# Scaling function for the number of alternating percolation clusters on self-dual finite square lattices

Chin-Kun Hu*
*Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan*

B. I. Halperin
*Department of Physics, Harvard University, Cambridge, Massachusetts 02138*
(Received 2 August 1996; revised manuscript received 24 September 1996)

We consider bond percolation with a bond probability $p$ on a $L_1 \times L_2$ self-dual square lattice with periodic boundary conditions in the horizontal direction and free boundary conditions in the vertical direction, terminated at the top and bottom by a row of vertical and horizontal bonds, respectively. We define the number $M$ of alternating percolation clusters as the minimum of $n_{\rm p}$ and $n_{\rm n}$, where $n_{\rm p}$ is the number of independent percolating clusters connecting sites on the top and bottom edges, and $n_{\rm n}$ is the number of percolating clusters in the complementary configuration on the dual lattice, a bond being present in the complementary configuration if and only if it is absent in the original configuration. We evaluate the probability $W_M^{\rm a}(L_1,L_2,p)$ for finding a given value of $M$ and find that, for a given aspect ratio $L_1/L_2$ all data of $W_M^{\rm a}(L_1,L_2,p)$ near the critical point $p_c$ fall on the same scaling function $F_M^{\rm a}$ which is symmetric with respect to the scaling variable for all $M$. [S0163-1829(97)02705-7]

Percolation[1] is an active research subject in recent years.[2–8] Important quantities in traditional studies of a percolation problem on a lattice $G$ include the existence probability $E_{\rm p}(G,p)$ and the percolation probability $P(G,p)$ with $p$ being the bond or site occupation probability. Here $E_{\rm p}(G,p)$ is the probability[6] that the system percolates and $P(G,p)$ is the probability that a given lattice site belongs to a percolating cluster. Other quantities which have been studied include the probability $W_n(G,p)$ for finding $n$ independent percolating clusters for a specified set of boundary conditions.[9,10] Of particular interest for all these quantities are their scaling behaviors in the limit $L \to \infty$ and $p \to p_c$, where $L$ is the linear dimension of the lattice and $p_c$ is the percolation threshold for the infinite system.

In the present paper, we consider the probability distribution of another quantity, the number of *alternating percolation clusters*, on self-dual finite square lattices, which will be described below. We define the number $M$ of alternating percolation clusters as

$$M = \min(n_{\rm p}, n_{\rm n}), \tag{1}$$

where $n_{\rm p}$ is the number of independent percolating clusters connecting sites on the top and bottom edges, and $n_{\rm n}$ is the number of percolating clusters in the complementary configuration on the dual lattice, a bond being present in the complementary configuration if and only if it is absent in the original configuration.

A typical $L_1 \times L_2$ square (sq) (Ref. 11) lattice considered in this paper is shown in Fig. 1, in which the sites and bonds of the original lattice $G$ are represented by black circles and solid lines, respectively, and the sites and bonds of the dual lattice $D$ are represented by open circles and dotted lines, respectively. Both the original lattice and the dual lattice have periodic boundary conditions in the horizontal direction of length $L_1$ and free boundary conditions in the vertical direction of length $L_2$. Therefore, the left-most black (open) circle and the right-most black (open) circle of the same horizontal line in Fig. 1 are identical lattice sites. Note that the original lattice is terminated at top and bottom by a row of vertical and horizontal bonds, respectively, while the reverse is true for the dual lattice. We say that a finite square lattice with these boundary conditions is *self-dual*, because the dual lattice is geometrically identical to the original lattice (rotated by 180°). Moreover, when the bond-occupation probability on the original lattice is $p$, the complementary configurations on the dual lattice correspond to the configurations of an identical bond percolation problem with bond-occupation probability $(1-p)$.

If clusters formed by occupied bonds on the original lattice are thought of as ''conducting regions'' and clusters formed by occupied bonds of the complementary configuration on the dual lattice are thought of as ''insulating,'' then we see that every bond belongs to either a conducting or an insulating cluster, and that conducting and insulating clusters cannot cross each other. We can now see why the quantity $M$ defined by Eq. (1) was denoted the number of alternating

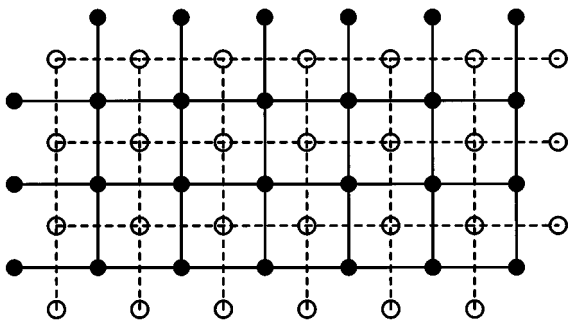

FIG. 1. A typical $L_1 \times L_2$ self-dual square lattice. In this example, $L_1 = 6$ and $L_2 = 4$.

 

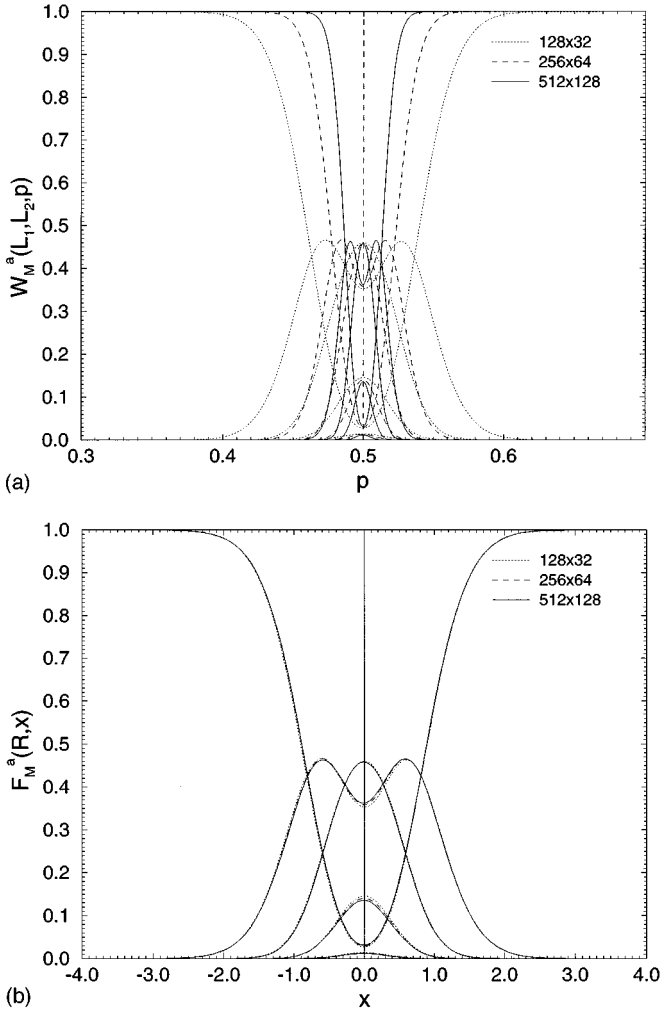


FIG. 2. (a) $W_M^a(L_1,L_2,p)$ for bond percolation on $128\times32$, $256\times64$, and $512\times128$ self-dual square lattices, which are represented by dotted, dashed, and solid lines, respectively. (b) The data of (a) are plotted as a function of $x=(p-p_c)L^{1/\nu}$. The scaling function for $W_M^a(L_1,L_2,p)$ is denoted by $F_M^a(R,x)$, where $R=L_1/L_2$. The $U$ shape curve is for $F_0^a(R,z)$. The $M$ shape curve is for $F_1^a(R,x)$. The bell shape curves from top to bottom are for $F_M^a(R,x)$ with $M$ being 2, 3, and 4, respectively.

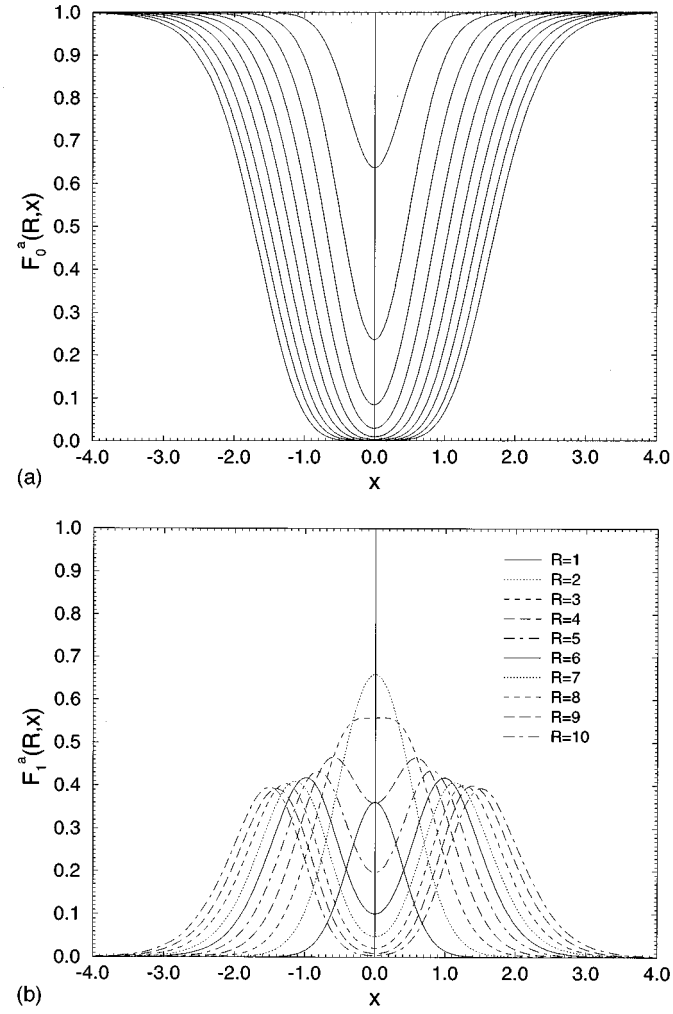


FIG. 3. $F_M^a(R,x)$ for $R=L_1/L_2=1,2,3,\ldots,10$. (a) $M=0$. (b) $M=1$.

clusters. The number $M$ will take on a value $>0$ if and only if there are precisely $M$ *conducting* clusters percolating from the top to the bottom of the sample, occurring in alternation with $M$ *insulating* percolation clusters, as one moves in a horizontal direction across the sample.

One motivation for studying the number of alternating percolation clusters comes from a recent analysis by Ruzin, Cooper, and Halperin[12] (RCH) of a problem in the quantum Hall effect. They argued that under certain circumstances, a quantum Hall sample at very low temperatures may be considered to be made up of randomly distributed microscopic regions belonging to two different quantum Hall plateaus. In each region, there is a local Hall conductivity $\sigma_{xy}$, which takes one of two values $\sigma_1$ or $\sigma_2$, while the local diagonal conductivity $\sigma_{xx}$ is neglibly small. If the macroscopic conductivity of such a sample is measured in the annular Corbino geometry, the observed value of the two-terminal conductance $G$ will be given, according to RCH (Ref. 12) by $G=M|\sigma_1-\sigma_2|$, where $M$ is the quantity which we have denoted here as the number of alternating percolation clusters which connect the inner and outer contacts. Therefore, the number of alternating percolation clusters in percolation problems is an interesting quantity and we will study this quantity in this paper to know more about its behavior.

We begin by recalling some results of previous studies of the scaling behavior of percolation clusters on two-dimensional lattices. With the free boundary conditions and in the limit $L\to\infty$, it was found that for site and bond percolation on the $L\times L$ square (sq) lattice, $E_p(G,p_c)=0.5$.[2–5] It was also proposed by Langlands, Pichet, Pouliot and Saint-Aubin[3] (LPPS) that for bond and site percolation on the plane triangular (pt) and honeycomb (hc) lattices with aspect ratios $\sqrt{3}/2$ and $\sqrt{3}$, respectively, $E_p(G,p_c)$ is also equal to 0.5. (The aspect ratio is defined as $N_1/N_2$ where $N_1$ and $N_2$ are the numbers of columns and rows of lattice sites in a rectangular representation.) Using a histogram Monte Carlo simulation method[6] (HMCSM) and the aspect ratios of LPPS,[3] Hu, Lin, and Chen[7,8] (HLC) found that for a given class of boundary conditions, and a fixed reduced aspect ratio $R^*$, all scaled data of $E_p$ and $P$ as a function of scaling variable $x$ fall on the same universal scaling functions $F(x)$ and $S(x)$, respectively, where

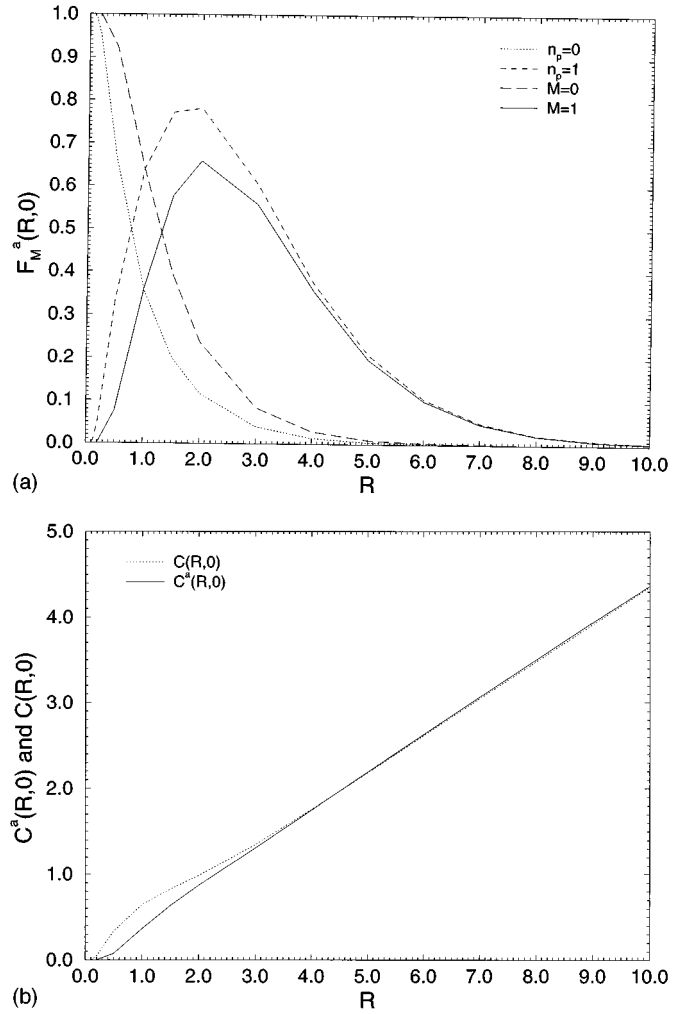


FIG. 4. (a) $F_M^a(R,0)$ and $F_M(R,0)$ as a function of $R=L_1/L_2$ for $M$ being 0 and 1. (b) $C^a(R,0)$ and $C(R,0)$ as a function of $R=L_1/L_2$.

$x=D_1(p-p_c)L^{1/\nu}$, $\nu=4/3$ is a correlation length exponent, and $D_1$ is a model-dependent nonuniversal metric factor.[13] The reduced aspect ratio $R^*$ for the different lattices may be defined as $R^*=gN_1/N_2$ where $g=1, 2/\sqrt{3}$, and $1/\sqrt{3}$ for sq, pt, and hc lattices, respectively. The constant $D_1$ is independent of the reduced aspect ratio $R^*$ and the boundary conditions.[7,8]

In a recent paper, Hu used the HMCSM (Ref. 6) to evaluate the probability $W_n(L_1,L_2,p)$ for the appearance of $n$ top-to-bottom percolating clusters of bond percolation with bond probability $p$ on finite $L_1\times L_2$ sq lattices.[11] When he plotted $W_n$ as a function of the scaling variable $z=(p-p_c)L^{1/\nu}$, all data of the same aspect ratio $L_1/L_2$ near the critical point $p_c$ fall on the same scaling function.[9] Using nonuniversal metric factor $D_1$ of Ref. 7, Hu and Lin showed that $W_n$ for bond and site percolations on sq, pt, and hc lattices with aspect ratios $L_1/L_2$, $\sqrt{3}L_1/2L_2$, and $\sqrt{3}L_1/L_2$ have universal scaling functions when they are plotted as functions of $x$, where $x=D_1z$.[10]

By a conformal transformation,[4,12] the $L_1\times L_2$ sq lattice mentioned above may be mapped into the annular Corbino geometry used in many experiments of quantum Hall effects, provided that the inner and outer radii $R_1$ and $R_2$ satisfy the relation $\ln(R_2/R_1)=2\pi L_2/L_1$. Therefore, at the critical point, where conformal invariance is valid, results of Hu and Lin[10] for rectangular domains may be applied to an annular geometry with appropriate aspect ratio. However, Hu and Lin[10] only calculated the number of percolating clusters, which we denoted in the present paper as $n_p$. The number of interest in the RCH theory[12] is the number of *alternating percolation clusters* defined by Eq. (1). For the Corbino disk and the corresponding lattice just discussed, $n_n=n_p=M$ for $n_p\geqslant 2$ (Ref. 14) and the results of Hu and Lin[10] for $n_p\geqslant 2$ may be used to compare with experimental data. However, for $n_p=1$ it is possible that $n_n=0$ and Hu and Lin's result for $n_p=1$ is only an upper bound of the result for $M=1$ alternating percolation cluster.

In the present paper, we use the HMCSM (Refs. 6,9) to evaluate the probability $W_M^a(L_1,L_2,p)$ for the appearance of $M$ top-to-bottom alternating percolation clusters of bond percolation with bond probability $p$ on finite $L_1\times L_2$ self-dual sq lattices shown in Fig. 1. We find that, for a given aspect ratio $L_1/L_2$ all scaled data of $W_M^a(L_1,L_2,p)$ near the critical point $p_c$ fall on the same scaling function $F_M^a(L_1/L_2,x)$, where $x=(p-p_c)L^{1/\nu}$, $F_M^a(L_1/L_2,-x)=F_M^a(L_1/L_2,x)$, and $\nu=4/3$ is the correlation-length exponent. In other words, we find the same type of scaling behavior as was previously found for the usual cluster number $n_p$. Such scaling behavior is, of course, necessary, if we hope to apply results of a lattice simulation to the continuum percolation problem of experimental interest.

In the bond percolation on a $L_1\times L_2$ sq (Ref. 11) lattice $G$ of $N$ sites and $E$ bonds, $N=L_1\times L_2$, each bond of $G$ is occupied with a probability $p$, where $0\leqslant p\leqslant 1$. If a bond of $G$ is occupied (unoccupied), then the dotted line of $D$ crossing this bond is unoccupied (occupied). For every subgraph $G'$ of $G$, there corresponds one and only one subgraph $D'$ of $D$. A cluster which extends from the top row of $G$ ($D$) to the bottom row is a percolating cluster. Between two neighboring percolating clusters of $G$ ($D$), there is one and only one percolating cluster of $D$ ($G$). If $G$ has two or more percolating clusters, then $D$ has the same number of percolating clusters, which is the number of alternating percolation clusters. If $G$ has only one percolating cluster, then the number of alternating percolation clusters is equal to the number of percolating clusters in $D$, which is 1 or 0.

The subgraph which contains at least one alternating percolation cluster is a percolating subgraph and denoted by $G'_p$ otherwise it is an nonpercolating subgraph. The percolating subgraph with $M$ alternating percolation clusters is denoted by $G'_M$ Now we have the definition

$$W_M^a(L_1,L_2,p)=\sum_{G'_M\subseteq G} p^{b(G'_M)}(1-p)^{E-b(G'_M)}, \qquad (2)$$

where $b(G'_M)$ is the number of occupied bonds in $G'_M$ The summation in (2) is over all subgraphs $G'_M$ of $G$. In the HMCSM, we choose $w$ different values of $p$. For a given $p=p_j$, $1\leqslant j\leqslant w$, we generate $N_R$ different subgraphs $G'$. The data obtained from the $wN_R$ different $G'$ are then used to construct arrays of numbers with elements $N_p(b)$, $N_f(b)$, and $N_M(b)$, $0\leqslant b\leqslant E$, which are, respectively, the

total numbers of percolating subgraphs with $b$ occupied bonds, nonpercolating subgraphs with $b$ occupied bonds, and the number of percolating subgraphs with $b$ occupied bonds and $M$ alternating percolation clusters. In the large number of simulations, the probability $W_M^a$ at any value of $p$ can then be calculated approximately from the following equation:[6,9]

$$W_M^a(L_1,L_2,p)=\sum_{b=0}^{E} p^b(1-p)^{E-b}C_b^E\frac{N_M(b)}{N_p(b)+N_f(b)}, \quad (3)$$

where $C_b^E=E!/(E-b)!b!$. It is obvious that $E_p=\Sigma_{M=1}^{\infty}W_M^a$.

We first use (3) to evaluate $W_M^a$ for bond percolation on $128\times32$, $256\times64$ and $512\times128$ sq (Ref. 11) lattices. The results are shown in Fig. 2(a), where $W_0^a=1-E_p$. Using the exact values of $\nu$ and $p_c$,[1] we obtain $W_M^a$ as a function of $x=(p-p_c)L^{1/\nu}$. The results are shown in Fig. 2(b), where very good scaling behavior of $W_M^a$ is observed and the corresponding scaling function is denoted by $F_M^a(R,x)$ with $R=L_1/L_2=4$. Figure 2(b) shows that $F_M^a(R,x)$ for any $M$ is a symmetric function of $x$. This is different from the function $F_n(R,x)$ studied in Ref. 10.

We have also calculated $F_M^a(R,x)$ for sq lattices for various other values of $R$. Plots of $F_0^a$ and $F_1^a$ as a function of $x$, for various values of $R$ are shown in Figs. 3(a) and 3(b). A plot of $F_M^a(R,0)$ as a function of $R$, for $M=0$ and 1, is shown in Fig. 4(a). Corresponding results of Ref. 10 are also shown for comparison. Please note that $F_1^a(R,0)$ of the present paper is smaller than that $F_1(R,0)$ of Ref. 10 as one would expect.

We may define the average number of alternating percolation clusters $C^a(R,x)$ via $C^a(R,x)=\Sigma_{M=1}^{\infty}F_M^a(R,x)M$. A plot of $C^a(R,0)$ as a function of $R$ is shown as a solid curve in Fig. 4(b). Results for the mean cluster number $C(R,0)$ defined in Ref. 10 are also shown for comparison. Please note that $C^a(R,0)$ and $C(R,0)$ approach each other for large $R$, and are nearly identical for $R\geqslant4$.

We note that our results are obtained from a random percolation model. Comparisons with experimental results on quantum Hall samples may be complicated, however, by the fact that there may be long-range correlations in the experimental system, such as a long-wave-length density gradient, which can destroy the expected universality of behavior near the percolation threshold. Another complication arises from the fact that unknown variations in the parent wafers from which samples are made mean that the actual value of $p$ may be different for different samples measured at a given value of the applied magnetic field, even if the samples have nominally the same electron density and other characteristics. These issues are discussed further in Ref. 12.

This work was supported by the National Science Council of the Republic of China (Taiwan) under Grants No. NSC 85-2112-M-001-007 Y and No. NSC 85-2112-M-001-045 and National Science Foundation of U.S.A. under Grant No. DMR 94-16910. The authors are grateful for helpful conversations with Nigel Cooper and Igor Ruzin. We thank Jau-Ann Chen for drawing Fig. 1 and Chai-Yu Lin for improving a computer program.

*Electronic address: huck@phys.sinica.edu.tw

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor and Francis, London, 1992).
[2] R. M. Ziff, Phys. Rev. Lett. **69**, 2670 (1992).
[3] R. P. Langlands, C. Pichet, Ph. Pouliot, and Y. Saint-Aubin, J. Stat. Phys. **67**, 553 (1992).
[4] J. L. Cardy, J. Phys. A **25**, L201 (1992).
[5] A. Aharony and J.-P. Hovi, Phys. Rev. Lett. **72**, 1941 (1994).
[6] C.-K. Hu, Phys. Rev. B **46**, 6592 (1992).
[7] C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Phys. Rev. Lett. **75**, 193 (1995); **75**, 2786E (1995).
[8] C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Physica A **221**, 80 (1995).
[9] C.-K. Hu, J. Korean Phys. Soc. (Proc. Suppl.) **29**, 97 (1996).
[10] C.-K. Hu and C.-Y. Lin, Phys. Rev. Lett. **77**, 8 (1996).
[11] Here ''square'' means a primitive unit cell of the lattice is square rather than $L_1=L_2$.
[12] I. M. Ruzin, N. R. Cooper, and B. I. Halperin, Phys. Rev. B **53**, 1558 (1996); see also N. R. Cooper, B. I. Halperin, C.-K. Hu, and I. M. Ruzin, Phys. Rev. B (to be published).
[13] V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).
[14] When $n_p\geqslant2$, two percolating conducting clusters are always separated by a percolating nonconducting cluster, therefore $n_p=n_n$ and $F_M^a(R,x)=F_M^a(R,-x)$ due to periodic boundary conditions and symmetry with respect to $p\rightarrow(1-p)$.