Non-Abelian bosonization of the frustrated antiferromagnetic spin-1/2 chain

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We study the spin-1/2 chain with nearest-neighbor (κ_1) and next-nearest-neighbor (κ_2) interactions in the regime $\kappa_2 \gg \kappa_1$, which is equivalent to two chains with a ''zigzag'' interaction. In the continuum limit, this system is described in terms of two coupled level-1 Wess-Zumino-Witten field theories. We illustrate its equivalence with four off-critical Ising models (Majorana fermions). This description is used to investigate the opening of a gap as a function of κ_1 and the associated spontaneous breakdown of parity. We calculate the dynamic spin structure factor near the wave vectors accessible to the continuum limit. We comment on the nonzero string order parameter and show the presence of a hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry via a nonlocal transformation on the microscopic Hamiltonian. For a ferromagnetic interchain coupling, the model is conjectured to be critical, with different velocities for the spin-singlet and spin-triplet excitations. [S0163-1829(97)00401-3]

I. INTRODUCTION

One-dimensional quantum antiferromagnets have peculiar properties (exotic ground states, gapped excitations, etc.) which are not accessible to traditional methods like spin-wave theory or perturbation theory, but require the use of variational, numerical, or field-theoretical approaches. In particular, field-theoretical methods have been used successfully to predict the existence of an excitation gap in the spin-1 Heisenberg chain[1] and the scaling behavior of the spin-1/2 Heisenberg chain.[2,3] In the latter case, Witten's non-Abelian bosonization[4] was used to express the spin-1/2 Heisenberg chain as a Wess-Zumino-Witten (WZW) model perturbed by irrelevant interactions.

In this work, we apply non-Abelian bosonization to the spin-$\frac{1}{2}$ Heisenberg chain with nearest-neighbor (NN) coupling κ_1 and next-nearest-neighbor (NNN) coupling κ_2 in the regime $\kappa_2 \gg \kappa_1$. This system may also be viewed as two spin-$\frac{1}{2}$ chains coupled with a ''zigzag'' interaction κ_1 (Fig. 1). This latter representation makes sense physically, since such arrangements of atoms occur frequently (see, for instance, Ref. 5). The Hamiltonian of this system is

$$H = \kappa_1 \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \kappa_2 \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+2}, \qquad (1)$$

wherein the spins are indexed consecutively along the zigzag. This model has recently been studied by White and Affleck.[6] We shall extend the somewhat brief theoretical analysis of Ref. 6. The originality of the approach followed in this paper lies mostly in the description of two coupled spin-$\frac{1}{2}$ chains in terms of four Majorana fermions (or Ising models), obtained directly from the non-Abelian description without going through an intermediate Abelian bosonization of the theory. This approach makes the symmetry of the system more evident, and the representation of the staggered magnetization density by order-disorder operators may be proven directly; it also allows for a simple description of the magnetic excitations, in terms of which the Majorana fermions have a direct interpretation. The fermions and associated order and disorder fields are also very useful in calculating the spin structure factor. We also discuss the occurrence of a string (topological) order parameter and the associated $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

Let us summarize here the main results of this paper and explain its organization. In Sec. II we set up the description of two coupled spin-$\frac{1}{2}$ chains in terms of four Majorana fermions. This implies a quick review of the non-Abelian bosonization of a single spin chain (its representation as a level-1 WZW model). The main result of this section is the representation (8) and (9) of the fields of the level-1 WZW model in terms of four Majorana fermions and their associated order and disorder fields. This representation allows for a representation of the spin operator and interchain interaction with the help of Eq. (4). In Sec. III, we write the interchain interaction (and the marginal intrachain perturbation) in terms of the four fermions and discuss the renormalization of the couplings and velocities. A mass scale $m \sim \kappa_2 \exp{-(\kappa_2/\kappa_1)}$ appears dynamically and provides a mass for the fermions, accompanied by a spontaneous breaking of parity. Lorentz invariance is explicitly broken by the interchain interaction and one of the four fermions acquires a distinct mass and velocity (the velocity renormalization is calculated explicitly). In Sec. IV we set up the calculation of the spin structure factor $S(q,\omega)$ (the imaginary part of the dynamic spin susceptibility) near the four wave vectors available to the continuum limit $q = 0, \pi, \pm \pi/2$ [when considering wave vectors, we regard the system (1) as a single, frustrated chain and not as two coupled chains]. The single-spin excitations appear at a frequency $\omega \sim m$ near
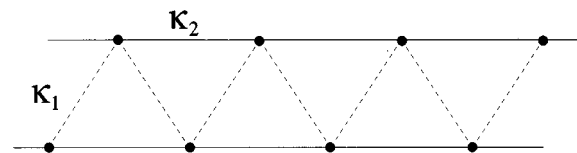

FIG. 1. Zigzag chain, with interchain coupling κ_1 and intrachain coupling κ_2, also equivalent to a single chain with NN coupling κ_1 and NNN coupling κ_2.

$q = \pm \pi/2$ while a two-particle continuum appears near $q = 0$ and $q = \pi$, like for the spin ladder (in calculating the spin structure factor, we treat the four fermions as free fields). In Sec. V we show that the nonlocal string order parameter of Ref. 7 is nonzero in the ground state, and how this breaks down a hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the system. We also perform an exact (i.e., discrete) nonlocal transformation Eq. (1) that reveals this $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. In Sec. VI we discuss the difference between this system and the usual spin ladder and address the case of ferromagnetic interchain coupling. A quick, largely notational review of WZW models and of the Ising model (Majorana fermion) is given in Appendixes A and B.

II. CONTINUUM DESCRIPTION OF TWO SPIN CHAINS

A. Non-Abelian bosonization

From the Bethe ansatz solution we know that the spin-$\frac{1}{2}$ Heisenberg chain Hamiltonian

$$H = \kappa \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} \tag{2}$$

is critical. It was also argued by Affleck[2] that this critical point is well described by a level-1 Wess-Zumino-Witten conformal field theory [unless said otherwise, we will mean by ''WZW model'' the SU(2) WZW model, at a specified level k] (cf. Appendix A). This equivalence is demonstrated by starting from the half-filled Hubbard model with hopping integral t and on-site repulsion U and taking the continuum limit. The charge degrees of freedom are then described by a Bose field φ which becomes massive for arbitrary small U, while the spin degrees of freedom are described by the level-1 WZW model. At $U = 0$ the characteristic velocity v of the WZW model is simply the Fermi velocity $v_F = |t| a_0$ (a_0 is the lattice spacing). For $U > 0$ the velocity v of the spin degrees of freedom is renormalized by U and differs from the velocity of the charge excitations (spin-charge separation). Moreover, the continuum limit of the Hubbard Lagrangian contains an additional term:

$$\mathcal{L}_1 = -\lambda J^a \bar{J}^a, \tag{3}$$

where J^a and $\bar{J}^a$ ($a = 1,2,3$) are the left and right components of the SU(2) currents of the level-1 WZW model and $\lambda \sim U/|t|$ (we will work in the imaginary-time Lagrangian formalism; $\mathcal{L}$ denotes the Lagrangian density). This perturbation is marginally irrelevant. Thus, at long distances, the spin degrees of freedom are exactly described by the level-1 WZW model.[2]

Additional perturbations to the Heisenberg Hamiltonian (2) may be expressed in terms of WZW fields by using the following continuum-limit expression for the spin operator $\mathbf{S}_i$ [Note that expression (4) cannot be substituted into Eq. (2) to find the WZW model Hamiltonian. This incorrect procedure yields the wrong sign for the marginal perturbation (3); Eq. (4) should be used only to evaluate correlation functions or express perturbations added to the half-filled Hubbard model]: (in this section, the system will be regarded as just one chain with NNN interactions and not as two chains with a zigzag interaction):

$$\frac{S^a(x)}{a_0} = \frac{1}{2\pi}[J^a(x) + \bar{J}^a(x)] + (-1)^{x/a_0} \Theta \, \mathrm{Tr}[g(x)\tau^a], \tag{4}$$

where τ^a are the usual Pauli matrices and $g(x)$ the fundamental WZW field [an SU(2) matrix]. The factor $(-1)^{x/a_0}$ alternates from one site to the next and Θ is a nonuniversal constant. The first two terms of Eq. (4) constitute the local magnetization and the last term is the local staggered magnetization.

Let us now turn our attention to the system (1). In the regime $\kappa_1 \ll \kappa_2$ and in the continuum limit, it may be regarded as two level-1 WZW models, plus some perturbations. Let J^a and $\bar{J}^a$ denote the SU(2) currents on one chain and J'^a and $\bar{J}'^a$ the corresponding currents on the other chain. The first perturbation is marginally irrelevant and given by two copies of Eq. (3):

$$\mathcal{L}_2 = -2\lambda_2(J^a\bar{J}^a + J'^a\bar{J}'^a), \tag{5}$$

where $\lambda_2 \sim U/|t|$. The second perturbation is the interchain interaction (κ_1). In the continuum limit and using Eq. (4), it can be shown without difficulty to be

$$\mathcal{L}_1 = 2\lambda_1(J^a + \bar{J}^a)(J'^a + \bar{J}'^a), \tag{6}$$

where $\lambda_1 \sim \kappa_1/|t| > 0$.

The relevance or irrelevance of a perturbation is determined, as a first approximation, from the scaling dimensions of the various fields at the WZW fixed point. In terms the conformal dimensions $(h, \bar{h})$ appearing in Eq. (A8), the scaling dimension is $\Delta = h + \bar{h}$ and the planar spin is $h - \bar{h}$. Since the conformal dimensions of J^a and $\bar{J}^a$ are, respectively, $(1,0)$ and $(0,1)$, a perturbation of the form $J^a\bar{J}^a$ [like Eq. (5)] is marginal, while a perturbation of the form J^aJ^a violates Lorentz (or rotation) invariance. In fact, it renormalizes the characteristic velocity of the theory (see below). The interaction (6) is marginal, except for a velocity renormalization.

WZW models, although they possess conformal invariance, are not easy to deal with, especially in what regards the calculation of correlation functions. In some cases (i.e., for some values of the level k) the WZW model is equivalent to a theory of free fields. Then the calculation of correlation functions becomes an almost trivial task and the overall analysis is much simplified, in particular the study of the vicinity of the fixed point. Such a free-field description is possible in the case of two coupled level-1 WZW models: Two such models are equivalent to one level-2 WZW model, plus one Ising model (or real fermion: see Appendix B). This equivalence was already used in Ref. 8 to describe the spin ladder with bond alternation. Moreover, the level-2 WZW model is equivalent to three Ising models.[9] We thus have three different ways of describing the system (1) in the continuum limit:

$$\mathrm{WZW}_{k=1} \otimes \mathrm{WZW}_{k=1}, \tag{7a}$$

$$\mathrm{WZW}_{k=2} \otimes \mathrm{Ising}, \tag{7b}$$

$$(\mathrm{Ising})^4. \tag{7c}$$

The representation (7b) may be useful from the point of view of symmetry since the interacting terms break down the SU(2)×SU(2) symmetry to SU(2). However, the representation (7c) is more practical for calculations since it is made entirely of free fields and its off-critical ($\kappa_1 \neq 0$) behavior may be characterized by ordinary fermion mass terms. All is not trivial, however, since the Ising model contains order and disorder fields in addition to a real fermion field, and these three fields cannot be expressed locally in terms of each other. Nevertheless, their correlation functions are known. An additional difficulty comes from the breaking of Lorentz invariance by the perturbation (6).

We identify operators in two different representations (7) by requiring their operator product expansions (OPE) to be compatible. The OPE for the WZW models and the Ising model are given in appendixes A and B. The correspondence of operators belonging to the pictures (7a) and (7c) is the object of the next subsection.

B. Description in terms of four Ising models

The $\mathrm{WZW}_{k=1}$ model cannot be simply represented in terms of two Majorana fermions, even if the central charge is the same in both cases ($c=1$). The reason is the nonexistence of a real, spin-$\frac{1}{2}$ representation of SU(2). However, two copies of $\mathrm{WZW}_{k=1}$ is equivalent to an SO(4) WZW model, and the latter group admits a representation in terms of four real fermions. A representation of the WZW currents J and J' in terms of four Majorana fermions $\psi_{1,2,3,0}$ follows immediately and its structure bears a strong resemblance with the chiral generators of the Lorentz group:

$$J^1 = \tfrac{1}{2} i(\psi_1 \psi_0 - \psi_2 \psi_3), \tag{8a}$$

$$J^2 = \tfrac{1}{2} i(\psi_2 \psi_0 - \psi_3 \psi_1), \tag{8b}$$

$$J^3 = \tfrac{1}{2} i(\psi_3 \psi_0 - \psi_1 \psi_2) \tag{8c}$$

(the corresponding expressions for J'^a are obtained by reversing the sign of ψ_0). Using the OPE's (B17) and Wick's theorem, it is a simple task to check that the OPE's (A12) are satisfied.

A representation of the matrix fields g and g' (the staggered magnetizations of the two chains) in terms of Ising fields is also needed in order to calculate correlation functions, and may be found in the following fashion. First, since g and g' have conformal dimensions $(\frac{1}{4}, \frac{1}{4})$, they must be products of four order and disorder fields, such as $\sigma_1 \sigma_2 \sigma_3 \sigma_0$, $\sigma_1 \mu_2 \sigma_3 \mu_0$, and so on (there are $2^4 = 16$ such products). Second, the action of each of the currents J^a, $\bar{J}^a$, J'^a, and $\bar{J}'^a$ may be calculated on these 16 products, using the OPE's (B3e)–(B3h). The result is a 16-dimensional matrix representation of the currents. According to the OPE (A11), the field $g_{1/2,1/2}$ is an eigenvector of J^3 with eigenvalue $-\frac{1}{2}$. Once such an eigenvector is found, one may apply on it the other components of the currents J and $\bar{J}$ and thus obtain the other components of g. Only one eigenvector allows a nontrivial solution (i.e., nonzero values of all the components of g). The same procedure is used for g', with the currents $J', \bar{J}'$. At last, one finds the following representation [we used the decomposition (A15)]:

$$g_0 = \sigma_1 \sigma_2 \sigma_3 \sigma_0 - \mu_1 \mu_2 \mu_3 \mu_0, \tag{9a}$$

$$g_1 = \mu_1 \sigma_2 \sigma_3 \mu_0 - \sigma_1 \mu_2 \mu_3 \sigma_0, \tag{9b}$$

$$g_2 = \sigma_1 \mu_2 \sigma_3 \mu_0 + \mu_1 \sigma_2 \mu_3 \sigma_0, \tag{9c}$$

$$g_3 = \sigma_1 \sigma_2 \mu_3 \mu_0 - \mu_1 \mu_2 \sigma_3 \sigma_0, \tag{9d}$$

$$g_0' = \sigma_1 \sigma_2 \sigma_3 \sigma_0 + \mu_1 \mu_2 \mu_3 \mu_0, \tag{9e}$$

$$g_1' = -\mu_1 \sigma_2 \sigma_3 \mu_0 - \sigma_1 \mu_2 \mu_3 \sigma_0, \tag{9f}$$

$$g_2' = -\sigma_1 \mu_2 \sigma_3 \mu_0 + \mu_1 \sigma_2 \mu_3 \sigma_0, \tag{9g}$$

$$g_3' = -\sigma_1 \sigma_2 \mu_3 \mu_0 - \mu_1 \mu_2 \sigma_3 \sigma_0. \tag{9h}$$

Note that the OPE's $J^a(z) g_i'(w,\bar{w}) \sim 0$ and $J'^a(z) g_i(w,\bar{w}) \sim 0$ are satisfied, as they should: The two chains are independent at this stage.

It is also possible to calculate the OPE of g with itself, with the help of Eqs. (B3a)—(B3d). This is a bit tricky, since one must remember to anticommute the different disorder fields. With the normalization chosen above and omitting terms that do not diverge as $z \to w$, the end result coincides with Eq. (A16) for g and g', plus the OPE $g_i(z,\bar{z}) g_j'(w,\bar{w}) \sim 0$. Thus, the representation (8) and (9) is a complete and faithful representation of two independent copies of the $\mathrm{WZW}_{k=1}$ model.

III. VELOCITY RENORMALIZATION AND RG ANALYSIS

We are now able to write down the Lagrangian associated to the continuum limit of (1) solely in terms of real fermions. The noninteracting part $\mathcal{L}_0$, equivalent to the two level-1 WZW models, is the free-fermion Lagrangian

$$\mathcal{L}_0 = \frac{1}{2\pi} \sum_{i=0}^{3} v_i (\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i), \tag{10}$$

where $v_0 = \cdots = v_3 = v$ is the velocity of spin excitation in isolated chains. The 2π factor in Eq. (10) is needed for consistency with the OPE (B1).

The interacting terms (5) and (6) may be expressed in terms of the following operators:

$$O_1 = \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 + \psi_1 \bar{\psi}_1 \psi_3 \bar{\psi}_3 + \psi_2 \bar{\psi}_2 \psi_3 \bar{\psi}_3, \tag{11a}$$

$$O_2 = \psi_0 \bar{\psi}_0 (\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2 + \psi_3 \bar{\psi}_3). \tag{11b}$$

The interaction (5) is simply

$$\mathcal{L}_2 = -\lambda_2 (O_1 + O_2). \tag{12}$$

The translation of Eq. (6) requires more care, however, since it implies regularized products of identical fermions. The following OPE must be used to extract the regular terms

$$\psi_i(z) \psi_j(w) = \delta_{ij} \left\{ \frac{1}{z-w} + 2(z-w) T^{(i)}(w) + \cdots \right\}, \tag{13a}$$

$$\overline{\psi}_i(\overline{z})\overline{\psi}_j(\overline{w}) = \delta_{ij}\left\{\frac{1}{\overline{z}-\overline{w}} + 2(\overline{z}-\overline{w})\overline{T}^{(i)}(\overline{w}) + \cdots\right\},$$
(13b)

where $(T^{(i)},\overline{T}^{(i)})$ is the energy-momentum tensor (B2) of the ith Ising model. The result is

$$\mathcal{L}_1 = \lambda_1(O_1 - O_2) + \lambda_1\left[-3(T^{(0)}+\overline{T}^{(0)}) + \sum_{i=1}^{3}(T^{(i)}+\overline{T}^{(i)})\right].$$
(14)

Apart from the energy-momentum terms, this interaction coincides with the marginal interchain interaction obtained by Shelton *et al.*[10] using Abelian bosonization. The effect of the energy-momentum tensor is simply to renormalize the speeds v_i of the fermions. Indeed, consider the Lagrangian (we have restored the velocity v in the interaction term)

$$\mathcal{L} = \frac{v}{2\pi}(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi}) - \tfrac{1}{2}v\lambda(\psi\partial\psi + \overline{\psi}\partial\overline{\psi})$$
(15)

[cf. Eq. (B2)], where λ is a dimensionless parameter. One may combine the energy-momentum tensor with the kinetic term and this amounts to the following renormalizations of the speed and fields:

$$v \rightarrow v\frac{1-\pi\lambda}{1+\pi\lambda}, \quad (\psi,\overline{\psi}) \rightarrow \frac{1}{\sqrt{1+\pi\lambda}}(\psi,\overline{\psi}).$$
(16)

In the case at hand, the velocity v_0 is renormalized differently from v_1, v_2, and v_3:

$$v_0 \rightarrow v_0\frac{1+3\pi\lambda_1}{1-3\pi\lambda_1}, \quad v_i \rightarrow v_i\frac{1-\pi\lambda_1}{1+\pi\lambda_1} \quad (i=1,2,3).$$
(17)

After the field renormalization, the interaction Lagrangian takes the following form:

$$\mathcal{L}_{int} = \frac{\lambda_1-\lambda_2}{(1+\pi\lambda_1)^2}O_1 - \frac{\lambda_1+\lambda_2}{(1+\pi\lambda_1)(1-3\pi\lambda_1)}O_2.$$
(18)

The O(4) symmetry of the fixed point, obvious in the Lagrangian (10), is violated by the interchain coupling λ_1, both in the interaction (18) and by the distinct renormalization of v_0.

The interaction terms (18) are marginal, since they have conformal dimensions (1,1). Their behavior under renormalization-group (RG) flow is characterized by their β functions. Instead of calculating the latter in the usual way (a one-loop Feynman diagram), let us follow Polyakov,[11] who has shown that the β functions of a critical system perturbed by marginal terms are related to the coefficients of the operator algebra. Explicitly, consider the perturbed action

$$S = S_0 + \sum_i \lambda_i \int d^2x \ \phi_i(x),$$
(19)

where S_0 is the fixed-point action and the $\phi_i(x)$ are marginal operators ($h=\overline{h}=1$). Let the operator algebra be of the form

$$\phi_i(x)\phi_j(y) \sim \frac{C_{ijk}\phi_k(y)}{|x-y|^2}.$$
(20)

Then the renormalization-group flow of the couplings λ_i, characterized by the β functions $\beta_{ijk}(L)$, is (in Euclidian space-time)

$$\frac{d\lambda_i}{d\ln L} = \beta_{ijk}\lambda_j\lambda_k = -2\pi C_{ijk}\lambda_j\lambda_k.$$
(21)

If we apply this method for the perturbation (18), we must use the OPE

$$(\psi_i\overline{\psi}_i)(z)(\psi_j\overline{\psi}_j)(w) \sim -\frac{\delta_{ij}}{|z-w|^2}$$
(22)

and realize that the eigenmodes of the RG flow are the operators

$$\mathcal{K}_+ = O_1 + O_2, \quad \mathcal{K}_- = O_1 - O_2,$$
(23)

which have the OPE

$$\mathcal{K}_\pm(z)\mathcal{K}_\pm(w) \sim \frac{6}{|z-w|^4} - \frac{4}{|z-w|^2}\mathcal{K}_\pm(w)$$
$$+ \frac{1}{|z-w|^2}\mathcal{O}(T+\overline{T}) + \cdots, \quad (24a)$$

$$\mathcal{K}_+(z)\mathcal{K}_-(w) \sim \frac{1}{|z-w|^2}\mathcal{O}(T+\overline{T}) + \cdots, \quad (24b)$$

where $\mathcal{O}(T+\overline{T})$ stands for terms containing the energy-momentum tensor which, although they have the right scaling dimensions, also have nonzero planar spin and do not contribute to the β functions. The terms in $|z-w|^{-4}$ only contribute to a shift of the vacuum energy and will be ignored. The interaction (18) may be expressed as a linear combination of the operators $\mathcal{K}_\pm$:

$$\mathcal{L}_{int} = \lambda_+\mathcal{K}_+ + \lambda_-\mathcal{K}_-,$$
(25)

with

$$\lambda_\pm = \frac{1}{2(1+\pi\lambda_1)}\left\{\mp\frac{\lambda_1+\lambda_2}{1-3\pi\lambda_1} + \frac{\lambda_1-\lambda_2}{1+\pi\lambda_1}\right\}.$$
(26)

The RG equations obtained from (21) and (24) are

$$\frac{d\lambda_\pm}{d\ln L} = 8\pi\lambda_\pm^2.$$
(27)

If $\kappa_1 \ll \kappa_2$, the starting point is $\lambda_1 \ll 1$ and $\lambda_2 \sim 1$; thus λ_+ is negative and λ_- positive and small. Under this flow, λ_+ renormalizes to zero (it is marginally irrelevant, like for an isolated spin chain) and λ_- is marginally relevant. By following the RG flow until $L=\xi$ (the correlation length), we conclude that $\xi \sim \exp(1/\lambda_-)$: A dynamical length scale ξ has set in. If we concentrate on the Heisenberg model (1) without referring to the underlying Hubbard model, the characteristic spin velocity should be $v \sim \kappa_2 a_0$ and the dimensionless coupling constant λ_- should be κ_1/κ_2. Thus the dynamical length scale is $\xi \sim \exp-\kappa_2/\kappa_1$. This conclusion was reached in Ref. 6 in the (WZW$_{k=1}$)^2 representation. We shall conclude from this that the fermions have acquired a mass $m_i \sim v_i\xi^{-1}$. The first three fermions ($\psi_{1,2,3}$) have a common mass m, while ψ_0 has a slightly higher mass m_0 (in absolute

value), since $v_0 > v_1 = v_2 = v_3$. An additional velocity renormalization will take place during the RG flow, but cannot be calculated by the above method. In a diagrammatic technique, it would show up at two loops, in a self-energy correction. The overall velocity renormalization is important, since without it the long-distance theory would have an O(4) symmetry and all four fermions would have the same mass (up to a sign).

In order to clarify the significance of these dynamically generated mass scales, let us consider the following model:

$$\mathcal{L} = \frac{1}{2\pi} \sum_{i=1}^{N} v \ (\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i) + \frac{1}{2}\lambda \sum_{i \neq j} \psi_i \bar{\psi}_i \psi_j \bar{\psi}_j. \quad (28)$$

This model would be equivalent to the Lagrangians (10) and (18) if all velocities were equal, if $\lambda_+ = 0$, and if $N = 4$, with $\psi_4 = \psi_0$ and $\bar{\psi}_4 = -\bar{\psi}_0$ (Kramers-Wannier duality). The model (28) has O(N) symmetry and a mass gap arises nonperturbatively in the spectrum if $\lambda > 0$. To see this in a mean-field approach, we assume that $\langle \psi_i \bar{\psi}_i \rangle = i\varepsilon \neq 0$ (no sum over i) and determine ε self-consistently. Let us make the substitution

$$\psi_i \bar{\psi}_i \rightarrow i\varepsilon + \psi_i \bar{\psi}_i \quad (29)$$

in the Lagrangian, neglecting terms quartic in ψ, which is equivalent to a large-N approximation. We find the Lagrangian of N massive fermions:

$$\mathcal{L} = \sum_{i=1}^{N} \left\{ \frac{v}{2\pi} (\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i) + i\lambda(N-1)\varepsilon \psi_i \bar{\psi}_i \right\}, \quad (30)$$

where the mass is $m = 2\pi\lambda(N-1)\varepsilon$. This mass may be determined self-consistently, using the following expression for the Green's function of real fermions:[12]

$$\langle \psi(0)\psi(\mathbf{x}) \rangle = \partial \int \frac{d^2k}{\pi} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}, \quad (31a)$$

$$\langle \bar{\psi}(0)\bar{\psi}(\mathbf{x}) \rangle = \bar{\partial} \int \frac{d^2k}{\pi} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}, \quad (31b)$$

$$\langle \psi(0)\bar{\psi}(\mathbf{x}) \rangle = -i\frac{1}{2} m \int \frac{d^2k}{\pi} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}, \quad (31c)$$

where $\mathbf{x}$ and $\mathbf{k}$ stand, respectively, for $(v\tau, x)$ and $(i\omega/v, k)$. The mass m is determined from the self-consistency condition

$$\frac{m}{2\pi\lambda(N-1)} = m \int \frac{d^2k}{\pi} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}, \quad (32)$$

whose solution is, besides $m = 0$,

$$m = \pm v \Lambda \exp - \frac{1}{2\pi\lambda(N-1)}, \quad (33)$$

where Λ is a momentum cutoff. This solution exists only for positive λ.

Returning to the Lagrangians (10) and (18) with λ_+ renormalized to zero, all velocities equal, and $\lambda = \lambda_-$, this implies a mass gap $m \sim v\Lambda\exp(-1/6\pi\lambda_-)$ or

$m \sim \kappa_2 \exp(-\kappa_2/6\pi\kappa_1)$ if the characteristic velocity (of order κ_2) is restored. Since $(\psi_4, \bar{\psi}_4) = (\psi_0, -\bar{\psi}_0)$, the mass m_0 of the fourth Ising model is equal to $-m$, if $v_0 = v_i$. Since $v_0 > v_i$, we conclude that $-m_0 > m$. Of course, this conclusion is obtained in the large-N approximation, whereas $N = 3$. Therefore it is only approximately valid: One cannot reliably predict the value of the masses. Moreover, the massive fermions still interact through a renormalized ψ^4 term (we will neglect this effect when calculating the spin structure factor in the next section).

A short remark about the sign of the mass: From the Ising model viewpoint, this sign simply indicates on which side of the transition we stand. By convention, $m > 0$ in the disordered phase ($\langle \mu \rangle \neq 0$) and $m < 0$ in the ordered phase ($\langle \sigma \rangle \neq 0$). Of course, it is the absolute value $|m|$ which occurs in the dispersion relation of the fermions.

The appearance of fermion mass terms breaks the diagonal $\mathbb{Z}_2$ symmetry $(\psi_i, \bar{\psi}_i) \rightarrow (\psi_i, -\bar{\psi}_i)$ $(i = 0\text{–}4)$ of the full Lagrangians (10) and (18). Thus, the ground state must be doubly degenerate and the condensate $\langle \psi_i \bar{\psi}_i \rangle \neq 0$ picks one of these ground states, the theory of massive fermions describing excitations above that ground state only. This is consistent with the Lieb-Schultz-Mattis theorem, which states that a half-integer spin chain with local interactions and no explicit parity breaking has either no gap or else has degenerate ground states.

IV. SPIN STRUCTURE FACTOR

In a recent paper, Rao and Sen[13] have argued that dimerized spin chains with second nearest-neighbor interactions admit possibly three different phases [here we use the word ''phase'' to distinguish regions where the spin structure factor $S(q)$ is not peaked at the same value of q]. They name the three phases as follows (in this section, the system will be regarded as just one chain with NNN interactions and not as two chains with a zigzag interaction): a Néel phase [$S(q)$ is peaked at π], a spiral phase [$S(q)$ is peaked at an intermediate momentum between π and $\pi/2$], and a collinear phase [$S(q)$ is peaked at $\pi/2$]. In view of the numerical results from Chitra et al.,[14] the collinear phase should not be stable for the spin-$\frac{1}{2}$ chain. At first sight, there are two paths that $S(q)$ may follow to go from the Néel phase to the collinear phase. The first possibility is for the peak of $S(q)$ to move continuously from π to $\pi/2$, thus going through a spiral phase. The second possibility is for the peak of $S(q)$ at π to progressively decrease in amplitude while a second peak at $\pi/2$ progressively appears; the system might then go through a dimerized state. In view of this, the question of the existence or not of a spiral phase for the frustrated spin-$\frac{1}{2}$ chain arises. To answer this question, we need to know how the spin structure factor evolves as a function of the ratio κ_1/κ_2. Unfortunately, in the present continuum approach we can only calculate the spin-spin correlation function near $q = 0$, $q = \pi$, and $q = \pm\pi/2$. As seen below, this calculation is also interesting from the point of view of symmetry and allows us to relate the elementary spin excitations to the fermions ψ_i.

The main conclusion of Sec. II is that the system (1) may be described in the continuum limit by four real fermions:

three with mass $m > 0$ and velocity v, and one with mass $m_0 < -m$ and velocity $v_0 > v$. At this point we will neglect the residual interaction between these fermions and treat them as free in the calculation of correlation functions. The spin operator $\mathbf{S}_i$ is represented in terms of these fermions through the relations (4), (8) and (9). The z component of the spin density has the following expression near the wave vectors accessible to the continuum limit:

$$S^z_{q \sim 0} \propto \psi_1 \psi_2 + \overline{\psi}_1 \overline{\psi}_2, \tag{34a}$$

$$S^z_{q \sim \pi} \propto \psi_0 \psi_3 + \overline{\psi}_0 \overline{\psi}_3, \tag{34b}$$

$$S^z_{q \sim \pi/2} \propto \mu_1 \mu_2 \sigma_3 \sigma_0, \tag{34c}$$

$$S^z_{q \sim -\pi/2} \propto \sigma_1 \sigma_2 \mu_3 \mu_0. \tag{34d}$$

Thus, the spin-spin correlation function near $q = 0$ takes the form

$$\chi^{(0)}(x,\tau) \propto \langle (\psi_1 \psi_2 + \overline{\psi}_1 \overline{\psi}_2)(x,\tau)(\psi_1 \psi_2 + \overline{\psi}_1 \overline{\psi}_2)(0,0) \rangle, \tag{35}$$

while near $q = \pi$, it takes the form

$$\chi^{(\pi)}(x,\tau) \propto \langle (\psi_3 \psi_0 + \overline{\psi}_3 \overline{\psi}_0)(x,\tau)(\psi_3 \psi_0 + \overline{\psi}_3 \overline{\psi}_0)(0,0) \rangle. \tag{36}$$

In the first case (q near 0), the two fermions have the same mass and velocity, while in the second case (q near π) they have different masses and velocities. Consider the case of two fermions with different masses (m and m') but identical velocities (for simplicity). The imaginary part of the Fourier transform of the spin-spin correlation function, i.e., the imaginary part of the dynamic susceptibility or the spin structure factor $S(q,\omega)$, may be calculated from the propagators (31):

$$S(q,\omega) \propto \frac{1}{u} \left[\frac{q^2}{s^2}(m+m')^2 + \frac{\omega^2}{s^2}(m-m')^2 - \frac{\omega^2 + q^2}{s^4} \right.$$

$$\left. \times (m+m')^2(m-m')^2 \right], \tag{37}$$

where u and s are defined by

$$u^2 = (s^2 + m^2 - m'^2)^2 - 4m^2 s^2, \tag{38}$$

$$s^2 = \omega^2 - v^2 q^2 \tag{39}$$

(in this expression we have returned to real frequencies). In the case $m = m'$ this result becomes the spin structure factor near $q = 0$:

$$S^{(0)}(q,\omega) \propto \frac{m^2 q^2}{s^3 \sqrt{s^2 - 4m^2}}. \tag{40}$$

Neglecting velocity renormalization ($v_0 = v_i$), the model is O(4) symmetric at long distances and $m_0 = -m$. Then the dynamical susceptibility near $q = \pi$ would be given by the more general expression (37) with $m' = -m$. The expression of $S^{(\pi)}(q,\omega)$ appropriate for the more realistic case $v_0 \neq v_i$ can be obtained in closed form but is too cumbersome to display here.

The fact that $S^{(0)}(0,\omega) = 0$ reflects the conservation of the total magnetization. This is not the case for the total magnetization of each chain, since $m \neq m'$, unless the two chains are decoupled ($m = m' = 0$). Thus, $S^{(\pi)}(0,\omega) \neq 0$.

According to Eq. (34), the magnetic susceptibility near $q = \pm \pi/2$ is a product of four two-point functions of the Ising model, involving order and disorder fields. For instance, near $q = \pi/2$,

$$\chi^{(\pi/2)}(x,\tau) \propto \langle (\mu_1 \mu_2 \sigma_3 \sigma_0)(0,0)(\mu_1 \mu_2 \sigma_3 \sigma_0)(x,\tau) \rangle$$

$$= \widetilde{C}^2(mr) C(mr) C(m_0 r), \tag{41}$$

where $C(R)$ and $\widetilde{C}(R)$ are, respectively, the two-point functions of the order field and disorder field, as a function of the reduced distance $R = \sqrt{x^2 + v^2 \tau^2}/\xi = mr$. These functions are known[15] and their leading asymptotic behavior is, in the disordered phase of the Ising model,

$$C(R) = \frac{A}{\pi} K_0(R) + O(e^{-3R}),$$

$$\widetilde{C}(R) = A \left\{ 1 + \frac{1}{8\pi R^2} e^{-2R} + O(e^{-4R}) \right\}, \tag{42}$$

where A is some constant and $K_{0,1}$ are the modified Bessel functions. If the argument of C is negative (i.e., for a negative mass), we perform a Kramers-Wannier duality transformation and identify $C(-R)$ with $\widetilde{C}(R)$. Thus, the leading asymptotic behavior of the susceptibility near $q = \pi/2$ is

$$\chi^{(\pi/2)}(x,\tau) \propto \widetilde{C}^2(mr) \widetilde{C}(|m_0|r) C(mr) \propto K_0(mr) + O(e^{-2mr}). \tag{43}$$

Notice that $K_0(R)$ is the real-space propagator of a free boson of mass m. Thus, its Fourier transform is $\sim (\mathbf{k}^2 + m^2)^{-1}$. Going back to real frequencies, the imaginary part of the susceptibility has a pole at $\omega = \sqrt{(vk)^2 + m^2}$, plus an incoherent part starting at $\omega = 2m$:

$$S^{(\pi/2)}(q,\omega) \propto \frac{m}{|\omega|} \delta(\omega - \sqrt{(vk)^2 + m^2}) + \text{incoherent part.} \tag{44}$$

The magnetic susceptibility near $-\pi/2$ is obtained by Kramers-Wannier duality:

$$\chi^{(-\pi/2)}(x,\tau) \propto \widetilde{C}(mr) C^2(mr) C(|m_0|r) \propto K_0(mr)^2 K_0(|m_0|r)$$

$$+ O(e^{-2mr}). \tag{45}$$

The associated spin structure factor has no single-particle peak, but instead a continuum starting at $\omega = 2m + |m_0|$. Thus, the single-particle magnetic excitations live around $k = \pi/2$, whereas the excitations near $k = 0, \pi$ have a two-particle behavior and those near $k = -\pi/2$ have a three-particle behavior.

The residual interactions between the four Majorana fermions should not greatly affect the asymptotics of Eq. (43) since it is governed by a single-particle function. However, these interactions might significantly modify the continuum appearing in Eq. (45).

The above analysis assumed that the mass m was positive, which amounts to choosing one of the degenerate ground states, characterized by a short-range order around $k = \pi/2$. If the other ground state were chosen, the mass m would be negative and the above analysis could be repeated by interchanging the roles of $k = \pi/2$ and $k = -\pi/2$. Thus, the spontaneous breakdown of parity is reflected in the nonequivalence of $S(k, \omega)$ and $S(-k, \omega)$. However, this would be unobservable in practice because of domain effects.

V. STRING ORDER PARAMETER AND $\mathbb{Z}_2 \times \mathbb{Z}_2$ SYMMETRY

Kohmoto and Tasaki have shown[16] that, for a spin-$\frac{1}{2}$ chain with dimerization, a string order parameter may be defined as den Nijs and Rommelse have previously done for the spin-1 chain.[7] A nonlocal unitary transformation is introduced to show that the nonzero value of this string order parameter is related to the breakdown of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. More recently, Shelton et al.[10] showed that the string order parameter

$$O^z(n,m) = \exp\left\{ i\pi \sum_{j=n}^{m} (S_j^z + \widetilde{S}_j^z) \right\} \qquad (46)$$

becomes, in the continuum limit,

$$\lim_{|x-y| \to \infty} \langle O^z(x,y) \rangle \sim \langle \sigma_1 \rangle^2 \langle \sigma_2 \rangle^2 - \langle \mu_1 \rangle^2 \langle \mu_2 \rangle^2. \qquad (47)$$

It was also argued in Ref. 10 that the nonzero value of this order parameter is related to the breakdown of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry which, in the continuum limit, is given by the invariance under sign inversion of both chiral components of each Majorana spinor: $\psi_a \to -\psi_a$ and $\overline{\psi}_a \to -\overline{\psi}_a$ $(a=1,2)$. This must be accompanied by an inversion of both order and disorder fields: $\sigma_a \to -\sigma_a$ and $\mu_a \to -\mu_a$.

Here, we expect a nonzero value of the string order parameter for two reasons. First, since the SU(2) symmetry cannot be spontaneously broken—according to the Mermin-Wagner theorem—the mass of the first Ising model (m_1) must be the same as that of the second Ising model (m_2), in order to keep the symmetry under the exchange of the labels 1, 2, and 3. It naturally implies that they must have the same sign. So, if $\langle \sigma_1 \rangle \neq 0$ ($m_1 > 0$), then $\langle \sigma_2 \rangle \neq 0$ ($m_2 > 0$). Similarly, if $\mu_1 \neq 0$ ($m_1 < 0$), then $\langle \mu_2 \rangle \neq 0$ ($m_2 < 0$). Second, since a gap open by the introduction of the interchain coupling, the masses m_1 and m_2 must be nonzero.

We can also reveal the presence of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry without going to the continuum limit, i.e., directly from the Hamiltonian (1). The unitary transformation U introduced in Ref. 16 consists of many transformations applied in succession. Explicitly, we have

$$U = (D^\tau)^{-1} R D G, \qquad (48)$$

where G performs a rotation of π about the y axis on some of the spins:

$$G = \overset{L/2}{\underset{j=1}{\otimes}} \exp\left[\frac{i\pi}{2} (S_{4j-1}^y + S_{4j}^y) \right]. \qquad (49)$$

D is a duality transformation (see Appendix A of Ref. 16) which introduces intersite spins. It is followed by a translation

$$R : r \to \tfrac{1}{2}(r + \tfrac{1}{2}). \qquad (50)$$

The spin on integer sites will be noted σ and those on the half-odd integer sites will be noted τ. The final operation is to make an inverse duality transformation for the τ spins. We refer the reader to the work of Kohmoto and Tasaki[16] for a full description of this unitary transformation. If we apply this transformation to the Hamiltonian (1), we find

$$UHU^{-1} = \sum_{j=1}^{N} \left\{ \kappa_1 [\tau_j^x + \sigma_j^x - \sigma_j^x \tau_j^x - \sigma_j^z \sigma_{j+1}^z - \tau_j^z \tau_{j+1}^z - \sigma_j^z \tau_j^z \tau_{j+1}^z \sigma_{j+1}^z] + \kappa_2 [\sigma_j^z \tau_j^x \sigma_{j+1}^z + \sigma_j^y \tau_j^y \tau_{j+1}^z \sigma_{j+1}^z + \tau_j^z \sigma_j^x \tau_{j+1}^z \right.$$
$$\left. + \sigma_j^z \tau_{j+1}^x \sigma_{j+1}^z + i\sigma_j^z \tau_j^y \tau_{j+1}^y \sigma_{j+1}^y + \tau_j^z \sigma_{j+1}^x \tau_{j+1}^z] \right\}. \qquad (51)$$

The σ's and τ's are sets of Pauli matrices. The new Hamiltonian $\widetilde{H} = UHU^{-1}$ is clearly invariant under a rotation of π about the x axis applied to the σ spins alone or the τ spins alone. A fourfold degeneracy of the ground state of H in the thermodynamic limit does not follow from this broken symmetry since this is not a local symmetry.[17]

VI. DISCUSSION

The crucial difference between the Hamiltonian (1) and that of the more familiar spin ladder is the occurrence, in the latter, of an interaction term of the form

$$\mathcal{L}_{\text{ladder}} = \frac{\eta}{2\pi} \text{Tr}(g \tau^a) \text{Tr}(g' \tau^a) \qquad (52)$$

(η is some constant proportional to the interchain coupling). Since the matrix field g has conformal dimensions $(\frac{1}{4}, \frac{1}{4})$, the above perturbation has scaling dimension 1: It is relevant. If the representation (9) and the OPE's (B3a)—(B3d) are used to express this interaction in terms of fermions, one finds

$$\mathcal{L}_{\text{ladder}} = i\frac{\eta}{2\pi}(\psi_1 \overline{\psi}_1 + \psi_2 \overline{\psi}_2 + \psi_3 \overline{\psi}_3 - 3\psi_0 \overline{\psi}_0). \qquad (53)$$

This coincides with the conclusions of Ref. 10, obtained by Abelian bosonization. The mass terms now appear explicitly, with a triplet of mass η and a singlet of mass -3η. The interchain coupling explicitly breaks the invariance under parity that is spontaneously broken in the zigzag case. If the

two rungs of the zigzag had different couplings (κ_1 and κ_1'), an interaction like Eq. (52) would be generated and the gap would have a linear dependence on the interchain coupling. Of course, the marginal interaction (18) is always present and provides an additional renormalization of the masses. As $\kappa_1' \to \kappa_1$, the dependence of the gap on the interchain coupling should become more and more exponential because of this renormalization.

We were concerned in this work with the regime $\kappa_1 \ll \kappa_2$ and the conclusions are nominally valid only in this regime, although we expect them to be qualitatively correct even for $\kappa_1 \sim \kappa_2$. However, in the opposite regime ($\kappa_1 \gg \kappa_2$) the system should be treated as a single chain and we should perturb around a single $\text{WZW}_{k=1}$ model. This is explained in Ref. 3. The conclusion is that the perturbation is marginally irrelevant if the ratio κ_2 / κ_1 is smaller than some critical value, and leads to an exponential gap above that critical value. In that regime the ground state is spontaneously dimerized (spontaneous breaking of parity). This conclusion is also valid in the regime $\kappa_1 \ll \kappa_2$. Indeed, the order parameter for dimerization,

$$d = \langle \mathbf{S}_{2i} \cdot \mathbf{S}_{2i-1} - \mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} \rangle, \tag{54}$$

coincides, in the continuum limit, with the ladder perturbation (52), up to terms that have a vanishing expectation value. Translated in terms of the bare interaction couplings $\lambda_\pm$ of Eq. (25) and of the masses m and m_0, the spontaneous dimerization becomes

$$d \propto m_0 [m(\lambda_- - \lambda_+) + m_0(\lambda_- + \lambda_+)]. \tag{55}$$

This is generically nonzero.

So far we have supposed that κ_1 is positive, corresponding to an antiferromagnetic interchain coupling. The ferromagnetic case may be treated just as well. In that case, both interaction constants $\lambda_\pm$ of Eq. (25) are negative and thus renormalize to zero: The model is equivalent to a theory of four free Majorana fermions, with different velocities. From Eq. (17) with negative λ_1, we expect the velocity v_0 of ψ_0 to be smaller than the velocity v of $\psi_{1,2,3}$. Thus, we conjecture that the ferromagnetic model is critical, albeit with two sectors having different velocities: a triplet sector equivalent to the $\text{WZW}_{k=2}$ theory and a singlet sector with a smaller velocity. This is not the same as saying that the two chains *decouple* at long distances, since in that case the structure of excitations would be different. This conjecture might be tested by exact diagonalizations on finite systems and some information on the velocity renormalization might be extracted this way.

ACKNOWLEDGMENTS

APPENDIX A: WZW MODELS

A systematic review of WZW models cannot take place in a regular paper. Here we simply recall basic concepts and a few definitions, in order to fix the notation and the normalization used in this work. We follow in this respect Ref. 18.

Wess-Zumino-Witten (WZW) models are defined in terms of a matrix-valued field g belonging to a unitary representation of SU(2) (more generally, of a Lie group $\mathfrak{g}$) with the action,[4,19]

$$S = \frac{k}{16\pi} \int d^2x \ \text{Tr}'(\partial^\mu g^{-1} \partial_\mu g)$$
$$- \frac{ik}{24\pi} \int_B d^3y \ \varepsilon^{\mu\nu\rho} \text{Tr}'(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g), \tag{A1}$$

where the trace Tr' is proportional to the usual trace operation:

$$\text{Tr}' = \frac{1}{x_s}\text{Tr}, \quad x_s = \tfrac{1}{3} s(s+1)(2s+1) \tag{A2}$$

(s is the spin of the representation). k is a positive integer called the *level* of the WZW model. The first term of Eq. (A1) is the usual nonlinear σ model. The second term is topological and is integrated on a three-dimensional manifold B of which two-dimensional space-time is the boundary. Its value is independent of the precise form of B (modulo 2π), provided k is an integer.

The fundamental property of the WZW model—enforced by the relative normalization of the two terms of the action (A1)—is its full conformal symmetry. For this reason, it is best described in the language of conformal field theory, with holomorphic (or left) and antiholomorphic (or right) coordinates

$$z = -i(x - vt) = v\tau - ix, \quad \bar{z} = i(x + vt) = v\tau + ix, \tag{A3}$$

where $\tau = it$ is the Euclidian time and v is the characteristic velocity of the model, implicit in the covariant notation of Eq. (A1). The left and right derivatives are commonly used:

$$\partial \equiv \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{1}{v}\frac{\partial}{\partial \tau}\right) \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{1}{v}\frac{\partial}{\partial \tau}\right). \tag{A4}$$

The WZW model has SU(2) symmetry and this entails the existence of a conserved current J_μ, expressed here in its left (z) and right ($\bar{z}$) components:

$$J \equiv J_z = \partial g g^{-1}, \quad \bar{J} \equiv J_{\bar{z}} = g^{-1}\bar{\partial} g. \tag{A5}$$

Closely related to its conformal symmetry is the separate conservation of the left and right currents: $\partial \bar{J} = 0$ and $\bar{\partial} J = 0$ [the SU(2) symmetry is enlarged to a chiral symmetry $\text{SU}(2)_L \otimes \text{SU}(2)_R$]. Hence $J(z)$ depends only on z and $\bar{J}(\bar{z})$ on the $\bar{z}$. These matrix currents may be decomposed along a basis of spin-s generators. For spin $\frac{1}{2}$, we choose

$$J(z) = J^a(z)\tau^a, \quad \bar{J}(\bar{z}) = \bar{J}^a(\bar{z})\tau^a, \tag{A6}$$

where the τ^a are the usual Pauli matrices.

In practice, the action (A1) is not useful for practical calculations. The traceless, symmetric energy-momentum ten-

sor, which generates local conformal transformations (in particular space-time translations), is more useful. Its two nonzero components are given by the so-called Sugawara form

$$T(z) = \frac{1}{(k+2)}(J^a J^a), \quad \overline{T}(\overline{z}) = \frac{1}{(k+2)}(\overline{J}^a \overline{J}^a). \quad (A7)$$

The notation $(\cdots)$ above stands for a normal ordering (regularized product). The dynamics of the theory is determined by the short-distance product (operator-product expansion or OPE) of the various fields. The OPE of $T(z)$ with a local scaling (or *primary*) field $\phi(w, \overline{w})$ reflects the conformal (or scaling) properties of that field:

$$T(z)\phi(w) \sim \frac{h\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial_w \phi(w,\overline{w})}{z-w}, \quad (A8)$$

where h is the conformal dimension of the field ϕ and the symbol $\sim$ means an equality modulo terms which are regular as $z \rightarrow w$. A similar expression holds for $\overline{T}$ and the sum $h + \overline{h}$ is the usual scaling dimension. The OPE of T with itself is slightly different:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \quad (A9)$$

The constant c in the most singular term is the *central charge* of the conformal theory and measures the number of degrees of freedom of the theory; its value in the SU(2) WZW model is

$$c = \frac{3k}{k+2}. \quad (A10)$$

The OPE of the currents J and $\overline{J}$ with a local matrix field ϕ fields reflects its transformation properties under the action of SU(2):

$$J^a(z)g(w,\overline{w}) \sim -\frac{1}{2}\frac{\tau^a g(w,\overline{w})}{z-w},$$

$$\overline{J}^a(\overline{z})g(w,\overline{w}) \sim \frac{1}{2}\frac{g(w,\overline{w})\tau^a}{z-w}. \quad (A11)$$

The OPE of the current with itself constitutes the so-called *current algebra*:

$$J^a(z)J^b(w) \sim \frac{(k/2)\delta_{ab}}{(z-w)^2} + i\varepsilon_{abc}\frac{J^c(w)}{z-w}, \quad (A12a)$$

$$\overline{J}^a(\overline{z})\overline{J}^b(\overline{w}) \sim \frac{(k/2)\delta_{ab}}{(\overline{z}-\overline{w})^2} + i\varepsilon_{abc}\frac{\overline{J}^c(\overline{w})}{\overline{z}-\overline{w}}, \quad (A12b)$$

$$J^a(z)\overline{J}^b(\overline{w}) \sim 0. \quad (A12c)$$

The WZW at level k contains several matrix-valued scaling fields, one for each value of the spin s up to (and including) $s = k/2$. The conformal dimensions h and $\overline{h}$ of the spin-s field are

$$h_s = \overline{h}_s = \frac{s(s+1)}{k+2}. \quad (A13)$$

The OPE of the various matrix fields is governed by the rule of addition of angular momenta and by the constraint that no field of spin $s > k/2$ occurs in the operator algebra. These OPE's were calculated in Ref. 9. We shall only be concerned with the simplest case $(k=1)$.

The level-1 WZW model has central charge $c = 1$ and contains a single matrix field $g_{n\overline{n}}$ $(n, \overline{n} = \pm \frac{1}{2})$ of conformal dimensions $(\frac{1}{4}, \frac{1}{4})$. With the proper normalization, its OPE is

$$g_{n\overline{n}}(z,\overline{z})g_{m\overline{m}}(w,\overline{w}) \sim \frac{1}{|z-w|}\varepsilon_{nm}\varepsilon_{\overline{n}\overline{m}}, \quad (A14)$$

where ε_{nm} is the antisymmetric symbol. We may use the following decomposition:

$$g(z,\overline{z}) = \frac{1}{2}\sum_{i=0}^{3} g_a(z,\overline{z})\tau^a, \quad g_a = \text{Tr}(\tau^a g). \quad (A15)$$

The OPE of g with itself is then

$$g_a(z,\overline{z})g_b(w,\overline{w}) \sim \frac{2\delta_{ab}}{|z-w|}(-1)^{\delta_{a0}+1}. \quad (A16)$$

APPENDIX B: THE ISING MODEL

It is well known that the two-dimensional Ising model is equivalent to a real (Majorana) fermion in $1+1$ dimensions. The critical point of the Ising model corresponds to the massless point of the fermion theory and constitutes one of the simplest conformal field theories, of central charge $c = \frac{1}{2}$. This theory contains a two-component fermion $(\psi(z), \overline{\psi}(\overline{z}))$. The holomorphic field ψ has conformal dimensions $(\frac{1}{2}, 0)$ while its antiholomorphic counterpart $\overline{\psi}$ has conformal dimensions $(0, \frac{1}{2})$. Their OPE is

$$\psi(z)\psi(w) \sim \frac{1}{z-w}, \quad (B1a)$$

$$\overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) \sim \frac{1}{\overline{z}-\overline{w}}, \quad (B1b)$$

$$\psi(z)\overline{\psi}(\overline{w}) \sim 0. \quad (B1c)$$

The product $\varepsilon = i\overline{\psi}\psi$ has conformal dimensions $(\frac{1}{2}, \frac{1}{2})$ and is called the *energy* field; it is the mass term that takes the model away from its critical point. The energy-momentum tensor of the fermion theory is

$$T(z) = -\frac{1}{2}\psi\partial\psi, \quad \overline{T}(\overline{z}) = -\frac{1}{2}\overline{\psi}\partial\overline{\psi}. \quad (B2)$$

The critical Ising model also contains an *order* field $\sigma(z,\overline{z})$ which is the continuum limit of the Ising spin. This field has conformal dimensions $(\frac{1}{16}, \frac{1}{16})$ and is not locally related to the fermion field. Indeed, in the transfer-matrix description of the two-dimensional Ising model, the fermion field is introduced by a (nonlocal) Wigner-Jordan transformation. The Kramers-Wannier duality transformation of the Ising model

maps the order field σ into a disorder field μ which has essentially the same properties, except that $\langle\sigma\rangle\neq0$, $\langle\mu\rangle=0$ in the ordered phase and $\langle\sigma\rangle=0$, $\langle\mu\rangle\neq0$ in the disordered phase. At the (massless) critical point, both fields have a vanishing expectation value. All three fields (ψ,σ,μ) are mutually nonlocal, which is reflected in their OPE by the existence of branch cuts. These OPE's are given below:

$$\sigma(z,\bar{z})\sigma(w,\bar{w})\sim\frac{1}{|z-w|^{1/4}}+\frac{1}{2}|z-w|^{3/4}\varepsilon(w,\bar{w}),\qquad(\mathrm{B3a})$$

$$\mu(z,\bar{z})\mu(w,\bar{w})\sim\frac{1}{|z-w|^{1/4}}-\frac{1}{2}|z-w|^{3/4}\varepsilon(w,\bar{w}),\qquad(\mathrm{B3b})$$

$$\sigma(z,\bar{z})\mu(w,\bar{w})\sim\frac{\gamma(z-w)^{1/2}\psi(w)+\beta\gamma^*(\bar{z}-\bar{w})^{1/2}\bar{\psi}(\bar{w})}{\sqrt{2}|z-w|^{1/4}},\qquad(\mathrm{B3c})$$

$$\mu(z,\bar{z})\sigma(w,\bar{w})\sim\frac{\gamma^*(z-w)^{1/2}\psi(w)+\beta\gamma(\bar{z}-\bar{w})^{1/2}\bar{\psi}(\bar{w})}{\sqrt{2}|z-w|^{1/4}},\qquad(\mathrm{B3d})$$

$$\psi(z)\sigma(w,\bar{w})\sim\frac{\gamma\mu(w,\bar{w})}{\sqrt{2}(z-w)^{1/2}},\qquad(\mathrm{B3e})$$

$$\psi(z)\mu(w,\bar{w})\sim\frac{\gamma^*\sigma(w,\bar{w})}{\sqrt{2}(z-w)^{1/2}},\qquad(\mathrm{B3f})$$

$$\bar{\psi}(\bar{z})\sigma(w,\bar{w})\sim\frac{\beta\gamma^*\mu(w,\bar{w})}{\sqrt{2}(\bar{z}-\bar{w})^{1/2}},\qquad(\mathrm{B3g})$$

$$\bar{\psi}(\bar{z})\mu(w,\bar{w})\sim\frac{\beta\gamma\sigma(w,\bar{w})}{\sqrt{2}(\bar{z}-\bar{w})^{1/2}},\qquad(\mathrm{B3h})$$

where some arbitrariness remains in the constants β and γ because of the nonlocal character of these OPE's: $\beta=\pm1$ and $\gamma=\pm e^{\pm i\pi/4}$. In this work we choose $\gamma=e^{i\pi/4}$ and $\beta=1$.

Since the regularized product of σ with μ is a fermion, these operators must carry some anticommuting character. We shall assume that the disorder operator μ anticommutes with all fermion fields and other disorder operators, but not with itself nor with the order fields. This is a matter of convention (σ could have been chosen instead).

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