Stochastic wave growth in scattering media

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Stochastic growth of waves is studied in a scattering medium in which the wave propagation is diffusive. This analysis generalizes previous work on randomly growing waves in nonscattering media and on constant growth in scattering media. New results are obtained for the stochastic laser threshold, the distribution of photons in the medium, the impulse response of the medium, and the transmission and reflection coefficients of a slab of random amplifying medium. [S0163-1829(97)12517-6]

I. INTRODUCTION

The purpose of this paper is to study growth of waves in media in which they diffuse spatially due to scattering while growing at a randomly varying rate. Propagation, growth, and damping of waves in scattering media have been intensively studied, with considerable attention being paid to cases in which photons can be considered to diffuse in space.[1–6] In parallel, the theory of random growth of unscattered waves has been developed in the plasma physics literature.[7,8] The behavior of nonrandomly damped or growing diffusing waves has also been investigated.[3,5] However, the case of simultaneous random growth and spatial diffusion does not appear to have been treated to date.

Random growth of diffusing waves is expected to be relevant to the theory of laser action in scattering media, where one cannot realistically expect the growth rate to be uniform when the refractive properties of the medium are randomly varying. Such a medium could be realized by mixing powders of a lasing medium and an absorbing one, for example. Similarly, in plasma physics there are situations in which waves are simultaneously pumped and scattered, for example waves driven by electron beams in space and the laboratory. In such cases, the rate of wave growth or damping is expected to be a random function of position and/or time because of the effects of turbulence, for example.

In Sec. II of this paper we briefly outline the basic theory of random growth of scattered waves. The important role played by boundary conditions is seen in Sec. III in which the case of a finite slab source with open boundaries is considered, generalizing previous work for waves damped at a constant rate.[3] In this section we calculate the laser threshold and, below threshold, quantities such as the probability distribution of photons, the power output from the slab, and the reflection and transmission coefficients for incident light. Section IV outlines corresponding results for cases in which one or both boundaries are reflecting.

Before proceeding, it should be noted that this analysis is complementary to recent work on localized waves in media with random loss and gain, which has extended standard localization theory of undamped waves to incorporate a variety of types of random loss and gain.[9,10] Strong localization of light in a time-independent medium would preclude the diffusive propagation of photons assumed here, while weak localization would introduce phase-sensitive phenomena not considered here.

II. BASIC THEORY

In this section we consider waves propagating in a one-dimensional system in which they are scattered or their growth rate varies randomly, but not both. This lays the basis for subsequent sections in which these two effects are combined. Spatial diffusion in the one dimension considered should more generally be considered to approximate the one-dimensional projection of the true motion of photons in three dimensions. A more sophisticated analysis would incorporate the fixed velocity of the waves and allow only diffusion in angle, but the present approximation has proved to be adequate for many purposes.[1–6]

A. Spatial scattering, no growth or damping

If undamped waves are scattered, the mean velocity of their photons decays on the same scale as the scattering length, with propagation dominated by photon diffusion on longer scales. This behavior can be encapsulated in the following pair of stochastic differential equations for photon position x and velocity v:

$$\frac{dx}{dt}=v, \tag{1}$$

$$\frac{dv}{dt}=-k_v v+D_{vv}^{1/2}\xi(t), \tag{2}$$

where t is time, k_v is the scattering rate, D_{vv} is the diffusion coefficient that results from multiple scatterings, and $\xi(t)$ is a white noise signal (correlations between scattering centers are neglected here).

Equations (1) and (2) are the equations for Brownian motion. They can be solved exactly in an infinite medium giving the probability distribution[11]

$$p(x,v,t)=\frac{1}{2\pi(\det\sigma)^{1/2}}\exp\left[-\frac{1}{2\det\sigma}(\sigma_{xx}x'^2 -2\sigma_{xv}x'v'+\sigma_{vv}v'^2)\right], \tag{3}$$

with $x'=x-\langle x(t)\rangle$ and $v'=v-\langle v(t)\rangle$. The elements of the variance matrix σ are given by

$$\sigma_{xx}(t)=\frac{D_{vv}}{2k_v^3}(2k_vt-3+4e^{-k_vt}-e^{-2k_vt}), \tag{4}$$

$$\sigma_{xv}(t)=\frac{D_{vv}}{2k_v^2}(1-e^{-k_vt})^2, \tag{5}$$

$$\sigma_{vv}(t)=\frac{D_{vv}}{2k_v}(1-e^{-2k_vt}), \tag{6}$$

with $\det\sigma=\sigma_{xx}\sigma_{vv}-(\sigma_{xv})^2$. Relevant moments of this distribution, and its variance in x, are given by

$$\langle v(t)\rangle=v_0e^{-k_vt}, \tag{7}$$

$$\langle x(t)\rangle=(v_0/k_v)[1-\exp(-k_vt)], \tag{8}$$

$$\sigma^2(x,t)=\sigma_{xx}(t), \tag{9}$$

respectively, where v_0 is the initial velocity and the photons all start at $x=0$. Hence, for times $t\gtrsim 1/k$, diffusion dominates. This implies that a *diffusive approximation* can be made for $b\gg v_0/k_v$, in which photons are assumed to propagate ballistically to a point $x\approx v_0/k_v$, then undergo pure diffusion.

B. Stochastic growth, no spatial scattering

Robinson[7,8] analyzed a model in which radiation is not explicitly scattered, but undergoes random growth and damping. In this model, the energy in the waves undergoes a random walk in the logarithm G of the energy density W [i.e., the net gain relative to a reference level W_0 at $t=0$, with $G=\ln(W/W_0)$]. In the absence of saturation mechanisms, this results in a probability distribution $p(G,t)$ at time t given by

$$p(G,t)=\frac{1}{(2\pi)^{1/2}\sigma(G,t)}\exp\left[-\frac{[G-\langle G(t)\rangle]^2}{2\sigma^2(G,t)}\right], \tag{10}$$

$$\sigma^2(G,t)=2\sigma^2(\Gamma)t_\Gamma t, \tag{11}$$

$$\langle G(t)\rangle=\langle\Gamma\rangle t, \tag{12}$$

where $\langle\Gamma\rangle$ is the mean growth rate, $\sigma(\Gamma)$ its standard deviation, and t_Γ its coherence time. If there is no scattering, the reflection coefficient for waves incident normally on a slab at $0\leq x\leq b$ is zero (assuming no refractive index mismatch at the edge), and the relevant time is $t=b/v_0$, where v_0 is the incident wave velocity.

The mean energy density is obtained by averaging $W=W_0e^G$ over $p(G,b/v_0)$; likewise, the standard deviation $\sigma(W)$ can be calculated, giving[8]

$$\langle W(t)\rangle=W_0\langle e^G\rangle, \tag{13}$$

$$=W_0\exp[\langle G(t)\rangle+\tfrac{1}{2}\sigma^2(G,t)], \tag{14}$$

$$=W_0\exp(\Gamma_{\rm eff}t), \tag{15}$$

$$\Gamma_{\rm eff}=\langle\Gamma\rangle+\sigma^2(\Gamma)t_\Gamma, \tag{16}$$

$$\sigma^2(W)=\langle W(t)\rangle^2[\exp\{\sigma^2(G,t)\}-1]. \tag{17}$$

These results imply $\sigma(W)/\langle W\rangle\approx\exp[\sigma^2(G,t)/2]\gg 1$ for $\sigma(G)\gtrsim 2$. A key result is Eq. (16), which implies that net growth can occur even when the mean growth rate is negative, because the exponential factor in Eq. (13) weights growth more heavily than damping, resulting in the term $\sigma^2(\Gamma)t_\Gamma$ in Eq. (16). The net transmission coefficient is found by dividing Eq. (15) by W_0 and setting $t=b/v_0$.

For large W saturation effects may become important. We do not discuss these effects in this paper because their incorporation into stochastic growth theory has been discussed elsewhere.[8] Pumping and saturation effects in scattering media with nonrandom growth rates have also been studied previously.[3,5]

III. FINITE MEDIUM WITH OPEN BOUNDARIES

In this section we consider growth of waves in a slab medium lying between open boundaries at $x=0$ and $x=b$ and using the diffusion approximation discussed in Sec. II A. Section III A deals with scattering in the absence of growth or damping, while Sec. III B builds upon these results to incorporate a randomly varying growth rate.

A. Diffusion without growth or damping

In the case of pure diffusion on an interval $[0,b]$, with absorbing boundaries to account for the escape of radiation from the faces of the slab, we can write down the following Fokker-Planck equation for the evolution of the probability distribution of position

$$\partial_t p(x,t)=\tfrac{1}{2}D_{xx}\partial_x^2 p(x,t), \tag{18}$$

where D_{xx} is a constant spatial diffusion coefficient. This can be transformed into the equation

$$\partial_T p(X,T)=\tfrac{1}{2}\partial_X^2 p(X,T), \tag{19}$$

on the interval $[0,1]$ if we make the substitutions

$$X=x/b, \tag{20}$$

$$T=tD_{xx}/b^2. \tag{21}$$

The problem of diffusion on the interval $[0,1]$ with absorbing boundaries can be solved straightforwardly in closed form (Ref. 11, p. 132) for a distribution initially localized at X_0 in the interval. The result is

$$p(X,T)=2\sum_{n=1}^{\infty}e^{-\lambda_nT}\sin(n\pi X_0)\sin(n\pi X), \tag{22}$$

$$\lambda_n=n^2\pi^2/2. \tag{23}$$

Equations (22) and (23) express the Green function of Eq. (18) or, equivalently, the impulse response of the medium. The functions $\sin(n\pi X)$ are eigenfunctions that satisfy the open-boundary requirement $p(0,T)=p(1,T)=0$ for all T.

The corresponding probability current is given by

$$j(X,T)=-\tfrac{1}{2}\,\partial_X p(X,T), \tag{24}$$

$$=-\sum_{n=1}^{\infty} n\pi e^{-\lambda_n T}\sin(n\pi X_0)\cos(n\pi X). \tag{25}$$

This current equals the net power $P(X,T)$ crossing a plane at X in the positive direction at time T.

The expectation values of the energy U (proportional to the number of photons) passing rightward through the left and right faces of the slab (a negative quantity at the left face) are

$$U(X)=\int_0^\infty j(X,T)dT, \tag{26}$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi X_0)\cos(n\pi X), \tag{27}$$

with $X=0$ and $X=1$, respectively. This gives[12,13]

$$|U(1)|=\left|\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi X_0)\right|=X_0=x_0/b, \tag{28}$$

$$|U(0)|=\left|\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi X_0)\right|=1-X_0=1-x_0/b. \tag{29}$$

The reflection and transmission coefficients for waves incident at $x=0$ are given by Eqs. (28) and (29), with $x_0=v_0/k_v$ in the diffusive approximation. For $X_0=1/2$, $R=T=1/2$, as expected on physical grounds; likewise, for $X_0=0$, $R=1$ and $T=0$, and for $X_0=1$, $R=0$ and $T=1$.

We can calculate the probability $p(x)$ of finding a photon at a given x, without regard to t, by integrating Eq. (22) over t and noting $p(x,t)=p(X,T)/b$. This quantity is the energy density at x (proportional to the photon density) in the case of a constant, rather than impulsive, unit source. Using Eqs. (20)–(22) we find[12,13]

$$p(x)=\int_0^\infty dT p(X,T)/b \tag{30}$$

$$=\frac{4}{\pi^2 b}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi x_0/b)\sin(n\pi x/b) \tag{31}$$

$$=\frac{2}{b}\begin{cases}\dfrac{x_0}{b}\left(1-\dfrac{x}{b}\right), & x_0<x,\\[2ex] \dfrac{x}{b}\left(1-\dfrac{x_0}{b}\right), & x<x_0.\end{cases} \tag{32}$$

Note that $p(x)$ is a triangular function, which peaks at $x=x_0$ as expected for a source at x_0. Note also that

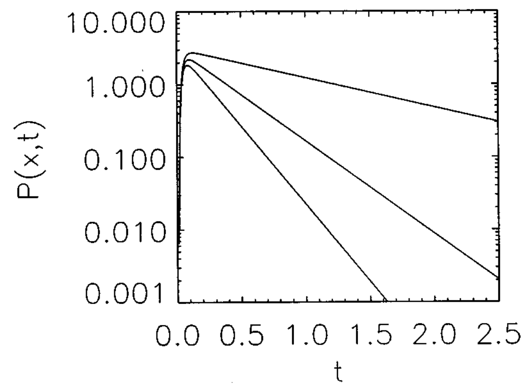

FIG. 1. Wave power $P(x,t)$ passing the point x in the rightward direction vs t, as given by Eq. (36), for $x=1$, $x_0=1/2$, $D_{xx}=1$, $b=1$, and $\sigma^2(\Gamma)t_\Gamma=2$. From top to bottom, the three curves have $\langle\Gamma\rangle=2,0,-2$, respectively.

$$\int_0^b dx p(x)=(x_0/b)(1-x_0/b)<1, \tag{33}$$

because of losses through the boundaries.

B. Scattering with growth and damping

If growth and damping are independent of scattering, the results of Secs. II B and III A can be combined to treat random growth of scattered waves. In general, the joint probability distribution of position and gain has the form

$$p(x,G,t)=p(x,t)p(G,t), \tag{34}$$

where $p(G,t)$ is given by Eq. (10) and $p(x,t)$ is the solution of Eqs. (1) and (2), or of Eq. (18) in the case of pure diffusion.

The power crossing a plane rightwards at fixed x is given by

$$P(x,t)=\int_{-\infty}^{\infty} dG e^G j(x,G,t), \tag{35}$$

where $j(x,G,t)=j(x,t)p(G,t)$. This yields

$$P(x,t)=j(x,t)\exp[\langle G(t)\rangle+\tfrac{1}{2}\sigma^2(G,t)], \tag{36}$$

where $\langle G(t)\rangle$ and $\sigma^2(G,t)$ are given by Eqs. (11) and (12), and $j(x,t)$ is given by Eq. (25) for pure diffusion.

Figure 1 shows $P(x,t)$ as a function of t for various $\langle\Gamma\rangle$. In all cases there is a rapid rise at small t, followed by an exponential falloff that is dominated by the $n=1$ term in Eq. (25). The rate of decrease with t is smaller for large $\langle\Gamma\rangle$, reflecting the generation of new photons within the medium, which becomes more effective as $\langle\Gamma\rangle$ increases.

Transmission and reflection coefficients can be obtained by the same procedure used in Sec. III C. In general, we find that the total integrated probability current at a gain G at x is

$$j(G,x)=\int_0^\infty j(x,t)p(G,t)dt, \tag{37}$$

with $j(x,t)=J(X,T)D_{xx}/b^2$. The total energy passing rightwards at x is then

$$U(x)=\int_{-\infty}^{\infty}e^{G}j(G,x)dG=\int_{0}^{\infty}P(x,t)dt, \tag{38}$$

where the reference energy W_0 introduced earlier has been set equal to unity without loss of generality. The reflection and transmission coefficients for waves incident at $x=0$ are $U(0)$ and $U(b)$, respectively, with $x_0=v_0/k_v$ in the diffusive approximation.

In the case of pure diffusion, we can substitute for $j(x,t)$ in Eq. (37) using Eqs. (25), (20), and (21). This yields

$$j(G,x)=-\frac{D_{xx}}{b^2}\sum_{n=1}^{\infty}\int_0^{\infty}dt\,n\pi e^{-\Lambda_n t}\sin(n\pi x_0/b)\times\cos(n\pi x/b)\frac{1}{\sigma(G,t)(2\pi)^{1/2}}\times\exp\left[-\frac{(G-\langle G(t)\rangle)^2}{2\sigma^2(G,t)}\right], \tag{39}$$

$$\Lambda_n=\frac{n^2\pi^2D_{xx}}{2b^2}. \tag{40}$$

Using Eqs. (11) and (12) one then obtains

$$j(G,x)=-\left(\frac{\pi}{2k_\sigma}\right)^{1/2}\frac{D_{xx}}{b^2}\times\sum_{n=1}^{\infty}n\sin(n\pi x_0/b)\cos(n\pi x/b)I_n, \tag{41}$$

$$k_\sigma=2\sigma^2(\Gamma)t_\Gamma, \tag{42}$$

$$I_n=\exp\left(\frac{G\langle\Gamma\rangle}{k_\sigma}\right)\int_0^{\infty}dt\,t^{-1/2}\exp\left[-\frac{\beta}{t}-\eta_n t\right] \tag{43}$$

$$=\sqrt{\frac{\pi}{\eta_n}}\exp\left(\frac{G\langle\Gamma\rangle}{k_\sigma}-2\sqrt{\beta\eta_n}\right), \tag{44}$$

$$\eta_n=\Lambda_n+\langle\Gamma\rangle^2/2k_\sigma, \tag{45}$$

$$\beta=G^2/2k_\sigma. \tag{46}$$

We can now write

$$U(x)=\frac{\pi D_{xx}}{b^2}\sum_{n=1}^{\infty}n\sin(n\pi x_0/b)\times\cos(n\pi x/b)\frac{J_n}{[2k_\sigma\Lambda_n+\langle\Gamma\rangle^2]^{1/2}}, \tag{47}$$

$$J_n=\int_{-\infty}^{\infty}dG\exp[G(1+x)-\{G^2(x^2+a_n^2)\}^{1/2}], \tag{48}$$

$$x=\langle\Gamma\rangle/k_\sigma, \tag{49}$$

$$a_n^2=2\Lambda_n/k_\sigma. \tag{50}$$

Hence,

$$J_n=\int_0^{\infty}dG[\exp\{-G(1+y+\sqrt{y^2+a_n^2})\}+\exp\{G(1+y-\sqrt{y^2+a_n^2})\}]. \tag{51}$$

For this integral to converge, both exponents must be negative; i.e.,

$$-(y^2+a_n^2)^{1/2}<1+y<(y^2+a_n^2)^{1/2}, \tag{52}$$

for all n, with the most stringent resulting condition occurring for $n=1$. The left-hand inequality in Eq. (52) is always satisfied, while the right-hand one requires

$$\Gamma_{\rm eff}=\langle\Gamma\rangle+\sigma^2(\Gamma)t_\Gamma<\Lambda_1. \tag{53}$$

Thus, the effective growth rate from standard stochastic growth theory must be less than the minimal diffusive loss rate from the slab if the integrals are to converge; otherwise, the slab lases and the analysis must be extended to incorporate pumping and saturation effects. The threshold (53) generalizes the one obtained Letokhov in the case of constant growth rate[3] and neatly combines standard stochastic growth theory with diffusive propagation. The threshold (53) is analogous to the criticality condition for a nuclear reactor[14] in which neutrons diffuse through a material of randomly varying fissile content. The effect of randomness is to introduce the term $\sigma^2(\Gamma)t_\Gamma$ into Eq. (53).

When the integral in Eq. (51) converges, we find

$$J_n=\frac{[\langle\Gamma\rangle^2+4\Lambda_n\sigma^2(\Gamma)t_\Gamma]^{1/2}}{\Lambda_n-\Gamma_{\rm eff}}, \tag{54}$$

and

$$U(x)=\frac{\pi D_{xx}}{b^2}\sum_{n=1}^{\infty}n\sin(n\pi x_0/b)\cos(n\pi x/b)\frac{J_n}{[2k_\sigma\Lambda_n+\langle\Gamma\rangle^2]^{1/2}}, \tag{55}$$

$$=\frac{\pi D_{xx}}{b^2}\sum_{n=1}^{\infty}\frac{n\sin(n\pi x_0/b)\cos(n\pi x/b)}{\Lambda_n-\Gamma_{\rm eff}} \tag{56}$$

$$=\frac{\pi D_{xx}}{2b^2\Lambda_1}\sum_{n=1}^{\infty}\left\{\frac{n\sin[n\pi(x_0-x)/b]}{n^2-\alpha^2}+\frac{n\sin[n\pi(x_0+x)/b]}{n^2-\alpha^2}\right\}, \tag{57}$$

$$\alpha^2 = \Gamma_{\rm eff}/\Lambda_1 < 1 \qquad (58)$$

[Note that α^2 is real but can be of either sign, subject to Eq. (58).] The standard result given by Eq. (3) on p. 99 of Ref. 12 enables Eq. (57) to be summed in closed form. Two cases must be distinguished, depending on the sign of $x_0 - x$. For $x_0 > x$, we obtain

$$U(x) = \csc(\alpha\pi)(\sin[\alpha\{\pi - \pi(x_0 - x)/b\}] + \sin[\alpha\{\pi - \pi(x_0 + x)/b\}]) \qquad (59)$$

$$= \frac{\sin[\pi\alpha(1 - x_0/b)]\cos(\pi\alpha x/b)}{\sin(\pi\alpha)}, \qquad (60)$$

Likewise, for $x_0 < x$, we find

$$U(x) = \csc(\alpha\pi)(-\sin[\alpha\{\pi - \pi(x - x_0)/b\}u] + \sin[\alpha\{\pi - \pi(x_0 + x)/b\}]) \qquad (61)$$

$$= \frac{\sin(\pi\alpha x_0/b)\cos[\pi\alpha(1 - x/b)]}{\sin(\pi\alpha)}. \qquad (62)$$

Figure 2 shows $U(x)$ vs x for $x_0 = 1/3$ and various values of α^2, both positive and negative. For $\alpha^2 < 0$, $U(x)$ decreases approximately exponentially with $|x - x_0|$, in accord with the rapid absorption of photons in this case. For $\alpha = 0$, $U(x)$ is piecewise constant in accord with conservation of energy and with Eqs. (28) and (29). For $0 < \alpha^2 < 1$, $U(x)$ increases in magnitude with increasing α and toward the edges of the medium, effects that result from wave amplification in these cases. In all cases, the discontinuity at $x = x_0$ is unity, reflecting the unit input of energy at this point at time zero.

Using the results in the previous paragraph, we can evaluate the transmission and reflection coefficients for waves incident at $x = 0$, obtaining

$$T = |U(b)| = \left|\frac{\sin(\pi\alpha x_0/b)}{\sin(\pi\alpha)}\right|, \qquad (63)$$

$$R = |U(0)| = \left|\frac{\sin[\pi\alpha(1 - x_0/b)]}{\sin(\pi\alpha)}\right|. \qquad (64)$$

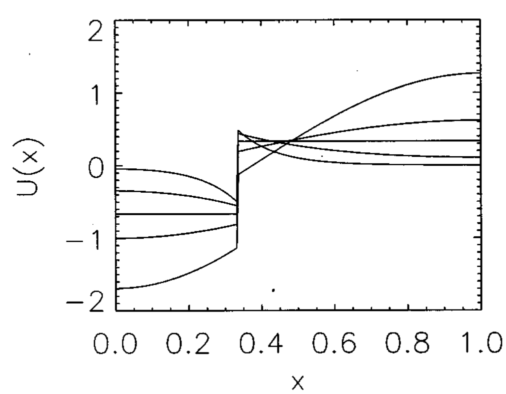

FIG. 2. Total wave energy $U(x)$ passing the point x in the rightward direction vs x, as given by Eqs. (60) and (62), for $x_0 = 1/3$ and $b = 1$. In order of increasing magnitude at the boundaries, the curves correspond to $\alpha^2 = -9, -1, 0, 0.36, 0.64$.

Physically, we can reproduce the case discussed in Sec. III A by taking the limit $\Gamma_{\rm eff} = 0$ (i.e., $\alpha = 0$), which leads again to Eqs. (28) and (29). This step immediately generalizes Eqs. (28) and (29) to the case $\Gamma_{\rm eff} = 0$ rather than the case with both $\langle\Gamma\rangle = 0$ and $\sigma^2(\Gamma)t_\Gamma = 0$, which was studied in Sec. III A. It also shows that $U(x)$ only depends only on the sign of $x_0 - x$ in this limit, as required by conservation of energy.

The sum of the reflection and transmission coefficients is

$$R + T = \left|\frac{2\sin(\pi\alpha/2)}{\sin(\pi\alpha)}\cos\left[\pi\alpha\left(\frac{1}{2} - \frac{x_0}{b}\right)\right]\right| \leq 1 \qquad (65)$$

(recall that $\alpha^2 < 1$ has been assumed), with $x_0 = v_0/k_v$ in the diffusive approximation. Equality holds in Eq. (65) only for $\alpha = 0$, in which case it holds for all x_0. For $\alpha^2 < 0$ and $0 < x_0 < b$, $R + T$ decreases exponentially with increasing $|\alpha|$. Denoting the limits of $U(x)$ as $x \to x_0$ from above and below as U_+ and U_-, respectively, we find $U_+ + U_- = 1$, as required physically.

The next quantity we calculate is $p(G,x)$, the time-averaged probability of finding a gain G at x. This also corresponds to the probability distribution of gain for a constant (rather than impulsive) unit source

$$p(G,x) = \int_0^\infty dt\, p(G,t)p(x,t), \qquad (66)$$

$$= \frac{D_{xx}}{b}\sum_{n=1}^{\infty}\frac{\sin(n\pi x_0/b)\sin(n\pi x/b)}{[2k_\sigma\Lambda_n + \langle\Gamma\rangle^2]^{1/2}} \times \exp\left[\frac{G\langle\Gamma\rangle}{k_\sigma} - \frac{2}{k_\sigma}(G^2\{2k_\sigma\Lambda_n + \langle\Gamma\rangle^2\})^{1/2}\right]. \qquad (67)$$

The probability $p(G,x)$ can be integrated over G to obtain $p(x)$, thereby reproducing Eq. (32). Note that Eq. (67) has exponential tails at both positive and negative G, with the longer tail occurring at positive G. This excess of probability relative to Eq. (10) occurs because scattering causes a small proportion of photons to spend long periods in the slab before escaping. This light has time to undergo large amounts of damping or growth,[3,5] particularly when net growth helps to maintain its intensity.

Figure 3 shows one instance of $p(G,x)$ vs G. The exponential tails at large $|G|$ are clearly visible, with the positive-G tail extending further than the one at negative G, as described above. In contrast to this, infinite-medium stochastic growth theory predicts a Gaussian dependence of $p(G,x)$ on G, with a much more rapid falloff at large $|G|$.[8]

The energy density $W(x,t)$ at x and t can also be calculated:

$$W(x,t) = p(x,t)\int_{-\infty}^{\infty} dG\, e^G p(G,t), \qquad (68)$$

$$= p(x,t)\exp(\Gamma_{\rm eff}t) \qquad (69)$$

from Eq. (15).

The time-integrated energy density $W(x)$ is equivalent to the energy density due to a constant source. It can be obtained as follows:

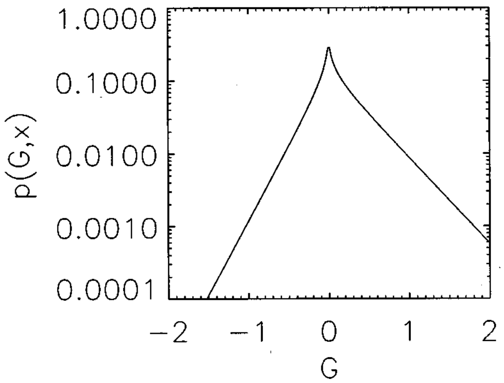

FIG. 3. Probability distribution $p(G,x)$ of gain G vs G, as given by Eq. (67), for $D_{xx}=1$, $b=1$, $k_\sigma=4$, $\langle\Gamma\rangle=4$, and $x=x_0=1/2$.

$$W(x)=\int_0^\infty dt W(x,t) \tag{70}$$

$$=\int_{-\infty}^\infty dG e^G p(G,x) \tag{71}$$

$$=\frac{4}{\pi^2 b}\sum_{n=1}^\infty \frac{\sin(n\pi x_0/b)\sin(n\pi x/b)}{n^2-\alpha^2} \tag{72}$$

$$=\frac{2}{\pi^2 b}\sum_{n=1}^\infty \frac{1}{n^2-\alpha^2}\{\cos[n\pi(x_0-x)/b] - \cos[n\pi(x_0+x)/b]\}, \tag{73}$$

subject to $\alpha^2<1$. Again, two cases must be distinguished, depending on the sign of x_0-x. For $x_0>x$, we find[12,13]

$$W(x)=\frac{2\sin[\pi\alpha(1-x_0/b)]\sin(\pi\alpha x/b)}{b\pi\alpha\sin(\pi\alpha)}. \tag{74}$$

Similarly, for $x_0<x$

$$W(x)=\frac{2\sin[\pi\alpha(1-x/b)]\sin(\pi\alpha x_0/b)}{b\pi\alpha\sin(\pi\alpha)}. \tag{75}$$

Equations (74) and (75) reproduce Eq. (32) in the limit $\alpha\to 0$ and generalize it to the case of random gain.

Figure 4 shows $W(x)$ vs x for $x_0=1/3$ and various values of α^2, both positive and negative. For $\alpha^2<0$, $W(x)$ decreases approximately exponentially with both α and $|x-x_0|$, as a result of the absorption of photons. For $\alpha=0$, $W(x)$ is a triangular function, in accord with Eq. (32). For $0<\alpha^2<1$, $W(x)$ increases rapidly as α increases, owing to amplification within the medium. As $\alpha\to 1$, the laser threshold, $W(x)$ approaches a sinusoidal form in which the location of x_0 becomes unimportant owing to the subsequent amplification in which most photons are generated elsewhere.

In the case of steady driving, we can integrate Eqs. (73) and (74) to find the total energy in the slab. For $\alpha^2<1$, this yields

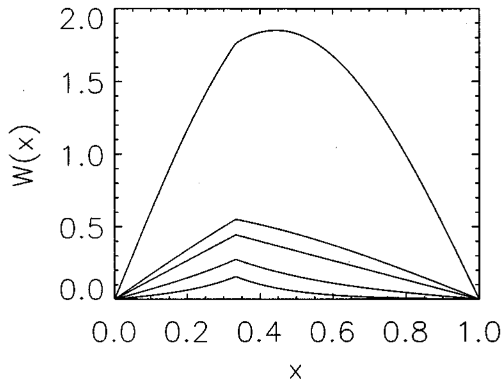

FIG. 4. Time-integrated wave energy density $W(x)$ vs x, as given by Eqs. (74) and (75), for $b=1$ and $x_0=1/3$. From top to bottom the curves correspond to $\alpha^2=0.81,0.25,0,-1,-4$, respectively.

$$\int_0^b W(x)dx=\frac{2}{\pi^2\alpha^2\sin(\pi\alpha)}[\sin(\pi\alpha x_0/b)-\sin(\pi\alpha) + \sin\pi\alpha(1-x_0/b)] \tag{76}$$

$$=\frac{4}{\pi^2\alpha^2\cos(\pi\alpha/2)}\sin(\pi\alpha x_0/2b) \times\sin[(\pi\alpha/2)(1-x_0/b)], \tag{77}$$

IV. OTHER CASES

Detailed results were derived in Sec. III for the case of a one-dimensional slab geometry with open boundary conditions. In this section we briefly outline the cases of one-dimensional media with one or two reflecting boundaries.

A. Finite medium with mixed boundary conditions

If we consider a one-dimensional medium with an open boundary at $x=b$ and a reflecting boundary at $x=0$, we must solve the diffusion equation (19) subject to the reflecting boundary condition $\partial_X p(X,T)=0$ at $X=0$ and the open boundary condition $p(1,T)=0$ for all T. The eigenfunctions that satisfy these conditions are of the form $\cos[(n-\frac{1}{2})\pi X]$ and the resulting spatial probability distribution is

$$p(X,T)=2\sum_{n=1}^\infty e^{-\lambda_n T}\cos\left[\left(n-\frac{1}{2}\right)\pi X_0\right]\cos\left[\left(n-\frac{1}{2}\right)\pi X\right], \tag{78}$$

$$\lambda_n=\left(n-\frac{1}{2}\right)^2\pi^2, \tag{79}$$

for the initial distribution $p(X,0)=\delta(X-X_0)$.

The analysis of stochastic growth proceeds from Eqs. (78) and (79) in a similar manner to that in Sec. III. The first point we note is that, on physical grounds, all radiation must emerge from the boundary at $x=b$. At large t it is the slowest decaying eigenfunction that determines the behavior and long-term stability of the system. Hence, for random growth in a scattering medium, Eq. (53) must be satisfied for stability, with $\Lambda_1 = \pi^2 D_{xx}/4b^2$. When this condition is fulfilled, we can calculate the power output as a function of time and other quantities in a similar manner to Sec. III.

B. Finite medium with reflecting boundary conditions

If both boundaries of a one-dimensional medium are reflecting, the diffusion equation (19) is solved subject to the derivative of $p(X,T)$ vanishing at $X=0,1$ for all T. The corresponding eigenfunctions are of the form $\cos(n\pi X)$ and the resulting probability distribution is

$$p(X,T) = 1 + 2\sum_{n=1}^{\infty} e^{-\lambda_n T}\cos(n\pi X_0)\cos(n\pi X), \quad (80)$$

$$\lambda_n = n^2\pi^2/2, \quad (81)$$

for the initial distribution $p(X,0) = \delta(X - X_0)$.

In this case, $p(X,T)$ evolves to a steady-state distribution as $t\to\infty$. Hence, the stability criterion (53) becomes $\Gamma_{\rm eff} < 0$, as in an infinite medium. Other properties of this system can be derived as in Sec. III.

V. SUMMARY AND DISCUSSION

The stochastic growth of waves in a strongly scattering medium has been studied, generalizing and unifying previous works in which either scattering or random growth was omitted. Among the main results are new expressions for the stochastic laser threshold, the reflection and transmission coefficients for incident waves, the spatial and gain probability distributions of light for impulsive and steady-state sources, and the dependence of the power leaving the medium on time. In particular, it was shown that the laser threshold is the point at which the effective growth rate from standard stochastic growth theory exceeds the minimal diffusive loss rate. It was also shown that scattering modifies the Gaussian gain distribution of standard stochastic growth theory to one with exponential tails at large positive and negative gain, with positive gain being particularly favored.

It should be noted that, because it ignores phase, the analysis here is only valid provided the localization length of the waves is much greater than the slab size or, in higher dimensional systems, if there is no localization of the waves in question. Complementary work on localization of randomly growing waves has been done elsewhere.[9,10] Also, the analysis here has not included saturation effects, which are required to treat global lasing of the medium.

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[1] A. M. Prokhorov, Sov. Phys. JETP **7**, 1140 (1958).
[2] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).
[3] V. S. Letokhov, Sov. Phys. JETP **26**, 835 (1968).
[4] A. Ishimaru, *Wave Scattering and Propagation in Random Media* (Academic, New York, 1978).
[5] D. S. Wiersma and A. Lagendijk, Phys. Rev. E **54**, 4256 (1996).
[6] W. Leutz and J. Rička, Opt. Commun. **126**, 260 (1996).
[7] P. A. Robinson, Solar Phys. **139**, 147 (1992).
[8] P. A. Robinson, Phys. Plasmas, **2**, 1466 (1995).
[9] A. A. Asatryan, N. A. Nicorovici, P. A. Robinson, C. M. de Sterke, and R. C. McPhedran, Phys. Rev. B **54**, 3916 (1996).
[10] A. A. Asatryan, N. A. Nicorovici, L. C. Botten, C. M. de Sterke, R. C. McPhedran, and P. A. Robinson (unpublished).
[11] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1990).
[12] V. Margulis, *Handbook of Series* (Academic, New York, 1965).
[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[14] B. Davison and J. B. Sykes, *Neutron Transport Theory* (Oxford University Press, Oxford, 1958).