PHYSICAL REVIEW B VOLUME 55, NUMBER 21 1 JUNE 1997-I

# Double transitions in the fully frustrated $XY$ model

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(Received 6 January 1997)

The fully frustrated $XY$ model is studied via the position-space renormalization group approach. The model is mapped into two coupled $XY$ models, for which the scaling equations are derived. By integrating directly the scaling equations, we observe that there exists a narrow temperature range in which both the vortex and coupling charge fugacities grow large, suggesting double transitions in the system. While the transition at lower temperature is identified to be of the Kosterlitz-Thouless type, the higher-temperature one appears not to be of the Ising universality class. [S0163-1829(97)03331-6]

The two-dimensional (2D) fully frustrated $XY$ (FF$XY$) model, which was originally proposed as an $XY$ version of the magnetic systems possessing frustration without disorder,[1] is now well known to describe the superconducting array under an external transverse magnetic field with the magnetic flux per plaquette given by half a flux quantum.[2] In this model there exists a discrete $Z_2$ symmetry in addition to the continuous $U(1)$ symmetry. The latter is inherent in the $XY$ model, related to the global rotation of the spins (phases); the former, introduced by the external magnetic field, describes the twofold degeneracy associated with the two types of the chirality ordering. Naturally, this $Z_2$ symmetry together with the U(1) symmetry leads to the interesting possibility of the two kinds of phase transitions: a Kosterlitz-Thouless (KT) type transition and an Ising-like one.

Since the observation of the KT-like and the Ising-like transitions at very close, if not equal, temperatures,[2] extensive works[3–15] have been performed to investigate the phase transition in this system. Still the nature of the phase transition is rather inconclusive, and in particular, there exists controversy as to whether the two kinds of transitions occur at the same temperature or separately at two different temperatures. Renormalization group (RG) approaches[3] have given results consistent with a single transition which is a combination of a KT-like one for spins and an Ising-like one for chirality. In the generalized uniformly frustrated $XY$ model, where the frustration is varied by changing the negative bond strength,[4] it has been concluded that the two transitions merge into a single transition in the fully frustrated case. Similar conclusion was reached in the Monte Carlo study of another generalized model.[5] While the FF$XY$ model was also argued to be equivalent to a superconformal field theory of central charge $c=3/2$,[6] subsequent Monte Carlo transfer-matrix studies[7] appear to yield larger values of the central charge and critical exponents which differ significantly from those of a pure Ising model. These exponents are in agreement with those on the single transition line of the coupled $XY$-Ising model,[8] which suggests a single transition of a new universality class in the FF$XY$ model. This single-transition scenario has also been favored by Monte Carlo simulations of the FF$XY$ model[9,10] and of the coupled $XY$-Ising model.[11]

In contrast to this single-transition scenario, finite-size scaling analysis of Monte Carlo results has demonstrated double transitions in the Coulomb gas system of half-integer charges,[12,13,16] which is believed to be in the same universality class as the FF$XY$ model. In particular, the higher-temperature transition has been found to be of the different universality class from the pure Ising one,[13] suggesting that the non-Ising exponents of the Ising-like order parameter may not be regarded as evidence for the single transition. High-precision Monte Carlo simulations of the FF$XY$ model[14] has also led to two transitions at slightly different temperatures. Further, the chirality-lattice melting transition at the higher transition temperature was suggested to belong to a new universality class rather than to the Ising one. On the other hand, the recent argument that the previously obtained non-Ising exponents are artifacts of the invalid scaling assumption[15] has raised another controversy.[17]

Here we investigate the phase transition of the 2D FF$XY$ model on a square lattice via the position-space RG study. We use the decomposition of the FF$XY$ model into two coupled $XY$ models, for which the Kosterlitz-like RG procedure yields the scaling equations. We integrate the scaling equations directly from the parameters corresponding to the original FF$XY$ model, and reexamine in detail the renormalization procedure. In particular, we observe the renormalization behavior of the vortex fugacity which diverges above the KT-type vortex unbinding transition temperature and that of the coupling charge fugacity which diverges below the Ising-like transition temperature. It is found that there exists a narrow temperature range where both fugacities grow large. This reveals that the two transitions occur at slightly different temperatures, which is consistent with the existing numerical results.[12–15]

The uniformly frustrated $XY$ model is described by the Hamiltonian

$$-\beta H_0=K_0\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}\cos(\phi_{\mathbf{r}}-\phi_{\mathbf{r}'}-A_{\mathbf{r}\mathbf{r}'}),\tag{1}$$

where $K_0\equiv\beta J\equiv J/k_BT$ with $J$ being the coupling strength, $\phi_{\mathbf{r}}$ is the angle of the $XY$ spin at site $\mathbf{r}$, and $A_{\mathbf{r}\mathbf{r}'}$ is the bond angle such that the plaquette sum is constant over the whole lattice:

$$\sum_P A_{\mathbf{r}\mathbf{r}'}=2\pi f,$$

 

with $f=1/2$ in the fully frustrated case. The Hamiltonian in Eq. (1) also describes the Josephson junction array under a transverse magnetic field in the high-capacitance limit, where $\phi_{\mathbf{r}}$ corresponds to the phase of the condensate wave function at site $\mathbf{r}$ and the bond angle $A_{\mathbf{rr}'}$ is given by the line integral of the vector potential $\mathbf{A}$:

$$A_{\mathbf{rr}'}=\frac{2e}{\hbar c}\int_{\mathbf{r}}^{\mathbf{r}'}\mathbf{A}\cdot d\mathbf{l}.$$

By constructing the corresponding Landau-Ginzburg-Wilson Hamiltonians, one can demonstrate that the FF$XY$ model may be decomposed into two coupled $XY$ models described by the Hamiltonian[18]

$$-\beta H=K\sum_{\langle\mathbf{r},\mathbf{r}'\rangle}[\cos(\theta_{\mathbf{r}}^{(1)}-\theta_{\mathbf{r}'}^{(1)})+\cos(\theta_{\mathbf{r}}^{(2)}-\theta_{\mathbf{r}'}^{(2)})]+h\sum_{\mathbf{r}}\cos[2(\theta_{\mathbf{r}}^{(1)}-\theta_{\mathbf{r}}^{(2)})], \tag{2}$$

where the effective interaction $K$ and the mode-coupling field $h$ depend on the original coupling $K_0$ in Eq. (1) via the relations $K=K_0/\sqrt{2}$ and $h=2K_0^4$. Note that the mode-coupling term in Eq. (2) leads to twofold degenerate ground states, in addition to the continuous degeneracy due to the global rotation, according to the parallel or antiparallel arrangement of $\theta^{(1)}$ and $\theta^{(2)}$. We thus expect an Ising-like transition associated with the coupling between $\theta^{(1)}$ and $\theta^{(2)}$, as well as the KT-type transition yielding quasi-long-range order for both $\theta^{(1)}$ and $\theta^{(2)}$.

The dual transformation allows us to write Eq. (2) in the form of a Coulomb gas Hamiltonian[3,19]

$$-\beta H_c=\pi K\sum_{\mathbf{R}\neq\mathbf{R}'}\left[m(\mathbf{R})\ln\left|\frac{\mathbf{R}-\mathbf{R}'}{a}\right|m(\mathbf{R}')+n(\mathbf{R})\ln\left|\frac{\mathbf{R}-\mathbf{R}'}{a}\right|n(\mathbf{R}')\right]+2\pi(K-\widetilde{K})\sum_{\mathbf{R}\neq\mathbf{R}'}m(\mathbf{R})\ln\left|\frac{\mathbf{R}-\mathbf{R}'}{a}\right|n(\mathbf{R}')$$
$$+\frac{2}{\pi\widetilde{K}}\sum_{\mathbf{r}\neq\mathbf{r}'}q(\mathbf{r})\ln\left|\frac{\mathbf{r}-\mathbf{r}'}{a}\right|q(\mathbf{r}')+2i\sum_{\mathbf{r},\mathbf{R}}q(\mathbf{r})\Theta(\mathbf{r}-\mathbf{R})[m(\mathbf{R})-n(\mathbf{R})]+\ln y\sum_{\mathbf{R}}[m^2(\mathbf{R})+n^2(\mathbf{R})]+\ln\widetilde{y}\sum_{\mathbf{r}}q^2(\mathbf{r}), \tag{3}$$

where $y$ and $\widetilde{y}$ denote the vortex fugacity and the coupling charge fugacity, respectively, and $\mathbf{R}$ represents the position of the dual lattice site. The function $\Theta(\mathbf{r})$ is given by

$$\Theta(\mathbf{r})\equiv\tan^{-1}(y/x)\quad\text{for }\mathbf{r}\equiv(x,y),$$

and the vortex charges $m(\mathbf{R})$ and $n(\mathbf{R})$ as well as the coupling charge $q(\mathbf{r})$ satisfy the charge neutrality conditions:

$$\sum_{\mathbf{R}}m(\mathbf{R})=\sum_{\mathbf{R}}n(\mathbf{R})=\sum_{\mathbf{r}}q(\mathbf{r})=0.$$

In this dual representation, the Hamiltonian has been generalized to include the off-diagonal coupling $K-\widetilde{K}$ between the two modes since such off-diagonal coupling is in general generated during the process of renormalization. The original model described by Eq. (2) corresponds to the case $\widetilde{K}=K$.

From the Hamiltonian in Eq. (3) one can derive the RG equations by means of the position-space RG technique. The procedure is entirely similar to that in Ref. 3, and will not be repeated here. The resulting scaling equations to the lowest order in the vortex fugacity $y$ and the coupling charge fugacity $\widetilde{y}$ read[19]

$$\frac{dK}{dl}=-4\pi^3[K^2+(K-\widetilde{K})^2]y^2+4\pi\widetilde{y}^2,$$

$$\frac{d\widetilde{K}}{dl}=-4\pi^3\widetilde{K}^2y^2+8\pi\widetilde{y}^2,$$

$$\frac{dy}{dl}=(2-\pi K)y,$$

$$\frac{d\widetilde{y}}{dl}=\left(2-\frac{2}{\pi\widetilde{K}}\right)\widetilde{y}, \tag{4}$$

where the physical line corresponding to the original FF$XY$ model is given by

$$K = \frac{1}{\sqrt{2}} K_0,$$

$$\tilde{K} = \frac{1}{\sqrt{2}} K_0,$$

$$y = \exp\left[ -\frac{\pi^2}{2\sqrt{2}} K_0 \right],$$

$$\tilde{y} = \exp\left[ -\frac{\sqrt{2}}{K_0} \right] \frac{I_1(2K_0^4)}{I_0(2K_0^4)}, \qquad (5)$$

with $I_n(x)$ being the $n$th modified Bessel function of the first kind. As is well known, the KT-type transition temperature $T_{\mathrm{KT}}$ is given by the minimum temperature above which the vortex fugacity $y$ renormalizes to infinity. Above $T_{\mathrm{KT}}$, consequently, free vortices condensate, which destroys quasi-long-range order in the system. Similarly, the Ising-like transition temperature $T_{\mathrm{c}}$ may be identified with the maximum temperature below which the coupling charge fugacity $\tilde{y}$ grows large as the renormalization proceeds. Here large fugacity $\tilde{y}$ corresponds to strong coupling between $\theta^{(1)}$ and $\theta^{(2)}$, which results in the long-range order for chirality.

The direct integration of the scaling equations (4) shows that at temperatures below $T_{\mathrm{KT}}$ the vortex fugacity $y$ decreases to zero while the coupling charge fugacity $\tilde{y}$ grows large with the renormalization process; this indicates that the system possesses both the quasi-long-range order for phases and the long-range order for chirality. On the other hand, at temperatures above $T_c$, the system is found to flow into a fixed point of the large vortex fugacity and the zero coupling charge fugacity, which corresponds to the absence of both phase ordering and chirality ordering. Here it should be noted that the scaling equations (4) are accurate only up to the lowest orders in the fugacities and that they cannot describe exact renormalization behavior as the fugacities become large. Further, it is practically impossible to prove numerically divergence of the fugacity. As a criterion for divergence, a cutoff for the fugacities is thus introduced and the fugacity is regarded as divergent if it grows steadily to the cutoff with the renormalization. Setting the cutoff equal to unity, we obtain $1.2857 < T_{\mathrm{KT}} < 1.2858$ and $1.2864 < T_c < 1.2865$ in units of $J/k_B$. These values of the transition temperatures depend rather insensitively on the values of the cutoff. In particular, it has been confirmed that the nature of the phase transition does not depend on the precise value of the cutoff.

It is thus concluded that the FF$XY$ model exhibits the ordered phase below $T_{\mathrm{KT}}$ and the disordered phase above $T_c$. Note that $T_{\mathrm{KT}}$ is lower than $T_c$, which raises a question as to the nature of the phase at temperatures between $T_{\mathrm{KT}}$ and $T_c$. Figure 1 displays the renormalization behaviors of the vortex fugacity and of the coupling charge fugacity at temperature $T = 1.2860$. It is evident that both fugacities grow large in a steady manner, indicating that in spite of the absence of phase ordering, chirality ordering still persists. It is found that these behaviors of the fugacities are ubiquitous at all temperatures between $T_{\mathrm{KT}}$ and $T_c$. This observation, which has been missed in the previous RG studies,[3] apparently suggests double transitions in the FF$XY$ system, in good agreement with the recent numerical studies.[12–15]

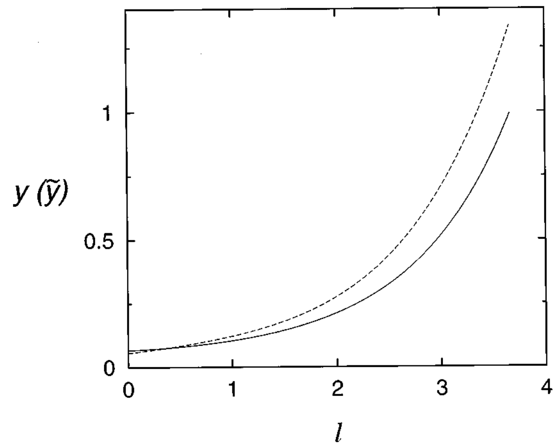


FIG. 1. Renormalization curves of the vortex fugacity $y$ and the coupling charge fugacity $\tilde{y}$ at temperature $T = 1.2860$ (in units of $J/k_B$). The solid line and the dashed line represent the renormalization behaviors of $y$ and $\tilde{y}$, respectively.

The lower-temperature transition is expected to be of the KT type since this transition concerns the breaking of the quasi-long-range order in the phases, driven by vortex pair dissociation. It is, however, not exactly the same as the standard KT transition. In the absence of the mode-coupling term, each mode would undergo a KT transition with the universal jump in the helicity modulus. This implies that the jump in the original FF$XY$ model is greater than the universal one since the coupling in the original model is given by the relation $K_0 = \sqrt{2}K$. Here the coupling field tends to suppress the jump, decreasing its value toward the universal value. Notwithstanding this, the jump is not restored to the universal one, resulting in the nonuniversal jump greater than the universal one. This nonuniversal jump was widely pointed out in existing works.[5,9,12–14] Interestingly, the nature of the intermediate phase in the two coupled $XY$ model also suggests the possibility that the higher-temperature transition associated with the chirality ordering may not be of the pure Ising type. In the intermediate phase, the mode-coupling field arranges the two modes $\theta_i^{(1)}$ and $\theta_i^{(2)}$ to be parallel or antiparallel to each other; still there is no phase ordering in each mode. Accordingly, the parallel or antiparallel arrangement of the phase at one site is more or less independent of the arrangement at other sites. This phase is clearly different from the ordered phase in the usual Ising model, which appears to suggest non-Ising behavior of the higher-temperature transition in the FF$XY$ model.

In summary, we have studied the phase transition of the fully frustrated $XY$ model through the use of the position-space renormalization group technique. We have used the decomposition of the fully frustrated $XY$ model into two coupled $XY$ models, which may be justified by constructing the Landau-Ginzburg-Wilson Hamiltonians. The direct integration of the scaling equations derived for the two coupled $XY$ models has shown that a Kosterlitz-Thouless type transition and an Ising-like one occur at different transition temperatures. The intermediate phase, which exists in a narrow temperature range between the two transition temperatures, is characterized by the absence of the spin (phase) order and the presence of the long-range order for chirality. It has been further pointed out that the lower-temperature transition is characterized by a larger-than-universal jump in the helicity modulus while the higher-temperature one displays non-Ising critical behaviors.

We acknowledge the partial support from the Basic Science Research Institute Program, Ministry of Education of Korea and from the Korea Science and Engineering Foundation through the SRC Program.


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[19] For more general RG study, one may introduce an additional term $\ln\alpha\Sigma_{\mathbf{R}}m(\mathbf{R})n(\mathbf{R})$ in Eq. (3), where $\alpha$ serves as the fugacity of combined vortices (Ref. 3). This in turn generates an additional term $-4\pi^3(2K-\widetilde{K})^2\alpha^2$ in the first of the scaling equations (4) and yields the scaling equation describing the renormalization of $\alpha$: $d\alpha/dl=[2-\pi(2K-\widetilde{K})]\alpha$, with the initial condition $\alpha=\exp[-(\pi^2/\sqrt{2})K_0]$. This generalization, however, has no effect on the nature of the transition and even does not change appreciably the estimated values of the transition temperatures.