv \Gamma_2 \equiv \Lambda_N$, the Landau-Lifshitz double-vector-product relaxation term becomes small and the Bloch equation is recovered. On the other hand, at low temperatures, when $\xi, \xi_0 \gg 1$, the magnitude of the vector \mathbf{m} in Eq. (2.14) saturates in most situations at $m = B(\xi) \equiv 1$, and the longitudinal relaxation term in Eq. (2.10) no longer plays a role. Here the usual Landau-Lifshitz equation is recovered. Using Eq. (2.14) one can derive the LLB equation for the spin polarization \mathbf{m} itself. The result can be written as

$$\begin{aligned} \dot{\mathbf{m}} = & \gamma [\mathbf{m} \times \mathbf{H}] - \Lambda_N \left(1 - \frac{m \xi_0}{m \xi} \right) \mathbf{m} \\ & - \gamma \lambda \left(1 - \frac{m}{\xi} \right) \frac{[\mathbf{m} \times [\mathbf{m} \times \mathbf{H}]]}{m^2} \end{aligned} \quad (2.16)$$

[cf. Eqs. (2.1) and (2.6)], where $\xi = \xi(m)$ is determined implicitly by the relation $m = B(\xi)$. Note that here at low temperatures, $\xi \gg 1$, the coefficient before the transverse relaxation term goes to $\gamma \lambda$, whereas the longitudinal one is nonessential, if m is saturated. At high temperatures, $\xi \ll 1$, the relaxation term in Eq. (2.16) acquires the Bloch form $\Lambda_N (\mathbf{m} - \mathbf{m}_0)$ with $\mathbf{m}_0 \equiv \xi_0/3$ [see also Eq. (2.15)]. The quantum generalization of the classical LLB equation written above was given in Ref. 9. The latter was applied in Ref. 12 to study the nonlinear dynamics of the RE sublattice of rare-earth ferrites garnets near the magnetization compensation point.

For small deviations from equilibrium, where $\xi \approx \xi_0$ and, accordingly, $\mathbf{m} \approx \mathbf{m}_0 \equiv B(\xi_0) \xi_0 / \xi_0$, one can put the LLB equation (2.16) [or, more conveniently, directly Eq. (2.10)] into the form

$$\dot{\mathbf{m}} = \gamma [\mathbf{m} \times \mathbf{H}] - \Gamma_1 \left(1 - \frac{m \mathbf{m}_0}{m^2} \right) \mathbf{m} - \Gamma_2 \frac{[\mathbf{m} \times [\mathbf{m} \times \mathbf{m}_0]]}{m^2}, \quad (2.17)$$

where the relaxation frequencies Γ_1 and Γ_2 are functions of ξ_0 . A kind of LLB equation similar to Eq. (2.17) was obtained by Gekht *et al.*,²⁴ who assumed, for the calculation of the linear transverse dynamic susceptibility, instead of Eq. (2.9) a distribution function of the form $f(\mathbf{N}, t) = f_0(\mathbf{N}) [1 + \alpha(t) \mathbf{N}]$, where f_0 is given by Eq. (2.4) and α corresponds to $\xi - \xi_0$ in our notations. Although Gekht *et al.* claimed that ‘‘the single-moment approximation is permissible for small deviations from equilibrium,’’ Eq. (2.17) is in fact only approximate, as well as the more general Eq. (2.16). The latter, in contrast, can be applied and has a rather good accuracy in situations where deviations from equilibrium are large, as was checked in Ref. 8. In Sec. III we will consider the solution of the FPE (2.3) for slowly varying field $\mathbf{H}(t)$. In this case the deviations from the instantaneous equilibrium state are small and the FPE can be solved exactly without assumptions about the form of the distribution function $f(\mathbf{N}, t)$.

III. THE ‘‘SLOW’’ LLB EQUATION

If the magnetic field \mathbf{H} slowly changes its magnitude and direction, the solution of the Fokker-Planck equation (2.3) slightly deviates from the instantaneous equilibrium one and can be searched for in the form

$$f(\mathbf{N}, t) \equiv \frac{\exp[\xi_0(t) \mathbf{N}]}{\mathcal{Z}(\xi_0)} [1 + Q(\mathbf{N}, t)], \quad Q \ll 1, \quad (3.1)$$

where $\xi_0(t) \equiv \mu_0 \mathbf{H}(t) / T$. The correction function $Q(\mathbf{N}, t) \propto |\dot{\mathbf{H}}|$ and, additionally, it depends slowly on time, so that $\dot{Q} \propto |\dot{\mathbf{H}}|^2$. Neglecting this small term, one obtains from Eq. (2.3) the equation for Q having the form

$$\begin{aligned} [\mathbf{N} \times \xi_0] \frac{\partial Q}{\partial \mathbf{N}} + \lambda \left(\frac{\partial}{\partial \mathbf{N}} + \xi_0 \right) \left[\mathbf{N} \times \left[\mathbf{N} \times \frac{\partial Q}{\partial \mathbf{N}} \right] \right] = \tau_0 (\mathbf{m}_0 - \mathbf{N}) \dot{\xi}_0, \\ \mathbf{m}_0 \equiv B(\xi_0) \frac{\mathbf{H}}{H}, \end{aligned} \quad (3.2)$$

where $\tau_0 \equiv \mu_0 / (\gamma T)$. One can see that in leading order the correction $Q(\mathbf{N}, t)$ is determined by the instantaneous values of the magnetic field $\mathbf{H}(t)$ and its first derivative $\dot{\mathbf{H}}$. The right-hand part of this equation can be separated into the terms describing the temporal changes of the magnitude and of the direction of \mathbf{H} as

$$(\mathbf{m}_0 - \mathbf{N}) \dot{\xi}_0 = \mathbf{N} [\xi_0 \times \boldsymbol{\Omega}] + \left[m_0 - \frac{\mathbf{N} \xi_0}{\xi_0} \right] \dot{\xi}_0, \quad (3.3)$$

where

$$\boldsymbol{\Omega} \equiv [\xi_0 \times \dot{\xi}_0] / \xi_0^2 \quad (3.4)$$

is the precession frequency of the vector ξ_0 . In the spherical coordinate system with z axis along ξ_0 Eq. (3.2) for $Q(x, \varphi)$, where $x \equiv \cos\theta$, takes on the form

$$\begin{aligned} \xi_0 \frac{\partial Q}{\partial \varphi} + \lambda \left\{ \left(\frac{\partial}{\partial x} + \xi_0 \right) (1-x^2) \frac{\partial}{\partial x} + \frac{1}{1-x^2} \frac{\partial^2}{\partial \varphi^2} \right\} Q \\ = \tau_0 \xi_0 \sqrt{1-x^2} [\Omega_y \cos\varphi - \Omega_x \sin\varphi] + \tau_0 (x - m_0) \xi_0, \end{aligned} \quad (3.5)$$

where Ω_x and Ω_y are x and y components of the vector Ω .

The solution of the linear differential equation (3.5) is a sum of two contributions induced by the transverse and longitudinal inhomogeneous terms: $Q = Q_{\perp} + Q_{\parallel}$. Using the substitution

$$Q_{\perp} = Q_x \cos\varphi + Q_y \sin\varphi, \quad Q_{\parallel} \equiv Q_x + iQ_y, \quad (3.6)$$

one comes to the equation

$$\begin{aligned} Q_{\perp} + \frac{i\lambda}{\xi_0} \left\{ \left(\frac{d}{dx} + \xi_0 \right) (1-x^2) \frac{d}{dx} - \frac{1}{1-x^2} \right\} Q_{\perp} \\ = \tau_0 \Omega_{\perp} \sqrt{1-x^2}, \end{aligned} \quad (3.7)$$

where $\Omega_{\perp} \equiv \Omega_1 + i\Omega_2$. This equation cannot, in general, be solved analytically, but the latter is possible in the typical case of the weak coupling to the bath, $\lambda \ll 1$. For $\lambda/\xi_0 \ll 1$ one can easily find the solution iteratively, which yields

$$Q_{\perp} \approx \tau_0 \Omega_{\perp} \sqrt{1-x^2} \left[1 + \frac{i\lambda}{\xi_0} (2 + \xi_0 x) + \dots \right]. \quad (3.8)$$

On the other hand, in the high-temperature region, where $\xi_0 \ll 1$, one can neglect ξ_0 in the round brackets in Eq. (3.7), after which Eq. (3.7) can be analytically solved to yield

$$Q_{\perp} \approx \tau_0 \Omega_{\perp} \sqrt{1-x^2} \frac{1 + 2i\lambda/\xi_0}{1 + (2\lambda/\xi_0)^2}. \quad (3.9)$$

These two solutions overlap in the region $\lambda \ll \xi_0 \ll 1$, and thus they can be sewn together in the whole range of temperatures into the formula, which can be obtained by replacing the numerator of the fraction in (3.9) by $1 + (i\lambda/\xi_0)(2 + \xi_0 x)$.

The equation for $Q_{\parallel}(x)$ can be written as

$$\left(\frac{d}{dx} + \xi_0 \right) (1-x^2) \frac{dQ_{\parallel}}{dx} = \Lambda_N^{-1} (x - m_0) \xi_0. \quad (3.10)$$

It can be solved in two steps with the help of the substitution $P(x) \equiv (1-x^2)dQ_{\parallel}/dx$. First, integrating Eq. (3.10) one gets

$$P(x) = \frac{\xi_0}{\Lambda_N \xi_0} \left[x - \coth\xi_0 + \frac{e^{-\xi_0 x}}{\sinh\xi_0} \right]. \quad (3.11)$$

Then, Q_{\parallel} is given by

$$Q_{\parallel}(x) = \int_{-1}^x \frac{dx'}{1-x'^2} P(x') + C, \quad (3.12)$$

where the constant C is determined from the normalization condition for the distribution function (3.1).

Now, the function $Q(\mathbf{N}, t)$ having been determined, one can calculate the spin polarization \mathbf{m} using Eqs. (2.5) and (3.1). Returning to vector designations, one comes to the result

$$\begin{aligned} \mathbf{m} \approx B(\xi_0) \left\{ \left(1 + \frac{\xi_0 B'}{\Gamma_{1,\text{int}} B} \frac{\mathbf{H}\dot{\mathbf{H}}}{H^2} \right) \frac{\mathbf{H}}{H} \right. \\ \left. + \frac{\gamma H}{(\gamma H)^2 + \Gamma_2^2} \left(\frac{[\mathbf{H} \times \dot{\mathbf{H}}]}{H^2} + \frac{\Gamma_2}{\gamma H} \frac{[\mathbf{H} \times [\mathbf{H} \times \dot{\mathbf{H}}]]}{H^3} \right) \right\}, \end{aligned} \quad (3.13)$$

where Γ_2 is the transverse relaxation rate given by Eq. (2.11) and $\Gamma_{1,\text{int}}$ is the inverse of the integral longitudinal relaxation time τ_{int} ,

$$\begin{aligned} \frac{1}{\Gamma_{1,\text{int}}} \equiv \tau_{\text{int}} = \frac{1}{\Lambda_N \xi_0 \sinh\xi_0 B'(\xi_0)} \\ \times \int_{-1}^1 dx \frac{e^{\xi_0 x}}{1-x^2} \left[x - \coth\xi_0 + \frac{e^{-\xi_0 x}}{\sinh\xi_0} \right]^2, \end{aligned} \quad (3.14)$$

which is determined as the area under the magnetization relaxation curve after an abrupt infinitesimal change of the longitudinal magnetic field.^{8,22} Equation (3.13) describes the lagging of the spin polarization \mathbf{m} from its quasiequilibrium value $\mathbf{m}_0(t)$ of Eq. (3.2), which is determined by the small derivative $\dot{\mathbf{H}}(t)$. The asymptotic forms of $\Gamma_{1,\text{int}}$ in Eq. (3.14) read

$$\Gamma_{1,\text{int}} \approx \begin{cases} \Lambda_N \left(1 + \frac{1}{9} \xi_0^2 \right), & \xi_0 \ll 1 \\ \Lambda_N \xi_0 \left(1 - \frac{1}{\xi_0} \right), & \xi_0 \gg 1. \end{cases} \quad (3.15)$$

Comparing Eqs. (3.15) and (2.12) one can see that $\Gamma_1 > \Gamma_{1,\text{int}}$. The relative deviation $\delta \equiv \Gamma_1/\Gamma_{1,\text{int}} - 1$ attains a value $\delta \approx 0.07$ at $\xi_0 \approx 3$.⁸

The next problem is to write down the equation of motion for \mathbf{m} , which has the solution (3.13). It is especially important if the spin-spin interactions are taken into account within the MFA (see the next section). In this case \mathbf{H} is replaced by the molecular field \mathbf{H}^{MFA} containing \mathbf{m} itself, and Eq. (3.13) is in fact a differential equation for $\dot{\mathbf{m}}$, which should be still simplified. It can be done differentiating Eq. (3.13) over time and neglecting terms of order \dot{H}^2 coming from the correction terms with $\dot{\mathbf{H}}$ in Eq. (3.13). This leads to

$$\dot{\mathbf{m}} \approx \xi_0 B'(\xi_0) \frac{(\mathbf{H}\dot{\mathbf{H}})\mathbf{H}}{H^3} - B(\xi_0) \frac{[\mathbf{H} \times [\mathbf{H} \times \dot{\mathbf{H}}]]}{H^3}. \quad (3.16)$$

Now $\dot{\mathbf{H}}$ in this relation should be expressed through \mathbf{m} with the help of Eq. (3.13), which after some vector algebra leads to the ‘‘slow’’ LLB equation

$$\dot{\mathbf{m}} = \gamma [\mathbf{m} \times \mathbf{H}] - \Gamma_{1,\text{int}} \left(1 - \frac{\mathbf{m}\mathbf{m}_0}{m^2} \right) \mathbf{m} - \Gamma_2 \frac{[\mathbf{m} \times [\mathbf{m} \times \mathbf{m}_0]]}{m^2}, \quad (3.17)$$

where \mathbf{m}_0 is given by Eq. (3.2) and which is the refinement of Eq. (2.17) in the slow-motion situation. The quantities Γ_1 of Eq. (2.11) and $\Gamma_{1,\text{int}}$ of Eq. (3.14) have the same leading high- and low-temperature asymptotes, and, as was said above, they differ by no more than 7% in the whole range of temperatures. The same order of magnitude also characterizes the difference between the Debye one-relaxator form of the longitudinal dynamic susceptibility $\chi_{\parallel}(\omega)$ following from Eq. (2.17) and the actual form of $\chi_{\parallel}(\omega)$ following from the solution of the exact Fokker-Planck equation (2.3) at intermediate temperatures.⁸ It should be noted that in the fast-motion situations Eq. (2.17) is better than Eq. (3.17), since it yields the exact leading (imaginary) term of the high-frequency expansion of $\chi_{\parallel}(\omega)$.⁸

IV. LLB EQUATION FOR FERROMAGNETS

For definiteness we consider the classical ferromagnetic model with the biaxially anisotropic exchange interaction

$$\mathcal{H} = -\mu_0 \sum_i \mathbf{H}_i \mathbf{s}_i - \frac{1}{2} \sum_{ij} J_{ij} (\eta_x s_{xi} s_{xj} + \eta_y s_{yi} s_{yj} + s_{zi} s_{zj}), \quad (4.1)$$

where $\eta_x \leq \eta_y \leq 1$ are the anisotropy coefficients. The dynamics of this model interacting with the bath is described by the stochastic Landau-Lifshitz equation

$$\dot{\mathbf{s}}_i = \gamma [\mathbf{s}_i \times (\mathbf{H}_{i,\text{tot}} + \boldsymbol{\zeta}_i)] - \gamma \lambda [\mathbf{s}_i \times [\mathbf{s}_i \times \mathbf{H}_{i,\text{tot}}]] \quad (4.2)$$

[cf. Eq. (2.1)], where $\boldsymbol{\zeta}_i$ are postulated to be uncorrelated on different lattice sites, and

$$\mathbf{H}_{i,\text{tot}} \equiv -\frac{1}{\mu_0} \frac{\partial \mathcal{H}}{\partial \mathbf{s}_i} = \mathbf{H}_i + \frac{1}{\mu_0} \sum_j J_{ij} (\eta_x \mathbf{s}_{xj} + \eta_y \mathbf{s}_{yj} + \mathbf{s}_{zj}) \quad (4.3)$$

is the total field acting on a given spin at the site i , which depends on the orientation of spins on the neighboring sites j . In Eq. (4.3) $\mathbf{s}_{\alpha j} \equiv s_{\alpha j} \mathbf{e}_{\alpha}$, $\alpha = x, y, z$, and \mathbf{e}_{α} are the ords of the Descarte coordinate system.

The Fokker-Planck equation for the distribution function

$$f_{\text{tot}}(\{\mathbf{N}_i\}, t) = \left\langle \prod_{i=1}^{\mathcal{N}} \delta(\mathbf{N}_i - \mathbf{s}_i(t)) \right\rangle_{\boldsymbol{\zeta}} \quad (4.4)$$

of the whole system consisting of \mathcal{N} spins can be derived in the same way as Eq. (2.3) and has the form

$$\begin{aligned} \frac{\partial f_{\text{tot}}}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{N}_i} \left\{ \gamma [\mathbf{N}_i \times \mathbf{H}_{i,\text{tot}}] - \gamma \lambda [\mathbf{N}_i \times [\mathbf{N}_i \times \mathbf{H}_{i,\text{tot}}]] \right. \\ \left. + \frac{\gamma \lambda T}{\mu_0} \left[\mathbf{N}_i \times \left[\mathbf{N}_i \times \frac{\partial}{\partial \mathbf{N}_i} \right] \right] \right\} f_{\text{tot}} = 0. \end{aligned} \quad (4.5)$$

One can check that the static solution of this equation is

$$f_{\text{tot},0}(\{\mathbf{N}_i\}) \propto \exp[-\mathcal{H}(\{\mathbf{N}_i\})/T], \quad (4.6)$$

where \mathcal{H} is given by Eq. (4.1). Solving Eq. (4.5) is a formidable task that goes beyond the scope of this paper. It is in any case not simpler than calculating averages with the distribution function (4.6) at an equilibrium and requires application of some kind of many-body perturbation theory, as the

diagram technique for classical spin systems (see, e.g., Ref. 25), which has proved to be rather efficient for description of their static properties. Here we resort to the mean-field approximation with respect to spin-spin interactions, which means, however, dropping their contribution into the relaxation rates. In MFA the distribution function of the system (4.4) is multiplicative, and one can use the distribution functions f_i for each spin on the site i , which satisfy the Fokker-Planck equation (2.3) with $\mathbf{H} \Rightarrow \mathbf{H}_i^{\text{MFA}}$, where $\mathbf{H}_i^{\text{MFA}}$ is given by Eq. (4.3) with the replacement $\mathbf{s}_i \Rightarrow \mathbf{m}_i \equiv \langle \mathbf{s}_i \rangle$. Solution of such mean-field FPE's similar to that of Sec. II or Sec. III leads to the set of coupled LLB equations for \mathbf{m}_i , $i = 1, 2, \dots, \mathcal{N}$ of the type (2.16) in a general nonlinear situation or Eq. (3.17) for slow motions. The static solution of these LLB equations satisfies the inhomogeneous Curie-Weiss equation,

$$\mathbf{m}_i = B(\xi_{0i}) \frac{\xi_{0i}}{\xi_{0i}}, \quad \xi_{0i} \equiv \frac{\mu_0 \mathbf{H}_i^{\text{MFA}}}{T}, \quad (4.7)$$

which describes within the MFA both the homogeneous state and such configurations as domain walls with account of thermal effects (see, e.g., Ref. 26 and references therein).

For the most of ferromagnetic substances the small-anisotropy case, i.e., $\eta'_{x,y} \equiv 1 - \eta_{x,y} \ll 1$, is realized. In this case the spatial inhomogeneity of magnetization at a distance of the lattice spacing is small, and one can use the continuous approximation. For $\mathbf{H}_i^{\text{MFA}}$ the latter means

$$\begin{aligned} \mathbf{H}^{\text{MFA}}(\mathbf{r}) \equiv \mathbf{H}_E + \mathbf{H}'_{\text{eff}}, \quad \mathbf{H}_E = \frac{J_0}{\mu_0} \mathbf{m}, \\ \mathbf{H}'_{\text{eff}} = \mathbf{H} + \frac{J_0}{\mu_0} [\alpha \Delta \mathbf{m} - \eta'_x \mathbf{m}_x - \eta'_y \mathbf{m}_y], \end{aligned} \quad (4.8)$$

where J_0 is the zero Fourier component of the exchange interaction, Δ is the Laplace operator, and α is a lattice-dependent constant (for the simple cubic lattice $\alpha = a_0^2/6$, where a_0 is the lattice spacing). The most important for ferromagnets is the case of the strong homogeneous exchange field, $|\mathbf{H}_E| \gg |\mathbf{H}'_{\text{eff}}|$, which is realized below $T_C = (1/3)J_0$, where there is a spontaneous magnetization, and also in the region just above T_C , where the longitudinal susceptibility is large. As in this case the external field $\mathbf{H}(t)$ that can drive the system off the equilibrium is a relatively small quantity, one can use Eq. (2.17) [or, for slow motions, Eq. (3.17)] and expand $\mathbf{m}_0 = B(\beta \mu_0 H^{\text{MFA}}) \mathbf{H}^{\text{MFA}}/H^{\text{MFA}}$, where $\beta \equiv 1/T$, up to the first order in \mathbf{H}'_{eff} . This leads to the equation

$$\begin{aligned} \dot{\mathbf{m}} = \gamma [\mathbf{m} \times \mathbf{H}'_{\text{eff}}] - \gamma \lambda_1 \left(\frac{1 - B/m}{\mu_0 \beta B'} - \frac{\mathbf{m} \mathbf{H}'_{\text{eff}}}{m^2} \right) \mathbf{m} \\ - \gamma \lambda_2 \frac{[\mathbf{m} \times [\mathbf{m} \times \mathbf{H}'_{\text{eff}}]]}{m^2}, \end{aligned} \quad (4.9)$$

where $B = B(m \beta J_0)$,

$$\lambda_1 = 2\lambda \frac{T}{J_0}, \quad \lambda_2 = \lambda \left(1 - \frac{T}{J_0} \right), \quad (4.10)$$

if Eq. (2.17) was used, and the same with $\lambda_1 \Rightarrow \lambda_1 \Gamma_{1,\text{int}}/\Gamma_1$ for the ‘‘slow’’ LLB equation (3.17). The difference

$1 - B/m$ in Eq. (4.9) is a small quantity proportional to the deviation from the equilibrium. It can be further simplified to

$$\frac{1 - B/m}{\mu_0 \beta B'} \cong \begin{cases} \frac{1}{2\chi_{\parallel}} \left(\frac{m^2}{m_e^2} - 1 \right), & T < T_c \\ \frac{J_0}{\mu_0} \left(\frac{3}{5} m^2 - \epsilon \right), & |\epsilon| \ll 1, \end{cases} \quad (4.11)$$

where $\epsilon \equiv 1 - T/T_c$, m_e is the equilibrium spin-polarization satisfying $m_e = B(m_e \beta J_0)$, and

$$\tilde{\chi}_{\parallel} = \frac{\partial m}{\partial H} = \frac{\mu_0}{J_0} \frac{B' \beta J_0}{1 - B' \beta J_0} \quad (4.12)$$

is the spin-polarization susceptibility, calculated for $m = m_e$. Using $B(\xi) \cong (1/3)\xi - (1/45)\xi^3 + \dots$ and $m_e^2 \cong (5/3)\epsilon$ near T_c , one can check that the two expressions in (4.11) overlap in this region.

The last step is to rewrite Eq. (4.9) for the macroscopic magnetization, $\mathbf{M} = \mu_0 \mathbf{m}/v_0$, where v_0 is the unit-cell volume. This leads to the final result

$$\dot{\mathbf{M}} = \gamma [\mathbf{M} \times \mathbf{H}_{\text{eff}}] + L_1 \frac{(\mathbf{M} \mathbf{H}_{\text{eff}}) \mathbf{M}}{M^2} - L_2 \frac{[\mathbf{M} \times [\mathbf{M} \times \mathbf{H}_{\text{eff}}]]}{M^2}, \quad (4.13)$$

where L_1 and L_2 are the longitudinal and transverse kinetic coefficients,

$$L_{1,2} = \gamma M_e \alpha_{1,2}, \quad \alpha_{1,2} = \lambda_{1,2}/m_e, \quad (4.14)$$

α_1 and α_2 are the corresponding Gilbert damping parameters, and the effective field \mathbf{H}_{eff} is given by

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \frac{1}{q_d} \Delta \mathbf{M} - \frac{1}{\chi_x} \mathbf{M}_x - \frac{1}{\chi_y} \mathbf{M}_y - \frac{1}{2\chi_{\parallel}} \left(\frac{M^2}{M_e^2} - 1 \right) \mathbf{M} \quad (4.15)$$

[cf. Eq. (4.8)]. In Eq. (4.15)

$$\frac{1}{q_d} \equiv \frac{\alpha J_0}{W_d}, \quad W_d \equiv \frac{\mu_0^2}{v_0}, \quad (4.16)$$

q_d and W_d are the characteristic dipolar wave number and dipolar energy, αJ_0 is the second moment of the exchange interaction, and the susceptibilities are given by

$$\chi_{\parallel} = \frac{W_d}{J_0} \frac{B' \beta J_0}{1 - B' \beta J_0}, \quad \chi_{x,y} = \frac{W_d}{J_0} \frac{1}{1 - \eta_{x,y}}. \quad (4.17)$$

The effective field \mathbf{H}_{eff} of Eq. (4.15) can be written as a variational derivative

$$\mathbf{H}_{\text{eff}}(\mathbf{r}) = \frac{\delta F}{\delta \mathbf{M}(\mathbf{r})}, \quad (4.18)$$

where F is the MFA free energy of a ferromagnet,

$$F = F_0 + \int d\mathbf{r} \left\{ -\mathbf{H} \mathbf{M} + \frac{1}{2q_d^2} (\nabla \mathbf{M})^2 + \frac{1}{2\chi_x} M_x^2 + \frac{1}{2\chi_y} M_y^2 + \frac{1}{8M_e^2 \chi_{\parallel}} (M^2 - M_e^2)^2 \right\}, \quad (4.19)$$

$(\nabla \mathbf{M})^2 \equiv (\nabla M_x)^2 + (\nabla M_y)^2 + (\nabla M_z)^2$, and F_0 is the equilibrium free energy in the absence of anisotropy and magnetic field. The direct derivation of this free energy from the mean-field theory is tricky and will be presented elsewhere. Equation (4.19) provides a link between the ‘‘micromagnetics,’’^{1,27} which ignores changes of the magnetization magnitude $|\mathbf{M}|$, and the Landau theory of phase transitions,^{28,29} which is a limiting form of the MFA pretending to be valid only in the vicinity of T_c where the order parameter $\mathbf{M}(\mathbf{r})$ is small. In fact, for weakly anisotropic systems in a magnetic field smaller than the homogeneous exchange field H_E , the actual small quantity, which remains small in the whole temperature range, is not $M^2(\mathbf{r})$, but rather the difference $M^2(\mathbf{r}) - M_e^2$, where M_e is the equilibrium magnetization in the absence of anisotropy and magnetic field. Since in the MFA near T_c one has $M_s^2 \propto \chi_{\parallel}^{-1} \propto \epsilon \equiv 1 - T/T_c$, the last term of Eq. (4.19) takes on the Landau form $A M^2 + B M^4$ with $A = -\epsilon A_0$, and $A_0, B = \text{const}$. This shows, further, that Eq. (4.19) can be continued into the region $T > T_c$ as the usual Landau theory. The free energy Eq. (4.19) can be brought into the ‘‘micromagnetic’’ form by introducing the magnetization direction vector $\boldsymbol{\nu} \equiv \mathbf{M}/M$. One can then identify

$$\frac{1}{2\chi_{x,y}} M_{x,y}^2 = K_{x,y} \nu_{x,y}^2, \quad K_{x,y} = \frac{M^2}{2\chi_{x,y}}, \quad (4.20)$$

where $K_{x,y}$ are the anisotropy constants.

V. DISCUSSION

In this paper several forms of the Landau-Lifshitz-Bloch (LLB) equation of motion for a single classical spin interacting with the bath as well as for classical ferromagnets within the MFA have been obtained. These LLB equations are applicable for all temperatures and contain both transverse and longitudinal relaxation terms. The *nonlinear* response of a single spin to the arbitrary changing magnetic field $\mathbf{H}(t)$ is the most accurately described by the nonlinear LLB equation (2.16). For slowly varying $\mathbf{H}(t)$ the exact ‘‘slow’’ LLB equation (3.17) containing the integral longitudinal relaxation time Eq. (3.14) can be used. This case is the most important one for the domain-wall dynamics. For ferromagnets within the MFA the magnetic field \mathbf{H} in the LLB equation should be replaced by \mathbf{H}^{MFA} , which is given by Eq. (4.3) with $\mathbf{s}_i \Rightarrow \mathbf{m}_i \equiv \langle \mathbf{s}_i \rangle$ in a general case or by Eq. (4.8), if the continuous approximation is applicable. If, additionally, in Eq. (4.8) the homogeneous exchange field \mathbf{H}_E is dominant, which is typical for ferromagnets below and near above T_c , the LLB equation reduces to the form (4.13) with Eqs. (4.18) and (4.19). Equation (4.13) could be also written, without specifying the form of kinetic coefficients $L_{1,2}$ and that of the free energy Eq. (4.19), from general arguments. It is very close to the phenomenological Bar'yakhtar equation,^{5,6} which contains an additional relaxation term proportional to ΔH_{eff} . This term, whose microscopic origin is the spin-spin interaction or the correlation of the Langevin fields ξ_i in Eq. (4.2) on different lattice sites $i \neq j$, was shown,⁷ however, to yield a contribution into the domain-wall dynamics, which is negligible in comparison to that of the longitudinal relaxation term in Eq. (4.13).

The quantum generalization of the nonlinear LLB equation (2.16), which contains additional relaxation terms of a different symmetry, was derived in Ref. 9 by the approximate solution of the density matrix equation for a single spin interacting with an idealized phonon bath, which is based on choosing the distribution function of the type similar to Eq. (2.9). In the classical limit the density-matrix equation goes over to the FPE and, accordingly, the quantum LLB simplifies to Eq. (2.16) with the microscopically determined bath-coupling parameter λ . For ferromagnets with the dominant homogeneous exchange interaction the quantum LLB equation simplifies to the same macroscopic form (4.13). The main result of the present paper — the “slow” LLB equation (3.17) — can be obtained in the quantum case, too, by a perturbative solution of the density-matrix equation for a slowly changing magnetic field, which is similar to the derivation in Sec. III. The final result can be, however, obtained by replacing $\Gamma_1 \Rightarrow \Gamma_{1,\text{int}} \equiv \tau_{\text{int}}^{-1}$ in the longitudinal relaxation term. The analytical expression for $\Gamma_{1,\text{int}}$ in the quantum case without single-site anisotropy was already given in Ref. 9. Very recently it was generalized for the anisotropic case to describe the thermoactivation escape rate of quantum spin systems.³⁰

The most serious problem by the derivation of the LLB equation for ferromagnets, that has not been solved yet, is taking into account the spin-spin interactions. This is a rather difficult task, since one should consider the FPE (4.5) for the whole system, which describes all possible static and dynamic spin correlations. Even at an equilibrium, where the solution of the FPE (4.5) is known and given by Eq. (4.6), one faces the problem of a phase transition in a many-body system. Calculation of spin-spin contributions into the longitudinal and transverse kinetic coefficients L_1 and L_2 in the LLB equation for ferromagnets, Eq. (4.13), goes beyond the scope of this paper and is planned for the future.

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APPENDIX: DERIVATION OF THE FOKKER-PLANCK EQUATION

Here the derivation of the Fokker-Planck equation (2.3) is presented, which is more direct and simple than the original one by Brown²³ and which uses more advanced stochastic methods applied, in particular, in the dynamical renormalization-group (RG) theory.²⁻⁴ The RG considerations start, however, with “soft-spin” models with the formal Langevin *sources* (i.e., the inhomogeneous terms in the stochastic differential equations for the spin density), which cannot be interpreted as random *magnetic fields* acting on spins. For our purposes, we will derive the FPE for magnetic systems with the methods of Refs. 2 and 4 but starting from the more realistic stochastic Landau-Lifshitz equation (2.1). At first we introduce the probability distribution of the random Gaussian noise ζ ,

$$\mathcal{F}[\zeta(\tau)] = \frac{1}{\mathcal{Z}_\zeta} \exp\left[-\frac{1}{2a} \int_{-\infty}^{\infty} d\tau \zeta^2(\tau)\right], \quad (\text{A1})$$

where $\mathcal{Z}_\zeta = \int D\zeta \mathcal{F}$ is the noise partition function, $\int D\zeta$ denotes functional integration over realizations of $\zeta(\tau)$ and $a \equiv 2\lambda T / (\gamma\mu_0)$. With the help of Eq. (A1) the average of any noise functional $\mathcal{A}[\zeta]$ can be written as

$$\langle \mathcal{A}[\zeta] \rangle_\zeta = \int D\zeta \mathcal{A}[\zeta] \mathcal{F}[\zeta]. \quad (\text{A2})$$

With the use of the obvious identity

$$\frac{\delta \zeta_\alpha(\tau)}{\delta \zeta_\beta(t)} = \delta_{\alpha\beta} \delta(\tau - t), \quad (\text{A3})$$

one can calculate variations of $\mathcal{F}[\zeta]$ of Eq. (A1):

$$\begin{aligned} \frac{\delta \mathcal{F}[\zeta]}{\delta \zeta_\alpha(t)} &= -\frac{1}{a} \zeta_\alpha(t) \mathcal{F}[\zeta], \\ \frac{\delta^2 \mathcal{F}[\zeta]}{\delta \zeta_\alpha(t) \delta \zeta_\beta(t')} &= \left[\frac{1}{a^2} \zeta_\alpha(t) \zeta_\beta(t') - \frac{1}{a} \delta_{\alpha\beta} \delta(t - t') \right] \mathcal{F}[\zeta], \end{aligned} \quad (\text{A4})$$

etc. Since for all n one has

$$\int D\zeta \frac{\delta^n \mathcal{F}[\zeta]}{\delta \zeta_{\alpha_1}(t_1) \delta \zeta_{\alpha_2}(t_2) \cdots \delta \zeta_{\alpha_n}(t_n)} = 0, \quad (\text{A5})$$

the functional integration of Eq. (A4) leads to $\langle \zeta_\alpha(t) \rangle = 0$ and Eq. (2.2). Further, one can show that all averages of an odd number of ζ components are zero and those of an even number $n > 2$ of ζ 's decay pairwise and can be expressed through the pair average Eq. (2.2), i.e., the statistics of the random field $\zeta(t)$ is Gaussian.

The distribution function of spins f is determined as

$$f(\mathbf{N}, t) \equiv \langle \pi(t, [\zeta]) \rangle_\zeta, \quad \pi(t, [\zeta]) \equiv \delta(\mathbf{N} - \mathbf{s}(t)). \quad (\text{A6})$$

The time derivative of f can be calculated using

$$\dot{\pi} = -\frac{\partial \pi}{\partial \mathbf{N}} \dot{\mathbf{s}} \quad (\text{A7})$$

and the equation of motion (2.1), which yields

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial}{\partial \mathbf{N}} \{ \gamma [\mathbf{N} \times \mathbf{H}] f - \gamma \lambda [\mathbf{N} \times [\mathbf{N} \times \mathbf{H}]] f \\ &\quad + \gamma [\mathbf{N} \times \langle \zeta(t) \pi(t, [\zeta]) \rangle_\zeta] \}. \end{aligned} \quad (\text{A8})$$

Then the average $\langle \zeta(t) \pi(t, [\zeta]) \rangle_\zeta$ can be transformed with the use of the first of Eqs. (A4) and integration by parts,

$$\begin{aligned} \langle \zeta(t) \pi(t, [\zeta]) \rangle_\zeta &= -a \int D\zeta \pi(t, [\zeta]) \frac{\delta \mathcal{F}[\zeta]}{\delta \zeta(t)} = a \left\langle \frac{\delta \pi(t, [\zeta])}{\delta \zeta(t)} \right\rangle \\ &= -a \left\langle \frac{\partial \pi}{\partial N_\beta} \frac{\delta s_\beta(t, [\zeta])}{\delta \zeta_\alpha(t)} \right\rangle \mathbf{e}_\alpha, \end{aligned} \quad (\text{A9})$$

where \mathbf{e}_α with $\alpha = x, y, z$ are the ords of the Descarte coordinate system and summation over components α, β is implied. The variational derivative $\delta s_\beta / \delta \zeta_\alpha$ can be calculated,

if one writes down the formal solution of the stochastic Landau-Lifshitz equation (2.1),

$$s_{\beta}(t) = \gamma \int_{t_0}^t dt' e_{\beta\gamma\alpha} s_{\gamma}(t') [H_{\alpha}(t') + \zeta_{\alpha}(t')] + \dots, \quad (\text{A10})$$

where $e_{\beta\gamma\alpha}$ is the antisymmetric unit tensor. One can see that

$$\frac{\delta s_{\beta}(t, [\zeta])}{\delta \zeta_{\alpha}(t')} = \begin{cases} \gamma e_{\beta\gamma\alpha} s_{\gamma}(t'), & t' < t \\ 0, & t' > t. \end{cases} \quad (\text{A11})$$

For $t=t'$ the above calculation does not yield a definite value of $\delta s_{\beta}/\delta \zeta_{\alpha}$, but with the help of the usual arguments based on the regularization of δ functions⁴ the latter can be found to be $(1/2)\gamma e_{\alpha\beta\gamma} s_{\gamma}(t)$. Now Eq. (A9) can be finally written in the form

$$\langle \zeta(t) \pi(t, [\zeta]) \rangle_{\zeta} = \frac{\gamma a}{2} \left[\mathbf{N} \times \frac{\partial f}{\partial \mathbf{N}} \right]. \quad (\text{A12})$$

Adopting it in Eq. (A8), one comes to the Fokker-Planck equation (2.3).

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