Analytic calculations of trial wave functions of the fractional quantum Hall effect on the sphere

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We present a framework for analytic calculations of the hierarchical wave functions and the composite fermion wave functions in the fractional quantum Hall effect on the sphere by using projective coordinates. Then we calculate the overlaps between these two wave functions at various fillings and small numbers of electrons. We find that the overlaps are almost equal to 1. This gives further evidence that two theories of the fractional quantum Hall effect, the hierarchical theory and the composite fermion theory, are physically equivalent. [S0163-1829(97)02303-5]

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) at the Landau-level (LL) filling fraction $\nu=1/m$ with m an old integer is very well described by Laughlin's theory.[1,2] The Laughlin wave function is a very good approximation of the exact ground state of the quantum Hall effect (QHE) at $\nu=1/m$. However for the FQHE at $\nu\neq 1/m$, there exist *two* well-known theories (notice that we will only consider the case that the electron spins are polarized in this paper). One is the hierarchical theory. The states at $\nu\neq 1/m$ are formed due to the condensation of the anyonic quasiparticles of Laughlin states.[3–10] The trial wave functions constructed from this theory are called hierarchical wave functions. Another theory is based on the composite fermion (CF) approach proposed by Jain,[11] where the FQHE is due to the integer QHE of the composite fermions (CF's) (electrons bounded with an even magnetic flux quanta). The trial wave functions constructed from the CF theory are called CF wave functions (or Jain's wave functions). The overlaps of the exact states with the hierarchical wave functions and the CF wave functions are both excellent. It has also been shown that two theories predict the same topological excitations at the same ν.[7,8,12] The two theories must be physically equivalent if they both describe correctly the physics of the FQHE. Thus it would be very interesting to study the difference and equivalence of the two theories.

In this paper, we present a framework for analytic calculations of the two wave functions on the sphere by using projective (or stereographic) coordinates on the sphere. There are several advantages of using spherical geometry. As it is a compact surface, there will be no edge state if we are only interested in the bulk state. Also, the system has rotational invariance symmetries. On the torus, though the system has translational invariance and no boundaries, the hierarchical wave functions are very difficult to calculate and quite complicated due to its nontrivial topology,[13] and we do not even know how to construct CF wave functions with the correct center coordinate degeneracy on a torus.

Because the states considered in the FQHE are restricted to the lowest Landau level (LLL), the wave functions are only dependent on holomorphic coordinates (polynomials of the holomorphic coordinates) on the sphere. Therefore it is possible to use only holomorphic coordinates to do all calculations. Comparing the two types of hierarchical wave functions is the same as comparing the two polynomials of holomorphic coordinates on the sphere. We note that our ultimate goal is to expand those wave functions in polynomials, and calculate the overlaps of two wave functions or physical quantities (for example, the density-density correlations) at an *arbitrary number* of electrons by the method (Jack polynomials method) used in studying the Calogero model.[14] We do not know how to do this at the moment, and further progress on it will enhance our understanding of the theories of the FQHE.

We organize the paper as follows: first we review the Landau-level problem on the sphere. A self-contained derivation of eigenstates of an electron on a sphere with a monopole field, performed by using a simple geometric argument and projective coordinates, is given in the Appendix. Then we show how to classify the many-body eigenstates of the angular momentum in the LLL. We then construct wave functions based on the theory of the hierarchical states and the theory based on the CF picture. The wave functions constructed in this paper are easy to handle in the practical calculation. Finally we calculate the overlaps of the hierarchical wave functions and the CF wave functions at various fillings ν and some small numbers of electrons.

II. QUANTUM MECHANICS ON THE SPHERE

The electrons are constrained to move on the surface of a sphere of radius R having a magnetic monopole in its center. The total magnetic flux $4\pi R^2 B$ must be an integer multiple $\phi=2S$ of the magnetic flux quantum $\phi_0=2\hbar\pi c/e$ according to the Dirac quantization condition. Therefore, the sphere radius R is equal to $S^{1/2}l_0$, where $l_0=(\hbar c/eB)^{1/2}$ is the magnetic length. The eigenstates of an electron are given by monopole spherical harmonics.[3,15] First, we briefly review the old method to derive the wave functions of the Landau

levels (LL's), then rederive them by using algebraic geometry.

For simplicity, we take units $\hbar$ and c equal to 1 in the following formulas. The Hamiltonian of a single electron of mass m_e is given by $H=(1/2m_e)(\mathbf{P}+e\mathbf{A})^2$. However, since the electron is confined on the spherical surface, one shows

$$H=\frac{1}{2m_eR^2}[\mathbf{r}\times(\mathbf{P}+e\mathbf{A})]^2=\frac{\omega_c}{2S}\mathbf{\Lambda}^2, \tag{1}$$

where $\mathbf{\Lambda}=\mathbf{r}\times(\mathbf{P}+e\mathbf{A})$, ω_c is the cyclotron frequency, $\mathbf{P}=-i\nabla$, $\nabla\times\mathbf{A}=B\hat{\mathbf{\Omega}}$, and $\hat{\mathbf{\Omega}}=\mathbf{r}/R$.

The components of $\mathbf{\Lambda}$ obey the commutation relations $[\Lambda_i,\Lambda_j]=i\epsilon_{ijk}(\Lambda_k-S\Omega_k)$. The angular momentum operators $\mathbf{L}=\mathbf{\Lambda}+S\mathbf{\Omega}$, and their commutation relations are $[L_i,L_j]=i\epsilon_{ijk}L_k$. Since $\mathbf{\Lambda}$ is normal to the surface, we have $\hat{\mathbf{\Omega}}\cdot\mathbf{\Lambda}=\mathbf{\Lambda}\cdot\hat{\mathbf{\Omega}}=\mathbf{0}$, and $\mathbf{L}\cdot\hat{\mathbf{\Omega}}=\hat{\mathbf{\Omega}}\cdot\mathbf{L}=S$. Using these equations, the relation $|\mathbf{\Lambda}|^2=|\mathbf{L}|^2-S^2$ can be obtained. Thus the eigenvalues of $|\mathbf{\Lambda}|^2$ can be deduced from the usual angular momentum algebra $|\mathbf{\Lambda}|^2=|\mathbf{L}|^2-S^2=L(L+1)-S^2$, $L=S+n$, $n=0,1,2\ldots$, and the eigenstates of the Hamiltonian are the eigenstates of $|\mathbf{L}|^2$ and L_3, and they are given by monopole spherical harmonics. We choose a gauge field $\mathbf{A}=-(S/eR)[(1+\cos\theta)/\sin\theta]\hat{\boldsymbol{\varphi}}$, of which the singularity lies on the north pole (we choose a different gauge from the one used in Ref. 3). The wave functions at the LLL are given by

$$u^{S+m}v^{S-m}, \tag{2}$$

where $m=-S,-S+1,\ldots,S$, and

$$u=\cos(\tfrac{1}{2}\theta)e^{i\varphi},\quad v=\sin(\tfrac{1}{2}\theta). \tag{3}$$

All wave functions of the LL's can be derived by this way,[15] and we will not repeat the derivation here. In the following, all eigenstates will be obtained by using projective coordinates,[16,17] and the method developed in Ref. 18.

The projective coordinates are given by $z=2R\cot(\theta/2)e^{i\varphi}$ and its complex conjugate $\bar{z}$. We will take $R=\frac{1}{2}$ for simplicity. The measure on the sphere is $\int[dxdy/(1+z\bar{z})^2]$. The Hamiltonian of Eq. (1) in projective coordinates is now written by the following formula:[17]

$$H=\frac{2}{m_e}(1+z\bar{z})^2(P_z+eA_z)(P_{\bar{z}}+eA_{\bar{z}}), \tag{4}$$

where

$$P_z=-i\frac{\partial}{\partial z},\quad P_{\bar{z}}=-i\frac{\partial}{\partial\bar{z}},\quad eA_z=i\frac{\phi}{2}\frac{\bar{z}}{1+z\bar{z}} \tag{5}$$

and ϕ is the flux (in the unit of the fundamental flux ϕ_0) out of the surface. Note that the Hamiltonian given by Eq. (5) (we call this Hamiltonian H' in the Appendix) is different from the one given by Eq. (1) by a constant.

The ground states can be determined from the solutions of the equation $(P_{\bar{z}}+eA_{\bar{z}})\psi=0$, and they are (unnormalized)

$$\psi=\frac{z^l}{(1+z\bar{z})^{\phi/2}}, \tag{6}$$

where $l=0,\ldots,\phi$. At any Landau levels, the eigenstates (unnormalized) are given by (from the Appendix)

$$\psi_{n,l}=\left[\partial_z+\left(\frac{B}{2}+1\right)\partial_z\ln g\right]\left[\partial_z+\left(\frac{B}{2}+2\right)\partial_z\ln g\right]\cdots\left[\partial_z+\left(\frac{B}{2}+n-1\right)\partial_z\ln g\right]\psi_{n,l}^{(0)}, \tag{7}$$

where

$$g=\frac{1}{(1+z\bar{z})^2}, \tag{8a}$$

$$\psi_{n,l}^{(0)}=g^{B/2}\tilde{\psi}_{n,l}^{(0)}, \tag{8b}$$

$$B=\phi/2, \tag{8c}$$

$$\tilde{\psi}_{n,l}^{(0)}=1,z,\ldots,z^l,\ldots,z^{\phi+2n}. \tag{8d}$$

Under any finite rotations, the z coordinate is transformed as $z'=(az+b)/(cz+d)$. The rotation matrix $R=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ is generated by the rotations along the three Cartesian axes

$$R_x=\frac{1}{\sqrt{2}}\begin{pmatrix}(1+\cos\alpha)^{1/2} & i(1-\cos\alpha)^{1/2}\\ i(1-\cos\alpha)^{1/2} & (1+\cos\alpha)^{1/2}\end{pmatrix}, \tag{9a}$$

$$R_y=\frac{1}{\sqrt{2}}\begin{pmatrix}(1+\cos\beta)^{1/2} & (1-\cos\beta)^{1/2}\\ -(1-\cos\beta)^{1/2} & (1+\cos\beta)^{1/2}\end{pmatrix}, \tag{9b}$$

$$R_z=\begin{pmatrix}\exp(i\gamma/2) & 0\\ 0 & \exp(-i\gamma/2)\end{pmatrix}. \tag{9c}$$

The rotational invariance of the Hamiltonian is shown by the identity

$$OH(z')O^{-1}=H(z), \tag{10}$$

where

$$O=\left(\frac{cz+d}{\overline{cz+d}}\right)^{\phi/2}. \tag{11}$$

The wave function is transformed under rotations as

$$\psi'=O\psi\left(\frac{az+b}{cz+b}\right). \tag{12}$$

We list some useful relations when we do a finite rotation on a many-body wave function.

$$d(z_i,z_j)=\frac{z_i-z_j}{\sqrt{1+z_i\bar{z}_i}\sqrt{1+z_j\bar{z}_j}}, \tag{13}$$

z_i-z_j, and $1+z_i\bar{z}_i$ are transformed under the finite rotation as

$$d(z_i',z_j')=\left(\frac{cz_i+d}{\overline{cz_i+d}}\right)^{1/2}\left(\frac{cz_j+d}{\overline{cz_j+d}}\right)^{1/2}d(z_i,z_j), \tag{14a}$$

$$z_i' - z_j' = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}, \tag{14b}$$

$$1 + \overline{z_i' z_i'} = \frac{1 + z_i \overline{z_i}}{(cz_i + d)(\overline{cz} + \overline{d})}. \tag{14c}$$

Finally, the angular momentum operators for N electrons are

$$J_x = \sum_{i=1}^{N} J_x(i) = \tfrac{1}{2} \sum_{i=1}^{N} \left[(1 - z_i^2) \frac{\partial}{\partial z_i} - (1 - \overline{z}_i^2) \frac{\partial}{\partial \overline{z}} + \frac{\phi}{2} (z_i + \overline{z}_i) \right], \tag{15a}$$

$$J_y = \sum_{i=1}^{N} J_y(i) = \frac{i}{2} \sum_{i=1}^{N} \left[(1 + z_i^2) \frac{\partial}{\partial z_i} + (1 + \overline{z}_i^2) \frac{\partial}{\partial \overline{z}} + \frac{\phi}{2} (\overline{z}_i - z_i) \right], \tag{15b}$$

$$J_z = \sum_{i=1}^{N} J_z(i) = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z} - \overline{z}_i \frac{\partial}{\partial \overline{z}} - \frac{\phi}{2} \right). \tag{15c}$$

III. PROJECTIONS AND ANGULAR MOMENTUMS IN THE LLL

The FQH state is restricted to the LLL. In this section, we will discuss briefly how to project states to the LLL on the sphere (see Ref. 16, and for the detailed discussions in the case of a plane or a disk, see Ref. 19), and how to find the eigenstates of angular momentums when the particles are restricted to the LLL. Note that the construction of the CF wave functions involves higher LL's, we need to project the wave functions to the LLL (see Sec. V).

The normalized states with flux ϕ in the LLL are

$$|l\rangle = \left[\frac{(\phi + 1)!}{2\pi l! (\phi - l)!} \right]^{1/2} \frac{z^l}{(1 + z\overline{z})^{\phi/2}}, \tag{16}$$

and $l = 0,1,2, \ldots, \phi$. The projection operator to the LLL is $P = \Sigma_l |l\rangle\langle l|$, and it can be written also in the following form:

$$P\psi(z,\overline{z}) = \int \frac{dw d\overline{w}}{(1 + w\overline{w})^2} G(z,w) \psi(w,w), \tag{17a}$$

$$G(z,w) = \frac{\phi + 1}{2\pi} \frac{(1 + z\overline{w})^{\phi}}{(1 + z\overline{z})^{\phi/2} (1 + w\overline{w})^{\phi/2}}. \tag{17b}$$

For the many-body wave functions, P (or G) is equal to $\Pi_{i=1}^{N} P_i$ (or $\Pi_{i=1}^{N} G_i$) where P_i is the projection operator of the ith particle and N is the number of particles.

If the state is not in the LLL, the antiholomorphic coordinate $\overline{z}$ will appear. Typically, it appears as

$$\psi = \frac{\overline{z'} z^{i+l}}{(1 + z\overline{z})^{(\phi/2) + j}}, \tag{18}$$

and $P\psi$ is equal to

$$\frac{(\phi + 1)! (l + i)! (\phi + j - l - i)!}{l! (\phi - l)! (\phi + j + 1)!} \frac{z^l}{(1 + z\overline{z})^{\phi/2}}. \tag{19}$$

On the sphere, if the interactions between electrons are rotationally invariant, the eigenstates of the many-body Hamiltonian should also be the eigenstates of rotational operators J^2 and J_z. The FQH ground states on the sphere are rotationally invariant and nondegenerated. In order to find the ground states, we can thus use the rotational invariant states to diagonalize the Hamiltonian. As the number of all possible rotational invariant states is much less than the number of all possible states, it is thus much easier to find the ground states by using the rotational invariant states to diagonalize the Hamiltonian than by using all possible states. It could be also interesting to find the eigenstates of $J^2 \neq 0$ (which are not rotationally invariant). The excited states in the FQH are not rotationally invariant. For Fermi-liquid-like systems in a half-filled Landau level, one can have ground states which are not rotationally invariant.[20]

Now we are going to find the many-body wave functions on the LLL which are the eigenstates of J^2 and J_z. In the LLL, the many-body wave functions Ψ have the form

$$\Psi = \prod_{i=1}^{N} \frac{1}{(1 + z_i \overline{z_i})^{\phi/2}} F(z_1, z_2, \ldots, z_N), \tag{20}$$

where $F(z_1, z_2, \ldots, z_N)$ is an antisymmetric holomorphic function. When $J_+ = J_x + iJ_y, J_- = J_x - iJ_y, J_z$ act on Ψ, we have

$$J_- \Psi = \prod_{i=1}^{N} \frac{1}{(1 + z_i \overline{z_i})^{\phi/2}} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} F, \tag{21a}$$

$$J_+ \Psi = \prod_{i=1}^{N} \frac{1}{(1 + z_i \overline{z_i})^{\phi/2}} \sum_{i=1}^{N} \left(-z_i^2 \frac{\partial}{\partial z_i} + \phi z_i \right) F, \tag{21b}$$

$$J_z \Psi = \prod_{i=1}^{N} \frac{1}{(1 + z_i \overline{z_i})^{\phi/2}} \left[\left(\sum_{i=1}^{N} z_i \frac{\partial}{\partial z_i} \right) - \frac{N\phi}{2} \right] F. \tag{21c}$$

Thus the projected **J** operators are

$$J_-' = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}, \tag{22a}$$

$$J_+' = \sum_{i=1}^{N} -z_i^2 \frac{\partial}{\partial z_i} + \phi z_i, \tag{22b}$$

$$J_z' = \left(\sum_{i=1}^{N} z_i \frac{\partial}{\partial z_i} \right) - \frac{N\phi}{2}, \tag{22c}$$

where they act only on F. The angular momentum eigenstates of the many-body wave functions restricted to the LLL can be obtained by solving

$$J_-' F(-J) = 0, \tag{23a}$$

$$J_z' F(-J) = -JF(-J), \tag{23b}$$

where $F(-J)$ is the lowest weight eigenstate with weight $-J$. Other states can be obtained by applying J_+' repeatedly on $F(-J)'$. Equation (23) leads to

$$\sum_{i=1}^{N} z_i \frac{\partial}{\partial z_i} F(-J) = \left(\frac{N\phi}{2} - J \right) F(-J), \tag{24a}$$

$$\sum_{i=1}^{N} \frac{\partial}{\partial z_i} F(-J) = 0. \tag{24b}$$

The first equation in Eq. (24) means that $F(-J)$ is a homogeneous polynomial with degree $(N\phi/2)-J$. As $F(-J)$ is an antisymmetric function of holomorphic coordinates, it can be factorized as $F(-J)=\Pi_{i<j}^{N}(z_i-z_j)F'(-J)$. One can check that

$$J_-' F(-J) = \prod_{i<j}^{N} (z_i - z_j) J_-' F'(-J), \tag{25a}$$

$$J_z' F(-J) = \frac{N(N-1)}{2} F(-J) + \prod_{i<j}^{N} (z_i - z_j) J_z' F'(-J). \tag{25b}$$

Thus $F'(-J)$ is a symmetric function with degree $L=(N\phi/2)-J-[N(N-1)/2]$, and the power of every coordinate in $F'(-J)$ shall be less or equal than ϕ' where $\phi'=\phi-(N-1)$. By using Eqs. (24) and (25), one finds that $F'(-J)$ satisfies the conditions

$$J_-' F'(-J) = 0, \tag{26a}$$

$$J_z' F'(-J) = \left(\frac{N\phi'}{2} - J \right) F'(-J). \tag{26b}$$

Define symmetric polynomials σ_i:

$$P(z_i) = \prod_{i=1}^{N} (z - z_i) = \sum_{i=0}^{N} (-1)^i \sigma_i z^{N-i}, \tag{27}$$

where

$$\sigma_0 = 1, \quad \sigma_1 = \sum_{i=1}^{N} z_i, \ldots, \sigma_N = \prod_{i=1}^{N} z_i. \tag{28}$$

F' can be expanded as

$$\sum_{s_i} C(s_i) \prod_{i=1}^{N} \sigma_i^{s_i}, \tag{29}$$

where s_i is a non-negative integer. By using Eq. (26), we obtain equations which $C(s_i)$ and s_i must obey. One of them is

$$\sum_{i=1}^{N} i s_i = L = \frac{N\phi'}{2} - J. \tag{30}$$

The condition

$$\sum_{i=1}^{N} s_i \leqslant \phi' \tag{31}$$

must be satisfied in order that the wave function is normalizable. $C(s_i)$ shall also satisfy the equation

$$J_-' F'(-J) = \sum_{s_i'} C'(s_i') \prod_{i=1}^{N} \sigma_i^{s_i'} = 0, \tag{32}$$

where $C'(s_i')$ is a linear combinations of $C(s_i)$, and it shall be equal to 0. Because $[J_z, J_-] = -J_-$, s_i' shall satisfy

$$\sum_{i=1}^{N} i s_i' = L - 1. \tag{33}$$

Apparently $\Sigma_{i=1}^{N} s_i'$ shall be also less then or equal to ϕ'. The number of linear independent solutions for $C(s_i)$ is equal to the number of solutions of Eq. (30) minus the number of solutions of Eq. (33), and it is also equal to $M(J,N,\phi')$, which is the number of states with spin J.

The generation function for the number of solutions of Eq. (30) or (33) is

$$G(t,q) = \prod_{i=1}^{N} \frac{1}{1 - tq^i}. \tag{34}$$

The number of solutions of Eq. (30) is equal to the sum of the coefficient of term $t^j q^L$ with $0 \leqslant j \leqslant \phi'$ in $G(t,q)$. Thus $M(J,N,\phi')$ is equal to

TABLE I. In this table, we list the number of rotational invariant states at various ν and a small number of electrons. N_t is the dimension of the total Hilbert space (in the LLL) and N_r is the number of the rotational invariant states.

ν	N	N_t	N_r	ϕ (formula)	ϕ	$\frac{N\phi}{2}$
$\frac{2}{5}$	4	5	1	$\frac{5}{2}N-4$	6	12
	6	58	3		11	33
	8	910	8		16	64
$\frac{2}{7}$	4	43	2	$\frac{7}{2}N-2$	12	24
	6	1.242	10		19	57
	8	46.029	80		26	104
$\frac{2}{9}$	4	43	2	$\frac{9}{2}N-6$	12	24
	6	2.137	13		21	63
	8	139.143	164		30	120
$\frac{2}{11}$	4	150	3	$\frac{11}{2}N-4$	18	36
	6	11.963	29		29	87
	8	1.229.093	702		40	160
$\frac{2}{13}$	4	150	3	$\frac{13}{2}N-8$	18	36
	6	17.002	34		31	93
	8	2.502.617	1.137		44	176
$\frac{3}{7}$	9	910	8	$\frac{7}{3}N-5$	16	27
$\frac{3}{11}$	6	2.137	13	$\frac{11}{3}N-1$	21	63
	9	610.358	506		32	144
$\frac{3}{17}$	6	17.002	34	$\frac{17}{3}N-3$	31	93
$\frac{5}{17}$	4	33	2	$\frac{17}{5}N-\frac{13}{5}$	11	22
	9	184.717	217		28	126

$$\oint \oint \frac{dt}{2\pi i t}\frac{dq}{2\pi i q}G(t,q)\left(\frac{1}{q^L}-\frac{1}{q^{L-1}}\right)\sum_{i=0}^{\phi'}\frac{1}{t^i}. \quad (35)$$

We can also use a generation function of one variable,[21]

$$G(t)=\frac{\Pi_{k=1}^{N+\phi'}(1-t^k)}{\Pi_{k=1}^{N}(1-t^k)\Pi_{k=1}^{\phi'}(1-t^k)}. \quad (36)$$

The number of solutions of Eq. (30) is then given by the coefficient of t^L of function $G(t)$. Thus

$$M(J,N,\phi')=\oint \frac{dt}{2\pi i t}G(t)\frac{1-t}{t^L}. \quad (37)$$

The asymptotic behavior of $M(J,N,\phi')$ can be obtained by using the steepest-descent method.[22] When L and ϕ' are both large, $M(J,N,\phi')$ is equal to $\exp(S(J,N,\phi'))$ approximately, and $\exp(S(J,N,\phi'))$ is determined by the following equations:

$$L=-\frac{\exp(-\rho)}{1-\exp(-\rho)}+\frac{1}{\rho^2}\left(-\int_0^{(N+\phi')\rho}+\int_0^{N\rho}+\int_0^{\phi'\rho}\right)du\frac{u\exp(-\rho u)}{1-\exp(-\rho u)}, \quad (38a)$$

$$S(J,N,\phi')=L\rho+\ln(1-\exp(-\rho))+\frac{1}{\rho}\left(\int_0^{(N+\phi')\rho}-\int_0^{N\rho}+\int_0^{\phi'\rho}\right)du\ln[1-\exp(-u)]. \quad (38b)$$

We list the number of rotational invariance states at various fillings in Table I.

IV. HIERARCHICAL WAVE FUNCTIONS

In this section, we discuss the construction of the hierarchical wave functions.[2–4] The quasiparticles satisfy fractional statistics, and the condensation of quasiparticles gives rise to the FQH state with $\nu\neq 1/m$.

Define

$$\Psi_m=\prod_{i<j}^{N}(u_iv_j-u_jv_i)^m, \quad (39)$$

where m is a positive integer. For $\nu=1/m$, with m being an old positive integer, the FQH wave function will be Ψ_m (the Laughlin wave function). The flux ϕ is equal to $\phi_m=m(N-1)$.[3] Or, in projective coordinates, it is

$$\Psi_m=\prod_{i<j}^{N}d(z_i,z_j)^m. \quad (40)$$

The Laughlin wave function in the presence of quasiparticle excitations is given by the quasiparticle excitation operators acting on the original Laughlin wave function. The quasiparticle excitation operator is given by

$$A^\dagger(\alpha,\beta)=\prod_{i=1}^{N}(\beta u_i-\alpha v_i) \quad \text{(quasihole)}, \quad (41a)$$

$$A(\alpha,\beta)=\prod_{i=1}^{N}\left(\beta\frac{\partial}{\partial u_i}-\alpha\frac{\partial}{\partial v_i}\right) \quad \text{(quasielectron)}, \quad (41b)$$

where $\alpha=\cos(\theta/2)e^{i\varphi}$, and $\beta=\sin(\theta/2)$ are the quasiparticle coordinates. In the projective coordinates, the operators of the quasihole excitation and the quasielectron excitation are given in the following form:

$$A^\dagger(\omega,\overline{\omega})\Psi_m(z_i)=\prod_{i=1}^{N}d(z_i,\omega)\Psi_m(z_i), \quad (42a)$$

$$A(\omega,\overline{\omega})\Psi_m(z_i)=\frac{1}{(1+\omega\overline{\omega})^{N/2}}\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{(\phi_m-1)/2}}\times\prod_{i=1}^{N}[(1+z_i\overline{\omega})\partial_{z_i}-\phi_m\overline{\omega}]F_m, \quad (42b)$$

where $\omega,\overline{\omega}$ are the projective coordinates of the quasiparticle, and $F_m(z_i)=\Pi_{i<j}^{N}(z_i-z_j)^m$. The flux ϕ in the presence of a quasielectron (quasihole) is ϕ_m-1 (ϕ_m+1).

The slightly entangled appearance of $A(\omega)$ hides, indeed, a form which is analogous to $A^\dagger(\omega)$. To unveil the similarities, one can show that

$$P(\phi_m-1,z_i)\prod_{i=1}^{N}d(\overline{z_i},\overline{\omega})\Psi_m \quad (43)$$

gives the wave function of the Laughlin state in the presence of a quasihole such as that in Eq. (42). $P(\phi,z_i)$ (here $\phi=\phi_m-1$) projects the wave function to the LLL with flux ϕ with respect to coordinates z_i. Thus the construction of the hierarchical wave functions due to the condensation of quasielectrons will *naturally* involve higher Landau levels as in the case of the CF wave functions (see Sec. V).

Instead using $A(\omega,\overline{\omega})\Psi_m(z_i)$, we can also create a quasielectron excitation using $\Psi_{m-2}AD$,[9,23] where $D=[\Psi_1(z_i)]^2$. AD is here equal to $P(\phi_2-1)\Pi_{i=1}^{N}d(\overline{z_i},\overline{\omega})D$. We call $(\Psi_1)^{m-2}AD$ a wave function by the hard-core construction.

In the case of many quasiparticle excitations, the operators of excitations are

$$A_{N_q}^\dagger=\prod_{j=1}^{N_q}A^\dagger(\omega_j,\overline{\omega_j}), \quad (44a)$$

$$A_{N_q}=\prod_{j=1}^{N_q}A(\omega_j,\overline{\omega_j}). \quad (44b)$$

When A_{N_q} acts on Ψ_m, one can show

$$A_{N_q}\Psi_m=\prod_{j=1}^{N_q}\frac{1}{(1+\omega_j\overline{\omega_j})^{N/2}}\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{(\phi_m-N_q)/2}}A'_{N_q}F_m(z_i), \tag{45a}$$

where

$$A'_{N_q}=\prod_{j=1}^{N_q}A'(j)=\prod_{i=1}^{N}[(1+z_i\overline{\omega_{N_q}})\partial_{z_i}-(\phi_m-N_q+1)\overline{\omega_{N_q}}]\prod_{i=1}^{N}[(1+z_i\overline{\omega_{N_q-1}})\partial_{z_i}$$
$$-(\phi_m-N_q+2)\overline{\omega_{N_q-1}}]\cdots\prod_{i=1}^{N}[(1+z_i\overline{\omega_j})\partial_{z_i}-(\phi_m-j+1)\overline{\omega_j}]\cdots\prod_{i=1}^{N}[(1+z_j\overline{\omega_1})\partial_{z_j}-\phi_m\overline{\omega_1}]. \tag{46a}$$

One should be careful about the ordering of $A'(j)$ in Eq. (46). $A'(j)$ in Eq. (46) is defined as

$$A'(j)=\prod_{i=1}^{N}[(1+z_i\overline{\omega_j})\partial_{z_i}-(\phi_m-j+1)\overline{\omega_j}]. \tag{47}$$

As in the case of a single quasihole excitation, the wave function in the presence of N_q quasiholes can be also written as

$$P(\phi)\prod_{i=1}^{N}\prod_{\alpha=1}^{N_q}d(\overline{z_i},\overline{\omega_\alpha})\Psi_m, \tag{48}$$

where $\phi=\Phi_m-N_q$.

To construct the hierarchical wave functions, we shall normalize the Laughlin wave functions in the presence of quasiparticles. One can show that $\Psi_{e,q}=[\Psi_1(\omega_\alpha)]^{1/m}A^\dagger_{N_q}\Psi_m$ and $\Psi_{e,q}=[\Psi_1(\omega_\alpha)]^{1/m}A_{N_q}\Psi_m$, or, for the hard-core-constructed wave function, $[\Psi_1(\omega_\alpha)]^{1/m}\Psi_{m-2}AD$, are normalized.[24] The hierarchical states are obtained if the quasiparticles are also condensed to the Laughlin states. The wave function of the quasiholes is $\Psi_q=[\Psi_1(\overline{\omega_\alpha}]^{p_2+(1/m)}$, the wave function of the quasielectrons is $\Psi_q=[\Psi_1(\omega_\alpha)]^{p_2-(1/m)}$, and p is a positive even integer. Quasiparticles satisfy fractional statistics,[5] and the wave functions here are in a singular gauge which shows fractional statistics explicitly. The hierarchical wave function for the electrons is then given by the following formula:

$$\int\prod_{\alpha=1}^{N_q}\frac{d\omega_\alpha d\overline{\omega_\alpha}}{(1+\omega_\alpha\overline{\omega_\alpha})^2}\Psi e,q\Psi_q, \tag{49}$$

or, explicitly,

$$\Psi_e(m,p)=\int\prod_{\alpha=1}^{N_q}\frac{d\omega_\alpha d\overline{\omega_\alpha}}{(1+\omega_\alpha\overline{\omega_\alpha})^2}\Psi_m(z_i)$$
$$\times\prod_{i=1}^{N}\prod_{\alpha=1}^{N_q}d(z_i,\omega_\alpha)|\Psi_1(\omega_\alpha)|^{2/m}\Psi_{p_2}(\overline{\omega_\alpha}) \tag{50}$$

is the hierarchical wave function due to the condensation of quasiholes, and the hierarchical wave functions due to the condensation of quasielectrons by the non-hard-core construction and the hard-core construction are given in the following formulas:

$$\Psi_e(m,-p)=P(\phi,z_i)\int\prod_{\alpha=1}^{N_q}\frac{d\omega_\alpha d\overline{\omega_\alpha}}{(1+\omega_\alpha\overline{\omega_\alpha})^2}\Psi_m(z_i)$$
$$\times\prod_{i=1}^{N}\prod_{\alpha=1}^{N_q}d(\overline{z_i},\overline{\omega_\alpha})\Psi_{p_2}(\omega_\alpha), \tag{51a}$$

$$\Psi_{e,\mathrm{hard}}(m,-p)=\Psi_{m-2}(z_i)$$
$$\times P(\phi_2-N_q,z_i)\int\prod_{\alpha=1}^{N_q}\frac{d\omega_\alpha d\overline{\omega_\alpha}}{(1+\omega_\alpha\overline{\omega_\alpha})^2}$$
$$\times\Psi_2(z_i)\prod_{i=1}^{N}\prod_{\alpha=1}^{N_q}d(\overline{z_i},\overline{\omega_\alpha})\Psi_{p_2}(\omega_\alpha). \tag{51b}$$

We also require the wave functions above to be rotationally invariant. This requirement leads to

$$m(N-1)+\xi_2N_q=\phi, \tag{52a}$$

$$p_2(N_q-1)=N. \tag{52b}$$

$\xi_2=\pm1$ in the case of the condensation of quasiholes and quasielectrons, respectively. The Landau-level filling fraction ν is equal to

$$\frac{1}{m+\dfrac{1}{\xi_2p_2}}. \tag{53}$$

For $m=1$ and $\xi_2=1$, the filling $\nu=1/[m+(1/p_2)]$ is equal to the filling of the charge conjugate state, $1-[1/(p_2+1)]$. Actually, the wave function $\Psi_e(m,p)$ is also the charge conjugate of the Laughlin wave function at filling $\nu=1/(p_2+1)$, and this shows that the construction of the wave function is consistent with the physical picture. When $m\neq1$, we notice that, in the formula for $\Psi_e(m,p)$, we cannot perform the integration exactly due to the term $|\Psi_1(\omega_\alpha)|^{2/m}$. We can approximate the trial wave function $\Psi_e(m,p)$ by omitting $|\Psi_1(\omega_\alpha)|^{2/m}$, and it becomes

$$\Psi_e(m,p)\approx \int \prod_{\alpha=1}^{N_q} \frac{d\omega_\alpha d\overline{\omega_\alpha}}{(1+\omega_\alpha\overline{\omega_\alpha})^2}\Psi_m(z_i) \times\prod_{i=1}^{N}\prod_{\alpha=1}^{N_q} d(z_i,\omega_\alpha)\Psi_{p_2}(\overline{\omega_\alpha}). \quad (54)$$

The wave function written in Eq. (54) is still rotationally invariant,[9,17] and we are able to integrate it. When $m=1$, the formula for $\Psi_e(m,p)$ in Eq. (50) is integrable. When $m=1$, we find that the overlap between the wave functions given in Eqs. (50) and (54) is excellent for a small number of electrons. In Ref. 9, it was also found that the overlapping of the wave functions given by Eq. (54) with the exact ground state of the FQH is almost equal to 1 for a small number of electrons. We note that the wave functions calculated in Sec. VI are based on the formulas written in Eqs. (52) and (54).

In the formula for $\Psi_e(m,-p)$ or $\Psi_{e,\mathrm{hard}}(m,-p)$, we note that one can perform the integration first, and then the projection, or vice versa. In Sec. VI, the overlap between these two wave functions will be calculated for a small number of electrons and it is found that the overlap is almost equal to 1.

We will call the above hierarchical states second-level hierarchical states, and the Laughlin states first-level hierarchical states. The higher-level hierarchical states can be built in a similar way.[7,8,17,9] We denote the k'th-level hierarchical states by $(p_1,\xi_2 p_2,\xi_3 p_3,\ldots,\xi_k p_k)$, where p_1 is an old positive integer, $p_i, i\neq 1$ are even positive integers, and $\xi_i=\pm$ indicate the quasihole condensation and quasielectron condensation from parent states.

For higher-level hierarchical wave functions involving the condensation of quasielectrons, we can make a *further* simplification. We take $(p_1,-p_2,-p_3)$ as an example. The wave function for this state is

$$\Psi_e=P(\phi,z_i)\int \prod_{\alpha=1}^{N_2}\frac{d\omega_\alpha^1 d\overline{\omega_\alpha^1}}{(1+\omega_\alpha^1\overline{\omega_\alpha^1})^2}\Psi_m(z_i) \times\prod_{i=1}^{N_1}\prod_{\alpha=1}^{N_2} d(\overline{z_i},\overline{\omega_\alpha^1})P(N_1,\omega_\alpha^1) \times\int\prod_{\alpha=1}^{N_3}\frac{d\omega_\alpha^2 d\overline{\omega_\alpha^2}}{(1+\omega_\alpha^2\overline{\omega_\alpha^2})^2}\Psi_{p_2}(\omega_\alpha^1) \times\prod_{\alpha=1}^{N_2}\prod_{\beta=1}^{N_3} d(\overline{\omega_\alpha^1},\overline{\omega_\beta^2})\Psi_{p_3}(\omega_\alpha^2), \quad (55)$$

where N_1 is the number of electrons, N_2 is the number of quasielectrons of the Laughlin state (p_1), N_3 is the number of quasielectrons of the hierarchical state $(p_1,-p_2)$, and ω_α^1 and ω_α^2 are the coordinates of quasiparticles of the two types, respectively. We can prove that $P(N_1,\omega_\alpha^1)$ can be *dropped* inside the formula. Thus the wave function can be written as

$$\Psi_e=P(\phi,z_i)\int\prod_{\alpha=1}^{N_2}\prod_{\beta=1}^{N_3}\frac{d\omega_\alpha^1 d\overline{\omega_\alpha^1}}{(1+\omega_\alpha^1\overline{\omega_\alpha^1})^2} \times\frac{d\omega_\beta^2 d\overline{\omega_\beta^2}}{(1+\omega_\beta^2\overline{\omega_\beta^2})^2}\Psi_m(z_i)\Psi_{p_2}(\omega_\alpha^1)\Psi_{p_3}(\omega_\alpha^2) \times\prod_{i=1}^{N_1}\prod_{\alpha=1}^{N_2} d(\overline{z_i},\overline{\omega_\alpha^1})\prod_{\alpha=1}^{N_2}\prod_{\beta=1}^{N_3} d(\overline{\omega_\alpha^1},\overline{\omega_\beta^2}). \quad (56)$$

The wave function in Eq. (56) is quite similar to the wave function constructed in Ref. 7,

$$\Psi_e=P(\phi,z_i)\int\prod_{\alpha=1}^{N_2}\prod_{\beta=1}^{N_3}\frac{d\omega_\alpha^1 d\overline{\omega_\alpha^1}}{(1+\omega_\alpha^1\overline{\omega_\alpha^1})^2} \times\frac{d\omega_\beta^2 d\overline{\omega_\beta^2}}{(1+\omega_\beta^2\overline{\omega_\beta^2})^2}\Psi_m(z_i)\Psi_{p_2}(\omega_\alpha^1)\Psi_{p_3}(\omega_\alpha^2) \times\prod_{i=1}^{N_1}\prod_{\alpha=1}^{N_2}\frac{1}{d(z_i,\omega_\alpha^1)}\prod_{\alpha=1}^{N_2}\prod_{\beta=1}^{N_3}\frac{1}{d(\omega_\alpha^1,\omega_\beta^2)}. \quad (57)$$

However, it is difficult to handle Eq. (57) in the practical calculation due to the singularities.

Finally by requiring the rotational invariance of the wave function Eqs. (56) or (57), one obtains

$$\begin{aligned} p_1(N_1-1)-N_2&=\phi,\\ N_1-p_2(N_2-1)+N_3&=0,\\ N_2-p_3(N_3-1)&=0, \end{aligned} \quad (58)$$

and Eq. (58) implies that the filling of the FQH state is equal to

$$\cfrac{1}{p_1+\cfrac{1}{p_2+\cfrac{1}{p_3}}}. \quad (59)$$

We point out that the wave function proposed in Ref. 7 had been also constructed on the torus.[13] It would be very interesting if we could generalize the construction of the wave function (56) to the torus.

V. COMPOSITE FERMION WAVE FUNCTIONS

The CF theory of the FQHE has significantly advanced the understanding of the FQHE recently.[11] The FQHE is due to the integer QHE of the CF's, where a CF is the bound state of an electron and an even number of vortices. We will discuss in this section how to calculate the CF wave functions in our framework.

Jain proposed that all trial wave functions of the FQHE (note again in this paper the spin is polarized) can be obtained by using two operations, D and C, the composite fermionization and charge conjugation, respectively, on the wave functions of the integer QHE of the CF's. For example, the trial wave function of electrons at $\nu=n/(2n+1)$ can be written as $PD\chi_n$, where χ_n is the wave function of the CF's

which fill completely the first n Landau levels with flux ϕ^* (P is the projection operator to the LLL as in the previous sections). The flux of the state $PD\chi_n$ is equal to $2(N-1)+\phi^*$, where $\phi^*=(N/n)-n$. We can also use $\Psi_1 P(\phi-N+1)\Psi_1\chi_n$ as the trial wave function and we call this wave function the wave function by the hard-core construction. The charge conjugation of $PD\chi_n$ (or $\Psi_1 P(\phi-N+1)\Psi_1\chi_n$) is then the trial wave function at $\nu=1-[n/(2n+1)]=(n+1)/(2n+1)$. The trial wave function at other fillings can be obtained by acting repeatedly D and C on $PD\chi_n$ ($\Psi_1 P(\phi-N+1)\Psi_1\chi_n$) (each state can be obtained only in a unique way in this picture except the ordering of operator P).

χ_n is given by the determinant $\chi_n=\det(\psi_{s,k}(z_i))$, where $s=0,1,\ldots,n-1$, $k=\phi^*+2s+1$, $i=1,2,\ldots,N$, $N=n\phi^*+n^2$. $\det(\psi_{s,k}(z_i))$ can be simplified and it is given by the formula

$$\chi_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ z_1^{N'-1} & z_2^{N'-1} & \cdots & z_N^{N'-1} \\ \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_N \\ \bar{z}_1 z_1 & \bar{z}_2 z_2 & \cdots & \bar{z}_N z_N \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \bar{z}_1 z_1^{N'-1} & \bar{z}_2 z_2^{N'-1} & \cdots & \bar{z}_N z_N^{N'-1} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \bar{z}_1^{n-1} & \bar{z}_2^{n-1} & \cdots & \bar{z}_N^{n-1} \\ \bar{z}_1^{n-1} z_1 & \bar{z}_2^{n-1} z_2 & \cdots & \bar{z}_N^{n-1} z_N \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \bar{z}_1^{n-1} z_1^{N'-1} & \bar{z}_2^{n-1} z_2^{N'-1} & \cdots & \bar{z}_N^{n-1} z_N^{N'-1} \end{vmatrix} \times \prod_{j=1}^{N} \frac{1}{(1+z_j\bar{z}_j)^{\phi^*/2+n-1}}, \tag{60}$$

where $N'=N/n=\phi^*+n$. We divide N electrons into n groups. The set of the original coordinates z_i can be mapped to $z_{s,k}$ with $s=0,1,\ldots,n-1$, $k=1,2,\ldots,N'$. The determinant is proportional to

$$\chi_n=\mathbf{AN}\prod_{s=0}^{n-1}[e_s^{N'}]^s\Psi_{s,1}\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{(n-1)/2}}, \tag{61}$$

where

$$e_s^{N'}=\prod_{k=1}^{N'}\overline{z_{s,k}}, \tag{62a}$$

$$\Psi_{s,1}=\prod_{k_1<k_2}^{N'} d(z_{s,k_1}-z_{s,k_2}), \tag{62b}$$

and **AN** is the antisymmetrizing operator on all coordinates $z_{s,k}$. The wave functions $\Psi=P(\phi)D\chi_n$ and $\Psi_{\rm hard}=\Psi_1 P(\phi-N+1)\Psi_1\chi_n$ can be written in the following form:

$$\Psi=\mathbf{AN}P\prod_{s=0}^{n-1}[e_s^{N'}]^s\Psi_{s,2}\prod_{i<j}^{N}(z_i-z_j)^2 \times\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{\phi/2+n-1}}, \tag{63a}$$

$$\Psi_{\rm hard}=\Psi_1\mathbf{SY}P(\phi-N+1)\prod_{s}[e_s^{N'}]^s\Psi_{s,2}\prod_{i<j}^{N}(z_i-z_j)\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{[(\phi-N+1)/2]+n-1}}, \tag{63b}$$

where **SY** is the symmetrizing operator on the electron coordinates, and

$$\Psi_{s,2}=\prod_{k_1<k_2}^{N'}(z_{s,k_1}-z_{s,k_2}). \tag{64}$$

Before performing the antisymmetrizing or symmetrizing in the formulas above, it appears that there are n different groups of electrons, and there are correlations between different groups. The generic terms before doing the projection, for example, in the formula of Ψ, are

TABLE II. The overlaps between the hierarchical wave functions and the CF wave functions at some fillings for a small number of electrons.

ν	N	overlap				
$\frac{2}{5}$	6 4	$\langle PD\chi_2\|E\Psi_3\rangle$ 0.999 323 414 9 1	$\langle\Psi_1P\Psi_1\chi_2\|E\Psi_3\rangle$ 0.999 361 597 1 1	$\langle PD\chi_2\|\Psi_1E\Psi_1^2\rangle$ 0.999 833 152 3 1	$\langle\Psi_1P\Psi_1\chi_2\|\Psi_1E\Psi_1^2\rangle$ 0.999 945 645 7 1	
ν	N	overlap		ν	N	overlap
$\frac{2}{5}$	6 4	$\langle E\Psi_3\|\Psi_1E\Psi_1^2\rangle$ 0.999 647 900 1 1	$\langle PD\chi_2\|\Psi_1P\Psi_1\chi_2\rangle$ 0.999 928 898 7 1	$\frac{2}{7}$	6 4	$\langle H\Psi_3\|DC\Psi_3\rangle$ 0.999 376 257 4 1
ν	N	overlap				
$\frac{2}{9}$	4	$\langle E\Psi_5\|DPD\chi_2\rangle$ 0.999 961 486 9	$\langle E\Psi_5\|PD^2\chi_2\rangle$ 0.999 961 486 9	$\langle DPD\chi_2\|PD^2\chi_2\rangle$ 1		
ν	N	overlap	ν	N	overlap	
$\frac{3}{11}$	6	$\langle EH\Psi_3\|DCPD\chi_2\rangle$ 0.999 652 238 3	$\frac{2}{11}$	4	$\langle H\Psi_5\|D^2C\Psi_3\rangle$ 1	
ν	N	overlap	ν	N	overlap	
$\frac{2}{13}$	4	$\langle E\Psi_7\|DPD^2\chi_2\rangle$ 0.999 921 885 9	$\frac{5}{17}$	4	$\langle HH\Psi_3\|DCDC\Psi_3\rangle$ 0.999 999 999 7	

$$\frac{\overline{z_{s,k}^s}z_{s,k}^l}{(1+z_{s,k}\overline{z_{s,k}})^{(\phi/2)+n-1}}. \tag{65}$$

This will be projected to

$$\frac{(\phi+1)!l!(\phi+n-1-l)!}{(\phi+n)!(l-s)!(\phi-l+s)!}\frac{z_{s,k}^{l-s}}{(1+z_{s,k}\overline{z_{s,k}})^{\phi/2}}. \tag{66}$$

As $(\phi+1)!/(\phi+n)!$ is a constant and is not dependent on s and l, we can discard it in the process of the projection. Thus P will act in the following way [discarding constant $(\phi+1)!/(\phi+n)!$]:

$$P\frac{\overline{z_{s,k}^s}F(z_{s,k})}{(1+z_{s,k}\overline{z_{s,k}})^{\phi/2+n-1}}=\frac{1}{(1+z_{s,k}\overline{z_{s,k}})^{\phi/2}}\frac{1}{(\phi-z_{s,k}\partial_{z_{s,k}})!}\times\partial_{z_{s,k}}^s(\phi+n-1-z_{s,k}\partial_{z_{s,k}})!F(z_{s,k}). \tag{67}$$

For example, by applying this formula to $\Psi=PD\chi_2$, the wave function is then given by

$$\Psi=\mathbf{A}\mathbf{N}P\prod_{i=1}^{N}\frac{1}{(1+z_i\overline{z_i})^{\phi/2}}\prod_{k=1}^{N/2}(\phi+1-z_{0,k}\partial_{z_{0,k}})\partial_{z_{1,k}}\times\prod_{s=0}^{1}(\Psi_{s,2})^3\prod_{k_1=1}^{N/2}\prod_{k_2=1}^{N/2}(z_{0,k_1}-z_{1,k_2})^2. \tag{68}$$

The trial wave function Ψ_c for filling $1-\nu$ is related to the trial wave function Ψ at filling ν by charge conjugation,

$$\Psi_c=\int\prod_i^M\frac{dz_{N+i}d\overline{z_{N+i}}}{(1+z_{N+i}\overline{z_{N+i}})^2}\times\overline{\Psi}(z_{N+i}\ldots z_{N+M})\Psi_1(z_1\ldots z_{N+M}) \tag{69}$$

where M is the number of particles in the state Ψ, $\underline{N}$ is the number of electrons in Ψ_c, $N+M=\phi+1$, and $\overline{\Psi}$ is the complex conjugate of Ψ. Note again, If we use $PD\chi_n$ as the trial wave function Ψ, the projection operator P can be dropped in Eq. (69). However if one uses $\Psi_{\rm hard}=\Psi_1P(\phi-M+1)\Psi_1\chi_n$ in Eq. (69), then the operator P *cannot* be dropped in Eq. (69).

One can also act D on Ψ, and obtain another trial wave function of the FQH state at filling $1/(2+\nu)$, where ν is the filling of the state Ψ. Repeatedly acting D and C on $PD\chi_n$, we can obtain the trial wave functions at all observable fillings.

VI. OVERLAPS BETWEEN HIERARCHICAL WAVE FUNCTIONS AND CF WAVE FUNCTIONS

We perform calculations of the wave functions symbolically by using MAPLE. The overlaps between the hierarchical wave function and the CF wave functions are calculated, Some overlaps between the wave functions with or without the hard-core construction are also calculated. The formula of the trial wave functions for the FQHE in the previous sections need to be normalized before we calculate the overlaps. Table II lists some overlaps at some fillings for a small number of electrons. $E\Psi$ means a state formed by the condensation of quasielectrons of a parent state Ψ, and $H\Psi$ means a state formed by the condensation of quasiholes of parent state Ψ. In all cases listed in the table, p_i is equal to

2 for $i>1$ in the constructions of the hierarchical states. The wave functions which involve D, P, and C operations are the CF wave functions.

When $N=3$ or 4, and $\phi=6$, there is only one rotational invariant state, which must also be the ground state. This explains why some of the overlaps in the table are exactly equal to 1.

Because of the limited CPU time we were allowed to use, we were only able to calculate some hierarchical wave functions up to six electrons, and some CF wave functions up to ten electrons. The detailed calculations can be found in Ref. 25. In the future, we will calculate wave functions with greater numbers of electrons. From the calculations, we conclude that the hierarchical wave functions and the CF wave functions are almost the same in the case of a small number of electrons.

VII. CONCLUSIONS

In this paper, we presented a detailed discussion of the calculation of trial wave functions on the sphere. The projective coordinates are used in performing the calculations. A self-contained derivation of the LL's on the sphere (or any surfaces with a constant curvature) using a geometrical method was also given. The many-body wave function in the LLL are studied and classified on angular momentum bases. We also simplify the formulas for the hierarchical wave functions and the CF wave functions.

There are many interesting things we want to study in the future. We shall use theories of polynomials to study those wave functions.[26,27] It would be very interesting if we could obtain the polynomials explicitly for the wave functions at an arbitrary number of electrons.

There is a mapping between a trial wave function in the FQHE and a wave function in one-dimensional space.[28] Because of the existence of the mapping, one may apply the method used to study the Calogero model to study the trial wave functions in the FQHE, and then it may be possible to calculate some physical quantities from the trial wave functions at an arbitrary number of electrons.

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APPENDIX: LANDAU LEVELS ON COMPACT CLOSED SURFACES

In this appendix, we will study LL's on general compact closed surfaces, and work out the LL's on the sphere as an example.

If the magnetic field and the curvature are constant, the spectrum, the wave functions and the degeneracy of Landau levels (LL's) can be obtained by using a very simple geometric argument.[18] A self-contained presentation of an idea based on Ref. 18 will be found in this appendix, and some examples will be included.

In the case when the surface is a plane, a sphere, or a torus, the spectrum and eigenfunctions of the LL's can be solved exactly.[29] For example, LL's on a sphere with a Dirac monopole at the origin, were solved by Dirac a long time ago. The problem in the case when the surface is an open upper-half hyperbolic plane with a constant negative curvature was solved completely. In this case there exists a discrete spectrum (this is the spectrum of the LL's) in the low-energy sector, and a continuous spectrum in the high-energy sector.[29]

Reference 18 studied the problem of LL's on compact closed Riemann surfaces with Poincaré metric, and obtained discrete low-energy eigenvalues (or LL's), their multiplicity and wave functions. Prior to Ref. 18, a similar problem also was studied, and discrete low-energy eigenvalues and their multiplicity were obtained by using results from the mathematical literature, for example, by using the Selberg trace formula (see the references quoted in Ref. 18).

Why can the problem of LL's in all the surfaces mentioned above be solved completely? By closely following the observation in Ref. 18, it is quite clear that the method developed in Ref. 18 can be easily generalized to the case of any *constant* curvature surface with a *constant* magnetic field applied on the surface (the surface can be a compact and closed surface, or an open surface; for example, an open up-half hyperbolic plane), and thus the problem of the LL's can be solved exactly in such cases.

We will show that, if the curvature and magnetic field are constant, we can obtain much information about the spectrum and the degeneracy of the LL's without solving the wave functions of the LL's explicitly by using a simple geometric argument,[18] even though the surface can be a very complicated one. If the magnetic field is constant, the wave functions of the ground states turn out to be a holomorphic line bundle defined on the surface. If the curvature of the surfaces is also constant, for the high LL's, the wave functions of the LL's are obtained by repeatedly applying covariant derivatives on some holomorphic line bundles (which will be specified later). The spectrum is obtained without solving the wave functions explicitly, and the degeneracy of the LL's can be obtained by the Riemann-Roch theorem. If the sections of some holomorphic line bundles can be obtained, the wave functions of the LL's can be obtained explicitly.

We use two simple examples to demonstrate how to use this geometric approach to solve LL's. Examples used are LL's on the sphere and the open upper-half hyperbolic plane.

1. Ground states

We will show here that, when the magnetic field is constant, the ground states satisfy a first-order holomorphic (or anti-holomorphic) differential equation, and the ground states belong to sections of a holomorphic line bundle. We consider a particle on a surface interacting with a magnetic field. In complex coordinates, the metric is $ds^2=g_{z\bar{z}}dzd\bar{z}$ and the volume form is $dv=[ig_{z\bar{z}}/2]dz\wedge d\bar{z}=g_{z\bar{z}}dx\wedge dy$. The natural definition of the constant magnetic field to the high genus Riemann surface is

$$F=Bdv=(\partial_z A_{\bar{z}}-\partial_{\bar{z}}A_z)dz\wedge d\bar{z}, \tag{A1}$$

where B is a constant. Thus we have $\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = i g_{z\bar{z}} B/2$. If the surface is closed, the magnetic field is then called a ''monopole'' field, and subjected to the Dirac quantization condition. The flux ϕ (ϕ must be an integer) is given by $2\pi\phi = \int F = BV$, where V is the area of the surface, and here we assume $B>0$ ($\phi>0$) for simplicity. The Hamiltonian of a particle on a surface is given by the following equation:

$$\begin{aligned} H &= \frac{1}{2m\sqrt{g}}(P_\mu - A_\mu) g^{\mu\nu}\sqrt{g}(P_\nu - A_\nu) \\ &= \frac{g^{z\bar{z}}}{m}[(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + (P_{\bar{z}} - A_{\bar{z}})(P_z - A_z)] \\ &= \frac{2g^{z\bar{z}}}{m}(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + \frac{B}{2m}, \end{aligned} \tag{A2}$$

where $g^{z\bar{z}} = 1/g_{z\bar{z}}$, $P_z = -i\partial_z$, $P_{\bar{z}} = -i\partial_{\bar{z}}$, $\partial_z = (\partial_x - i\partial_y)/2$, and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$. We define the inner product between two wave functions as $\langle\psi_1|\psi_2\rangle = \int dv\, \psi_1 \times \psi_2$.

Define $H' = (2g^{z\bar{z}}/m)(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}})$. H' is a positive definite Hermitian operator because $\langle\psi|H'|\psi\rangle \geq 0$ for any ψ. If $H'\psi = 0$, ψ must satisfy $(P_{\bar{z}} - A_{\bar{z}})\psi = 0$. The solutions of this equation are the ground states of the Hamiltonian H or H'. In the case of a closed compact surface, the existence of solutions of this equation is guaranteed by the Riemann-Roch theorem.[30,31] The solutions belong to the sections of the holomorphic line bundle, with the connection given by the gauge field. The Riemann-Roch theorem tells us that

$$h^0(L) - h^1(L) = \deg(L) - h + 1, \tag{A3}$$

where h is the genus of the surface; $h^0(L)$ is the dimension of the sections of the holomorphic line bundle or the degeneracy of the ground states of the Hamiltonian H; $h^1(L)$ is the dimension of the holomorphic differential $(L^{-1}\times K)$, where K is the canonical bundle; and $\deg(L)$ is the degree of the line bundle, which is equal to the first Chern number of the gauge field, or the magnetic flux out of the surface, ϕ. When $\deg(L) > 2h-2$, $h^1(L)$ is equal to zero,[30] thus $h^0(L) = \phi - h + 1$. One finds that $h^0(L)$ indeed gives the right degeneracy of the ground states in the case of a particle on a sphere or a torus interacting with a magnetic-monopole field.

In the case of noncompact surfaces, for example, an infinite plane or an upper-half hyperbolic plane, the flux out of the surfaces are infinite, and the degeneracy is infinite too. The degeneracy of the LL's turns out be infinite. Thus Eq. (A3) also correctly gives the degeneracy, as when the flux is infinite, the equation implies that $h^0(L)$ becomes infinite. When the surface has a boundary, for example a disc, one would expect that Eq. (A3) is replaced by an index relation given by the boundary index theory. Note that, when the flux is much larger than 1, the degeneracy of the ground states is approximately equal to the flux ϕ out of the surface.

2. Higher Landau levels

We study higher LL's in the case when the curvature of the surface is constant. When the curvature is constant, $g^{z\bar{z}}\partial\bar{\partial}\ln g_{z\bar{z}} = C$, a Liouville-like integrable equation. For the flat surface, $C=0$, as in the case of a plan or a torus, the spectrum and the wave functions of the LL's can be completely solved. When the surface is flat, higher LL's are obtained by successively applying a first-order differential operator to the states in the LLL. Now we shall generalize such a construction of LL's in the case of a flat surface to the case of a curved surface.

Here we consider a *closed* and *curved* surface with constant (nonzero) curvatures. It is easy to generalize to the case of an open surface with a constant curvature, and we will demonstrate it in an example in the end of the Appendix. When C is not equal to zero, one has $g_{z\bar{z}} = (1/C)\partial\bar{\partial}\ln g_{z\bar{z}}$. As the magnetic field is constant, we can fix the gauge field as $A_z = -iB'\partial(\ln g_{z\bar{z}})/2$, and the magnetic field F is equal to $B\,dv$, where $B = 2B'C$. For example, in the case of the Poincaré metric, $ds^2 = y^{-2}(dx^2 + dy^2)$, $g_{z\bar{z}} = y^{-2}$, and $C = \frac{1}{2}$; thus $B = B'$. For a closed surface, by Gauss theorem, the flux ϕ out of the surface is equal to $\phi = B(h-1)/c = 2B'(h-1)$. B' must be a rational number as ϕ is an integer. For the negative curvature closed surface, according to the Gauss theorem, we should have $h \geq 2$. On the other hand, for the positive constant curvature surface, h must be equal to zero, and thus the surface is topologically equivalent to a sphere. Without losing any generalities, we assume in the following discussions that B is a positive number. For a negative B, the wave functions are the complex conjugate of the wave functions in the case of a positive B.

Any eigenfunctions of the Hamiltonian satisfy

$$H\psi = E\psi. \tag{A4}$$

If the domain of $\tilde{z}$ intersects the domain of z nontrivially, $g_{z\bar{z}}dz\,d\bar{z}$ is invariant under coordinate changes, or

$$g_{z\bar{z}}dz\,d\bar{z} = g_{\tilde{z}\bar{z}}d\tilde{z}\,q\,d\tilde{\bar{z}} \tag{A5}$$

on the intersection of the domains of z and $\tilde{z}$. Define

$$D = \partial - (B'/2)\partial\ln g_{z\bar{z}}, \quad \bar{D} = \bar{\partial} + (B'/2)\bar{\partial}\ln g_{z\bar{z}}. \tag{A6}$$

D and $\bar{D}$ are transformed as

$$\tilde{D} = (dz/d\tilde{z})U^{-1}DU, \quad \tilde{\bar{D}} = (d\bar{z}/d\tilde{\bar{z}})U^{-1}\bar{D}U, \tag{A7}$$

where $U(z,\tilde{z}) = (dz/d\tilde{z})^{-B'/2}(d\bar{z}/d\tilde{\bar{z}})^{B'/2}$.

We take $m=2$ in Eq. (A2) for simplicity. The Hamiltonian can be written in the form

$$H = -g^{z\bar{z}}D\bar{D} + (B/4). \tag{A8}$$

Thus the Hamiltonian in the domain z is transformed to the Hamiltonian in the domain $\tilde{z}$ as

$$\tilde{H} = U^{-1}HU, \tag{A9}$$

and the wave function is transformed as

$$\tilde{\psi} = U^{-1}\psi. \tag{A10}$$

Therefore $\psi(dz)^{B'/2}(d\bar{z})^{-B'/2}$ is invariant under the transformation, and it implies that ψ is a differential form of type $T^{B'/2}_{B'/2}$ where we use the following notation: if

$F(z,\overline{z})(dz)^X(d\overline{z})^Y$ is invariant under the transformation, then $F(z,\overline{z})$ is a differential form of type T_X^{-Y}

The ground states are given by the solutions of the equation $D\psi=0$. When the curvature is negative, C, ϕ, and B' are positive numbers. If $\phi>2h-2$, or $B'>1$, then the number of the solutions is $\phi-h+1$ according to the previous discussions. For smaller ϕ, some discussions can be found in Ref. 18. In the case of compact and closed Riemann surfaces with the Poincaré metric, the wave functions in the LLL were constructed by calculating the determinant of holomorphic sections of some bundle.[18]

When the curvature is positive, C and B' are negative numbers, and $h=0$, as shown in the previous discussions. Now ϕ is equal to $\phi=-2B'$. The LLL states are again given by the solutions of the equation $D\psi=0$. As $|\phi|>2h-2=-2$ ($h=0$ in this case), the number of the solutions is equal to $|\phi|-h+1=|\phi|+1$.

To obtain the spectrum and wave functions of the higher LL's, we introduce the covariant derivative[18] ∇_z, and its Hermitian conjugate $(\nabla_z)^\dagger=-\nabla^z$,

$$\nabla_z: T_k^l \to T_{k+1}^l, \nabla_z = g^k \partial g^{-k}, \tag{A11a}$$

$$(\nabla_z)^\dagger: T_k^l \to T_{k-1}^l, (\nabla_z)^\dagger = -g^{-l-1}\overline{\partial} g^l, \tag{A11b}$$

and we call $g=g_{z\overline{z}}$ for short. Note that D is the covariant operator ∇_z acting on $T_{B'/2}^{\overline{B'/2}}$ and $\overline{D}=g\nabla^z$, where ∇^z acts on $T_{B'/2}^{\overline{B'/2}}$ The Hamiltonian can be written by using the covariant operators

$$H-B/4=-\nabla_z\nabla^z. \tag{A12}$$

One can verify the commutation relation

$$[\nabla^z\nabla_z]T_n^m=-(m+n)C. \tag{A13}$$

Assume that ψ_1 is a state in the higher LL's and an eigenfunction of H with eigenvalue E_1, then $\psi_1=-(1/\epsilon_1)\nabla_z\nabla^z\psi_1$, where $\epsilon_1=E_1-B/4>0$. Therefore one can write $\psi_1=\nabla_z\Phi(1)$, where $\Phi(1)$ is a differential form of type $T_{B'/2-1}^{\overline{B'/2}}$. More explicitly, we have

$$\psi_1=(\partial-(B'/2-1)\partial\ln g)\Phi(1). \tag{A14}$$

Using the relation $\partial\overline{\partial}\ln g=gC$, one can show that

$$-\nabla_z\nabla^z\psi_1=(B'-1)C\psi_1+\nabla_z[-\nabla_z\nabla^z\Phi(1)]. \tag{A15}$$

We first discuss the case of a negative curvature surfaces. If $B'\geqslant 1$, one can show that $\langle\psi_1|\nabla_z[-\nabla_z\nabla^z\Phi(1)]\rangle\geqslant 0$. Thus one can conclude that states of the lowest excited level are obtained, if there exist Φ such that $\nabla_z\nabla^z\Phi(1)=0$. $\nabla_z\nabla^z\Phi(1)=0$ leads to $D\Phi(1)=0$. The solution of $D\Phi(1)=0$ is $\Phi(1)=g^{-B'/2}\widetilde{\Phi}(1)$, with $\overline{\partial}\widetilde{\Phi}(1)=0$, where $\widetilde{\Phi}(1)$ is of the form $T_{B'-1}$. By the Riemann-Roch theorem, there exist solutions of the equation $\overline{\partial}\widetilde{\Phi}(1)=0$ for $B'\geqslant 1$, and the number of the solutions or the degeneracy of this Landau level is the dimension of the sections of the holomorphic bundle $T_{B'-1}$, which is equal to $(2B'-3)(h-1)$ if $B'>2$. The energy of this LL or the lowest excited states is

$$E_1=\tfrac{3}{2}B'C-C. \tag{A16}$$

If $B'<1$, there is only the zeroth ''Landau level'' or the LLL (there exists a continuous spectrum in the high-energy sector, and the states in the continuous spectrum are not the states in the LL's).

We can generalize the above discussion to higher LL's. The wave function of the k'th LL is given by

$$\psi_k=(\nabla_z)^k\Phi(k)=(\partial-(B'/2-1)\partial\ln g)(\partial-(B'/2-2)\partial\ln g)\times\cdots(\partial-(B'/2-k)\partial\ln g)\Phi(k), \tag{A17}$$

with $\Phi(k)=g^{-B'/2}\widetilde{\Phi}(k)$ and $\overline{\partial}\widetilde{\Phi}(k)=0$. $\widetilde{\Phi}(k)$ is a differential form of the type $T_{B'-k}$. Notice that this construction generalizes the standard construction of the harmonic oscillator. The difference between the constructions of the high LL's among the case of the flat surfaces and the case of the curved surfaces is clear now. In the case of the surface being a plan or a torus, high LL's are obtained by successively applying a first-order differential operator to the ground states. However, the situation is different when the surface is curved. $\Phi(k)$ for $k\neq 0$ is *not* the ground state of the Hamiltonian H.

Using Eq. (A13), we calculate the eigenvalue of the corresponding wave function ψ_k, and it is equal to

$$E_k=CB'(k+\tfrac{1}{2})-\frac{k(k+1)C}{2}. \tag{A18}$$

The degeneracy of the k'th LL is given by the dimension of the sections of the holomorphic bundle of the type $T_{B'-k}$, which is equal to $(2B'-2k-1)(h-1)$ when $B'-k>1$. Because the dimension of T_n is zero when n is negative, k must not be greater than B'. Hence there is only a *finite* number of ''Landau levels.''

When B' is an integer, k can take value from 0 to B'. When $k=B'$, the corresponding $\widetilde{\Phi}(k)$ is the differential form of the type T_0. T_0 is a constant function on the surface, and the degeneracy of this LL is equal to 1. For the twisted boundary conditions, which would physically correspond to the presence of some magnetic flux through the handles, there does not exist a nonzero constant function which satisfies the twisted boundary condition; thus the dimension of T_0 is zero, and the degeneracy of this LL is equal to zero or there does not exist a B'th LL. When $k=B'-1$, the degeneracy of this LL is the dimension of the canonical bundle T_1, which is equal to h for the nontwisted boundary condition, and to $h-1$ for the twisted boundary condition (this result can be obtained by the Riemann-Roch theorem). B could be also a half-integer. Then k can take values from 0 to $B'-\frac{1}{2}$. When $k=B'-\frac{1}{2}$, the degeneracy of this LL is the dimension of the spin bundle $T_{1/2}$. The dimension of the holomorphic sections of the spin bundle generically is zero for the even-spin structures and one for the odd ones (or for twisted ones). It is possible that B' is fractional assuming that $2B'(h-1)$ is an integer, and k can take values from 0 to $[B']$, where $[B']$ is the biggest integer smaller than B'. When $k=[B']$, the degeneracy of this LL is the dimension of bundle $T_{B'-[B']}$. $B'-[B']$ is a fractional number between 0 and 1, and the discussions of such a case can be found in Ref. 18. Beyond those LL's, little is known about the continuous spectrum in the case of the complicated negative curvature surfaces.

To normalize ψ_k, we calculate the inner product $\langle\psi_k|\psi_k\rangle$ by using Eq. (A13). It is given by the following equation:

$$\begin{aligned}\langle\psi_k|\psi_k\rangle &= \langle\nabla_z^k\Phi(k)|\nabla_z^k\Phi(k)\rangle \\ &= \langle\Phi(k)|(\nabla_z^k)^\dagger\nabla_z^k\Phi(k)\rangle \\ &= \langle\Phi(k)|\Phi(k)\rangle C^k 2^{-k}k! \\ &\quad\times\prod_{i=1}^{k}(2B'-k-i),\end{aligned} \tag{A19}$$

where the inner product $\langle\Phi(k)|\Phi(k)\rangle$ is defined as

$$\langle\Phi(k)|\Phi(k)\rangle = \int dv\, g^k\overline{\Phi}(k)\times\Phi(k). \tag{A20}$$

The definition of the inner product between the two $\Phi(k)$'s given in Eq. (A20) is quite natural, because $\Phi(k)$ is a differential form of the type $T^{\overline{B'/2}}_{(B'/2)-k}$

If $\Phi(k)$ is normalized to 1, then

$$\psi_k \Bigg/ \left[C^k k! 2^{-k}\prod_{i=1}^{k}(2B'-k-i)\right]^{1/2} \tag{A21}$$

is also normalized to 1.

Now we come to the case of a closed surface with a positive curvature, which is slightly different from the case of a surface with a negative curvature. Now we have only $h=0$ according to the previous discussion. The wave function ψ is a differential form of type $T^{\overline{B'/2}}_{B'/2}$ with B' being a negative number. In the formula $-\nabla_z\nabla^z\psi_1 = (B'-1)C\psi_1+\nabla_z[-\nabla_z\nabla^z\Phi(1)]$, one can show that $\langle\psi_1|\nabla_z(-\nabla_z\nabla^z\Phi)\rangle\geq 0$ for any negative B'. By using the Riemann-Roch theorem, one finds that there *always* exists $\Phi(1)$ such that $\nabla_z\nabla^z\Phi(1)=0$, which leads to $D\Phi=0$. Therefore, for any B', there exists a higher LL. One can repeat the argument to obtain the states in the higher LL's, and obtain the *full* spectrum and wave functions.

The wave functions of the states in the k'th LL are again given by Eq. (A17), with $\widetilde{\Phi}(k)=g^{B'/2}\Phi(k)$ and $\partial\widetilde{\Phi}(k)=0$. $\widetilde{\Phi}(k)$ is a differential form of the type $T_{B'-k}$. The degeneracy of the k'th LL is equal to the dimension of the holomorphic line bundle $T_{B'-k}$, which is equal to $2(B'-k)(h-1)-h+1=-2(B'-k)+1$, as h is equal to zero.

The energy is again given by Eq. (A18). However, a higher LL has a *higher* degeneracy, and the number of the LL's is *infinite* in such case. Instead, in the case of a surface with a negative curvature, a higher LL has a *smaller* degeneracy. and the number of LL's is *finite*. From Eq. (A18), one notices that, in the case of a positive curvature surface, the energy gap in the neighboring LL's *increases* when the level increases, and, in the case of a negative curvature surface, the energy gap in the neighboring LL's *decreases* when the level increases.

It is easy to generalize the above discussions to noncompact surfaces, and we will work out an example in the following discussion.

3. Examples

a. Upper-half hyperbolic surface

We consider that the surface is a upper-half hyperbolic surface (also see Comtet and Dunne in Ref. 29). In the projective coordinates. the metric g is written as $1/(1-z\overline{z})^2$, where $|z|\leq 1$, The other quantities are $C=2$, $B=4B'$, and $A_z=-i[B'\overline{z}/(1-z\overline{z})^2]$. The wave functions are given by Eq. (A17). As the wave functions of the LL's shall be normalizable (opposite to the wave function of a state inside the continue spectrum), $\langle\psi_k|\psi_k\rangle$ shall be normalizable. A normalizable $\langle\psi_k|\psi_k\rangle$ is equivalent to a normalizable $\langle\Phi(k)|\Phi(k)\rangle$. A normalizable $\langle\Phi(k)|\Phi(k)\rangle$ leads the condition $B'-\frac{1}{2}>k\geq 0$. $\Phi(k)$ is given by function $g^{-B'/2}z^l$, where l is a non-negative integer. Thus the degeneracy is infinite for every LL. This is consistent with the Riemann-Roch theorem, as the flux out of the surface is infinite. Finally, the energy is given by Eq. (A18).

b. Sphere

Another example is that the surface is a sphere. In the projective coordinates. the metric g is written as $g=[1/(1+z\overline{z})^2]$. The other quantities are $A_z=i[B'\overline{z}/(1+z\overline{z})^2]$, $C=-2$, and $B=2B'C=-4B'$. Thus the flux $\phi=-2B'$ is a non-negative integer according to the Dirac quantization condition (note we always assume $B>0$ in this paper). The wave functions are again given by Eq. (A17), the energies are given by Eq. (A18), and $\Phi(k)$ is given by $g^{\phi/4}z^l$. The normalizable condition leads to $l=0,1,\ldots,2k+\phi$. Thus the degeneracy of the k'th LL is equal to $2k+\phi+1$. The degeneracy can be also obtained by the Riemann-Roch theorem, and the result is consistent with the result obtained by requiring the wave functions to be normalizable. In this way, we obtain the full spectrum and all wave functions on the sphere. The wave functions at the n'th Landau level ($n=0$ is the lowest Landau level) are given by Eq. (7).

From previous discussions, we can easily find the inner product $\langle\psi_{n,l}|\psi_{n,l}\rangle$ is equal to $\pi[l!(\phi+2n-l)!/(\phi+2n+1)(\phi+n)!]$. The inner product is as previously defined, $\langle\psi_1|\psi_2\rangle=\int dv\,\psi_1\times\psi_2$, where $dv=\int[dxdy/(1+z\overline{z})^2]$.

However, inside the paper, the definition of the inner product is *different* from the definition in the Appendix. The inner product in the paper is *defined* as

$$\langle\psi_1|\psi_2\rangle = \int\frac{dz\,d\overline{z}}{(1+z\overline{z})^2}\,\overline{\psi_1}\times\psi_2. \tag{A22}$$

As $dz\,d\overline{z}=2dx\,dy$, thus $\langle\psi_{n,l}|\psi_{n,l}\rangle$ is given by the following formula:

$$\langle\psi_{n,l}|\psi_{n,l}\rangle = 2\pi\frac{l!(\phi+2n-l)!}{(\phi+2n+1)(\phi+n)!}. \tag{A23}$$

This formula is used in the paper.

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