Radiation of linear waves by solitons in a Josephson transmission line with dispersion

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We report a method of using a distributed Josephson junction to generate and amplify electromagnetic waves, based on Cherenkov radiation of linear waves by Josephson solitons. The device by which this principle can be realized is essentially a distributed Josephson junction electromagnetically coupled to a dispersion waveguide system providing resonance interaction between moving solitons and a linear electromagnetic wave. The current-induced motion of Josephson vortices in a distributed junction excites, due to the Cherenkov effect, a synchronous electromagnetic wave in the dispersion line; by interacting with the radiation field the vortices bunch in the decelerating phase of the wave, thus providing the coherence contribution of a large number of solitons to radiation. These devices are actually the Josephson analog of the traveling-wave and the backward-wave tubes. A linear theory for these types of oscillators and amplifiers is developed, and the equations for the starting current and growth rate of instability are obtained. A study performed on the nonlinear effects has yielded straightforward estimates of the generation power. [S0163-1829(97)01409-4]

I. INTRODUCTION

The Josephson effect as a means for generation of electromagnetic radiation has long been attracting researchers worldwide. One promising trend in the development of a Josephson oscillator is the use of the motion of Josephson vortices in a long Josephson junction. Oscillators based on this principle — the so-called flux-flow oscillators (FFO) — is a well-addressed research subject today both theoretically and experimentally[1–5] and has already found application as local oscillator in integrated receivers.[6] However, the power and noise characteristics of radiation from such oscillators do not fully meet the desired standard and have to be improved. The drawback of the currently used smooth Josephson-junction FFO is a low efficiency of interaction between the solitons and linear modes, determined by the dispersion pattern of Swihart modes in the Josephson junction $\omega^2=\omega_j^2+v_s^2k^2$, where v_s is the Swihart velocity and ω_j the Josephson plasma frequency. This type of wave dispersion makes the resonance interaction of solitons and waves within the junction, observed at $v_{\mathrm{sol}}\approx v_{\mathrm{ph}}$, impossible, since the phase velocity of the wave $v_{\mathrm{ph}}>v_s$ and of the soliton $v_{\mathrm{sol}}<v_s$, so waves are radiated only at the end of the Josephson junction where it links the passive transmission line.

In this paper we propose a kind of the Josephson flux-flow oscillator in which radiation of electromagnetic waves proceeds within the entire junction. The main idea involved in this mechanism is to cause distortion of the dispersion characteristic of Swihart modes by electromagnetically coupling the Josephson junction to an external space-periodic waveguide system. The external view of such a device and its linear wave dispersion curve are exemplified in Fig. 1.

The operation principle of the proposed generator is similar to that underlying the traveling-wave tubes (TWT) and the backward-wave tubes (BWT) and is based on the well-known analogy between Josephson vortices and charged particles. The Josephson vortices moving under a bias current in a junction coupled to a transmission line with dispersion excite an electromagnetic wave due to the Cherenkov effect. Further, through interaction with the radiation field the vortices bunch in the decelerating phase of the wave, providing the coherence contribution of a large number of solitons to radiation.

The paper is concerned with the development of a theory for oscillators of this type. It is organized as follows. In Sec. II we provide derivation of the basic equations describing the dynamics of a Josephson junction coupled to an auxiliary electrodynamical system. Section III deals with the problem on radiation of waves by a single soliton. We show therein that once a wave and a soliton are synchronous, the field structure in the junction changes qualitatively as compared to that in a smooth Josephson junction. In Sec. IV a linear theory of interaction between solitons and an increasing wave is considered, a TWT gain and a bias current threshold required for inducing generation in a BWT are found. Section V describes a simple nonlinear theory that allows estimation of the radiation power of the oscillators in question. Finally, the obtained results are summarized in the Conclusion.

II. GENERAL EQUATIONS

Let us consider a simplest system that provides realization of the above principle and a sufficient detail of analytical description in terms of the perturbation theory. It is essentially a long Josephson junction electrodynamically coupled to a strip transmission line with dispersion, see Fig. 1. The line dispersion is caused by strip resonators periodically arranged along the sides of the junction, whose impedance is frequency dependent. The equations describing the dynamics of coupled lines follow from the Kirchhoff's law for an equivalent scheme (Fig. 2) and have the form:

$$L_1I_{1t}+U_{1x}=-MI_{2t}, \tag{1}$$

$$C_1U_{1t}+G_1U_1+I_{1x}=-C(U_{1t}-U_{2t})-j_s+j_{\mathrm{ext}}, \tag{2}$$

$$\hat{L}_2I_{2t}+U_{2x}=-MI_{1t}, \tag{3}$$

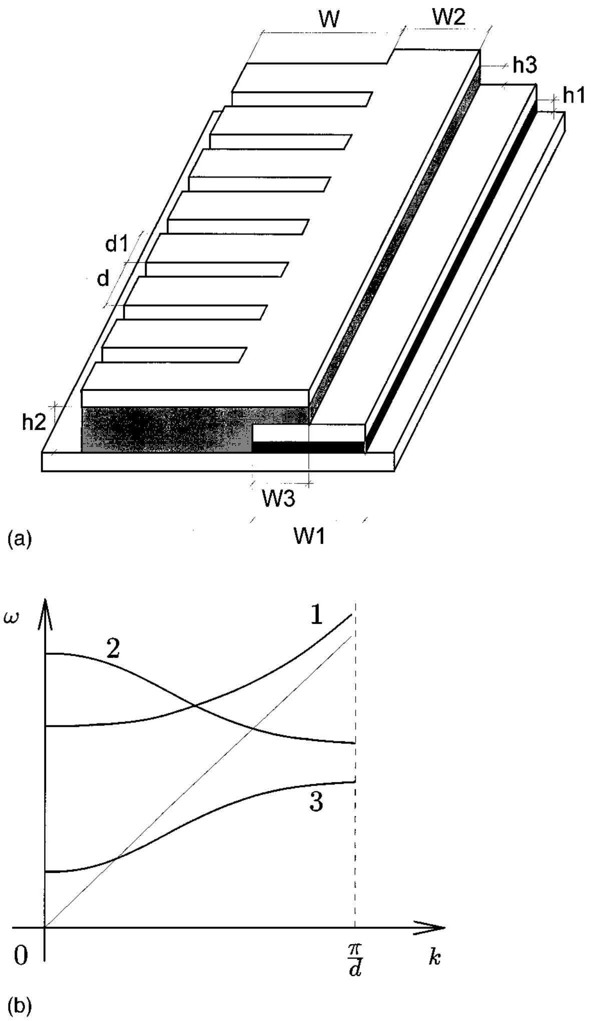

FIG. 1. (a) External view of the device under consideration. Josephson area is shown in black, dielectric interlayers in the line with dispersion and in the overlap area are shown in gray. (b) Dispersion characteristics of the system in question. Curve 1 is the dispersion characteristic of the linear waves in Josephson junction. Thin straight line is the asymptote for curve 1, and this is a dispersion characteristic for a soliton moving at its ultimate velocity. Curves 2 and 3 are the dispersion characteristics of the waves in the transmission line with dispersion.

$$\hat{C}_2 U_{2t} + I_{2x} = -C(U_{2t} - U_{1t}). \tag{4}$$

Here $I_{1,2}U_{1,2}$ are the currents and voltages in the long Josephson junction and in the transmission line with dispersion, respectively, j_s is the linear density of supercurrent, j_{ext} is the density of external current, L_1, C_1 are the inductance and capacitance per unit length of the Josephson junction, $\hat{L}_2, \hat{C}_2$ are the linear operators defining the coupling between the magnetic flux density and current,

$$S_2(t) = \hat{L}_2 I_2 = \int_0^\infty L(\tau) I_2(t-\tau)\, d\tau$$

and between the charge density and voltage in the external electrodynamical system,

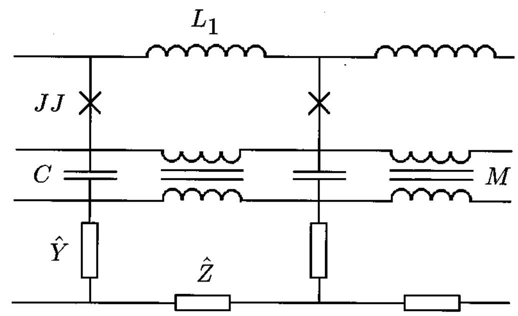

FIG. 2. Discrete electronic circuit of the system considered. The interaction is given by the mutual inductance M and the capacitance C. Elements JJ drawn by crosses symbolize Josephson conductance between the Josephson-junction electrodes, L_1 is the inductance of the Josephson junction, $\hat{Z}$ and $\hat{Y}$ are the impedance and the admittance of the transmission line with dispersion.

$$\rho_2(t) = \hat{C}_2 U_2 = \int_0^\infty C_2(\tau) U_2(t-\tau)\, d\tau,$$

respectively. By analogy with the no-dispersion case we shall further refer to them as the inductance and capacitance operators.[7] In the Fourier representation they become multiplying operators: $\hat{L}_2 = L_2(\omega,k)$, $\hat{C}_2 = C_2(\omega,k)$ and define the dispersion relation for linear waves in an external electrodynamical system:

$$-C_2(\omega,k)\omega^2 + L_2^{-1}(\omega,k)k^2 = 0.$$

For a smooth strip line without dispersion the values of $C_2 L_2$ are independent of frequency. The operators $\hat{C}_2 \hat{L}_2$ are related to the commonly used impedance, $\hat{Z}$, and conductivity, $\hat{Y}$, operators as the well-known equalities

$$\hat{Z} = \frac{\partial}{\partial t}\hat{L}_2, \quad \hat{Y} = \hat{C}_2 \frac{\partial}{\partial t}, \tag{5}$$

or, in the Fourier representation,

$$Z = -i\omega L_2, \quad Y = -i\omega C_2. \tag{6}$$

The real parts of $L_2(\omega), C_2(\omega)$ define the dispersion, the imaginary ones — the losses in the electrodynamical system. The coefficients M, C, i.e., the mutual inductance and capacitance, respectively, define the value of coupling between the Josephson junction and the external electrodynamical system. The subscripts t and x designate, as usual, the time and space partial derivatives. Taking into account the Josephson phase difference φ in the junction and the z component of the vector potential in the dispersive line ψ, which relate to the voltages $U_{1,2}$ and to the linear densities of magnetic fluxes, $S_{1,2}$, in the Josephson junction and in the line as

$$U_1 = \frac{\Phi_0}{2\pi}\varphi_t, \quad U_2 = \frac{\Phi_0}{2\pi}\psi_t, \quad S_1 = \frac{\Phi_0}{2\pi}\varphi_x, \quad S_2 = \frac{\Phi_0}{2\pi}\psi_x, \tag{7}$$

where Φ_0 is the magnetic flux quantum, and using the explicit expression for supercurrent: $j_s = j_c \sin\varphi$, where j_c is the critical density of the Josephson current, from Eqs. (1)–(4) we find a set of equations for the variables φ,ψ:

$$(C_1+C)\varphi_{tt}+G_1\varphi_t-\frac{\partial}{\partial x}(L_1-M^2\hat{L}_2^{-1})^{-1}\varphi_x+\frac{2\pi j_c}{\Phi_0}\sin\varphi$$
$$=-M\frac{\partial}{\partial x}(L_1-M^2\hat{L}_2^{-1})^{-1}\hat{L}_2^{-1}\psi_x+C\psi_{tt}+\frac{2\pi}{\Phi_0}j_{\mathrm{ext}}, \tag{8}$$

$$(\hat{C}_2+C)\psi_{tt}-\frac{\partial}{\partial x}\left(\hat{L}_2-\frac{M^2}{L_1}\right)^{-1}\psi_x$$
$$=-\frac{M}{L_1}\frac{\partial}{\partial x}\left(\hat{L}_2-\frac{M^2}{L_1}\right)^{-1}\varphi_x+C\varphi_{tt}, \tag{9}$$

which describes the dynamics of a Josephson junction inductively and capacitively coupled to a linear transmission line. Surface losses not written explicitly in Eqs. (8) and (9) may play an important part at high frequencies of waves and high velocities of solitons. Throughout the paper we will consider them to be small enough so that they can be taken into account as small anti-Hermitian parts of $\hat{L}_2,\hat{C}_2$ operators.

In this paper we are seeking analytical results, so we shall restrict our consideration to the case of a weak coupling between the junction and the electrodynamical system, assuming further $C\ll C_1,\hat{C}_2$ and $M^2\ll L_1,\hat{L}_2$. The "weak-coupling" approximation allows one to neglect the terms with C and M in the left-hand parts of the sets (8) and (9), leaving them only in the right-hand parts.

It is convenient to go over to the commonly used time and space dimensionless variables by introducing the value ω_j^{-1} as a unit of time $[\omega_j=(2\pi j_c/\Phi_0 C_1)^{1/2}$ is the Josephson plasma frequency$]$ and the Josephson length $\lambda_j=(\Phi_0/2\pi j_c L_1)^{1/2}$ as a unit of scale, the velocity unit $v_s=\lambda_j\omega_j$ is the Swihart velocity in a Josephson junction. With these dimensionless variables the sets (8) and (9) take the form

$$\varphi_{tt}-\varphi_{xx}+\sin\varphi=-\alpha\psi_{xx}+\beta\psi_{tt}+j_{\mathrm{ext}}-\gamma\varphi_t, \tag{10}$$

$$\hat{D}\psi=-\alpha\varphi_{xx}+\beta\varphi_{tt}, \tag{11}$$

where $\alpha=M/L_2$, $\beta=C/C_1$ are the coefficients of the magnetic and electric interaction between the Josephson junction and the transmission line, respectively, $\gamma=G_1/C_1\omega_j$ is the dimensionless damping factor describing the losses in the Josephson junction, the operator

$$\hat{D}=\frac{\hat{C}_2}{C_1}\frac{\partial^2}{\partial t^2}-L_1\frac{\partial}{\partial x}\hat{L}_2^{-1}\frac{\partial}{\partial x}$$

characterizes dispersion of waves in the electrodynamical system. The sets of equations similar to Eqs. (10) and (11) describe the dynamics of the whole class of Josephson systems; some particular cases have been widely discussed elsewhere.[2,8–11]

Note that by eliminating ψ we can write the systems (10) and (11) as one equation

$$\hat{K}\varphi+\sin\varphi=j_{\mathrm{ext}}, \tag{12}$$

having the form of the sine-Gordon equation where the D'Alamber operator is replaced by some linear operator $\hat{K}$ which defines the dispersion features of an electromagnetical system with a long Josephson junction. Equation of type (12) is the most general expression to describe the dynamics of a Josephson junction coupled to a linear electrodynamical system. Examples of how nonlocal equations of type (12) are used to describe Josephson-junction dynamics can be found in Refs. 11–14.

In conclusion, we provide the explicit expressions for the capacitance and inductance entering in the sets (1)–(4) via the geometrical dimensions of the strip lines shown in Fig. 1(a). For a weak coupling, if the long-wave approximation $\lambda\gg d$ (where λ is the wavelength, d the system period) is met, a simple computation yields the expressions

$$C_1=\epsilon\epsilon_0\frac{W_1}{h_1},\quad C_2=\epsilon\epsilon_0\left(\frac{W_2}{h_2}+\frac{d_1}{d}\frac{W}{h_2}\frac{\tan(\omega W/v)}{(\omega W/v)}\right),$$

$$C_3=\frac{\epsilon\epsilon_0 W_3}{h_3},$$

$$L_1=\mu\mu_0\frac{h_1+2\lambda}{W_1},\quad L_2=\mu\mu_0\frac{h_2+2\lambda}{W_2}, \tag{13}$$

$$M_3=\mu\mu_0\frac{W_3}{h_3+2\lambda}\frac{(h_1+2\lambda)(h_2+2\lambda)}{W_1W_2},$$

where h_1, h_2, h_3 are the thicknesses of the dielectric interlayers in the Josephson junction, in the line with dispersion, and in the overlap area, respectively; W_1, W_2 are the widths of the strip lines; W_3 is the width of the junction-strip line overlap area; d is the period of the structure; $v=1/\sqrt{\epsilon\epsilon_0\mu\mu_0}\sqrt{h_2/(h_2+2\lambda)}$ is the velocity of wave propagation in a passive electrodynamical system. The complicated frequency dependence of the capacitance C_2 is due to the resonance in the side outgrowths of the strip line. If the coupling between the Josephson junction and the dispersion line is strong, and the wavelength and the period of the system are comparable, the inductance and capacitance coefficients should be determined by way of more exact electrodynamic calculations.

In the conclusion of the section it is nesessary to make an important remark. Generally speaking, the periodicity of the Josephson transmission line leads to pinning of the solitons. But due to weakness of coupling between the Josephson junction and the dispersive line, accepted in the article, we will not take it into account. Besides, a periodical structure is not the only way to reach a dispersion.

III. THE CHERENKOV RADIATION OF A SOLITON MOVING IN A TRANSMISSION LINE WITH DISPERSION

In this section we consider spontaneous Cherenkov radiation of a soliton moving in a long Josephson junction coupled to a transmission line with dispersion. A likelihood of the effect of the Cherenkov radiation of solitons was also suggested earlier in some particular cases of stacked Josephson junctions,[15] in Josephson junctions with nonlocal electrodynamics,[14] in the discrete models of the sine-Gordon equation.[16–18] Here we provide only a general treatment valid for any electrodynamical system with an assigned dispersion law.

Regarding the right-hand sides of the systems (10) and (11) as perturbations, i.e., assuming the coupling parameters α,β and the damping factor γ to be small, in the zero approximation we shall have two independent equations, one describing the dynamics of a Josephson junction, the other –the linear waves in a line with dispersion. The first equation, referred to as the sine-Gordon one, has a well-known solution defining the Josephson vortex soliton

$$\varphi_{\text{sol}}(x,t)=4\arctan\exp\left(\frac{x-vt}{\sqrt{1-v^2}}\right),$$

where v is the dimensionless velocity of a soliton. Substitution of this expression into the right-hand side of Eq. (11) yields an equation describing a field created by a moving soliton in the electrodynamical system,

$$\hat{D}\psi=-\epsilon\frac{\partial^2\varphi_{\text{sol}}}{\partial x^2}, \tag{14}$$

where ϵ is the effective coupling coefficient, $\epsilon=(\alpha-\beta v^2)$.

A solution to this equation is sought via a Fourier transform with respect to coordinate and a Laplace transform with respect to time which are performed by a common procedure,[7]

$$\psi(\omega,k)=\int_0^\infty e^{i\omega t}dt\int_{-\infty}^\infty \psi(x,t)e^{-ikx}dx. \tag{15}$$

Assuming that there is no field in the external electrodynamical system at $t=0$, we find Eq. (15) after the transform (14) as

$$D(\omega,k)\psi(\omega,k)=-\frac{ikf(k)}{\omega-kv},$$

where $D(\omega,k)$ is the Fourier image of the operator $\hat{D}$, the function $f(k)$ is the Fourier representation of a magnetic field in a soliton

$$f(k)=\int_{-\infty}^{\infty}\frac{\partial\varphi_{\text{sol}}}{\partial x}e^{-ikx}dx=\frac{2\pi}{\cosh[(\pi k/2)\sqrt{1-v^2}]}. \tag{16}$$

From here follows an explicit expression for $\psi(\omega,k)$:

$$\psi(\omega,k)=-\frac{i\epsilon kf(k)}{(\omega-kv)D(\omega,k)},$$

and the problem on finding $\psi(x,t)$ reduces to calculations of integrals corresponding to the inverse Laplace transform in ω and the Fourier transform in k.

Let us first find $\psi(t,k)$ expressed by the integral

$$\psi(t,k)=-\int_C\frac{ik\epsilon f(k)e^{-i\omega t}}{(\omega-kv)D(\omega,k)}\frac{d\omega}{2\pi}, \tag{17}$$

where the path of integration C in the complex plane ω goes along the real axis above all singularities of the integrand. For $t>0$ we can complete the path on an infinite semicircle in the lower half-plane ω and find the integral as a sum of subtractions in the poles of the integrand, which are located in the points $\omega=kv$ and $\omega=\omega_l(k)$ determined by zero values of the denominator of the integrand with respect to ω, where $\omega_l(k)=\pm\omega_l'(k)-i\Gamma_l(k)$ are the complex eigenfrequencies of waves in the electrodynamical system, that can be found from the solution to the equation $D(\omega,k)=0$. Integration of Eq. (17) via residues yields

$$\psi(t,k)=\frac{ik\epsilon f(k)e^{-ikvt}}{D(\omega=kv,k)}+ik\epsilon f(k)\times\sum_l\frac{e^{-i\omega_l(k)t}}{[\omega_l(k)-kv][\partial D(\omega,k)/\partial\omega]|_{\omega=\omega_l(k)}},$$

where the sum is taken over all branches of the dispersion characteristic. Now, to find $\psi(t,x)$ we need to take the integral

$$\psi(x,t)=\int_C\psi(t,k)e^{ikx}\frac{dk}{2\pi}, \tag{18}$$

where the path of integration C in the complex plane k goes along the real axis.

Contributions to the integral (18) are determined by singularities of the integrand in the complex region k, which can be classified into two types. The singularities of type I relate to the poles of function $f(k)$, located in the points $k_n=(2n+1)i/\sqrt{1-v^2}$, and to the zero values of functions $\partial D(\omega,k)/\partial\omega|_{\omega=\omega_l}$. Type-II singularities are due to the zeroes in the expressions $D(\omega=kv,k)$ and $\omega_l(k)=kv$. The type I lie on the imaginary axis, and their contributions go down rapidly at $x\to\pm\infty$. These fields — further referred to as the local fields ψ_{loc} — are rigidly coupled to a soliton and are responsible for its deformation by motion. The contributions from the type-II singularities (if any) are of a qualitatively different nature; they determine the intensity of the Cherenkov radiation by linear waves.

We now look for this contribution, assuming damping of the eigenwaves to be relatively small. Let the straight line $\omega=kv$ cross some dispersion branch $\text{Re}\,\omega(k)$ in point k_c, i.e., k_c is the solution to the equation $\text{Re}\,\omega_l(k_c)=vk_c$. Further on we ignore the subscript l, implying thereby that only one mode is radiated. Then equations $D(\omega=kv,k)=0$ and $\omega_l=kv$ can be easily solved yielding a common root $k=k_c+i\Gamma(k_c)/[v_g(k_c)-v]$ which defines the position of the sought-for singularity of the integrand; here $\Gamma(k)=\text{Im}\,\omega_l(k)$ is the damping rate of the wave and $v_g(k)=\partial\omega_l(k)/\partial k|_{k=k_c}$ is the group velocity of this wave. Note that the damping rate coefficient Γ accounts for all losses in the Josephson transmission line.

We seek the contribution of this singularity by expanding $D(\omega=kv,k)$ and $\omega_l(k)$ near k_c in the Taylor series and considering only the first terms; this yields the expression for the radiation field:

$$\psi_{\text{rad}}(x,t)=-\frac{ik_c\epsilon f(k_c)e^{ik_c(x-vt)}}{\partial D/\partial\omega}\times\int_{-\infty}^{+\infty}\frac{e^{i\chi(x-vt)}-e^{i\chi[x-v_g(k_c)t]-\Gamma(k_c)t}}{[v-v_g(k_c)]\chi+i\Gamma(k_c)}\frac{d\chi}{2\pi}+\text{c.c.},$$

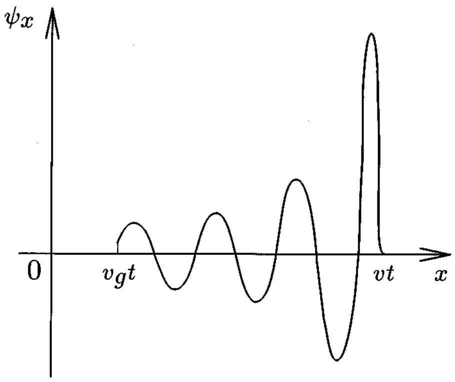

FIG. 3. The distribution of ψ_x (dimensionless magnetic field) in the transmission line with dispersion.

in which we have introduced a variable $\chi=k-k_c$; the values of the function $\partial D(\omega,k)/\partial k$ here and elsewhere through the paper are taken in the points $k=k_c$, $\omega=k_c v$. Integration using the theory of residues leads us to the final result for the radiation field:

$$\psi_{\rm rad}(x,t)=-\frac{k_c\epsilon f(k_c)e^{ik_c(x-vt)+\Gamma(x-vt)/[v-v_g(k_c)]}}{|\partial D/\partial\omega|[v-v_g(k_c)]}\times[\theta(x-vt)-\theta(x-v_g(k_c)t)]+\text{c.c.}; \quad (19)$$

here $\Theta(x)$ is a step function. The complex-conjugate contribution appears here because the equation describing the presence of the Cherenkov resonance $D(\omega=kv,k)=0$ has two solutions. The step function $\Theta(x)$ reflects a physically evident localization of the radiation field in the interval $v_g t<x<ut$. Expression (19), from a mathematical point of view is the contribution of eliminated singularities in the integrand (17).

So, we have found that a field created by solitons in an electrodynamical system has the form

$$\psi=\psi_{\rm loc}+\psi_{\rm rad}.$$

One should note that, of course, the far and near field separation is only possible provided damping rate is small enough $\Gamma(\omega_c,k_c)\ll\omega_c$, $\omega_c=k_c v$.

The qualitative dependence of ψ_x (dimensionless magnetic field) on the coordinates for the case $v>v_g(k_c)$ is given in Fig. 3. The radiation field is nonzero in the region $v_g(k_c)t<x<vt$. In calculations we neglected the terms like $\partial^2\omega/\partial k^2$ and, therefore, the obtained expression for the radiation field holds at the times

$$t\ll\frac{\partial^2\omega/\partial k^2}{[v-v_g(k_c)]^2},$$

when the dispersion spreading of the wave packet can be ignored. To find the radiation field at the times larger than the above it is necessary to retain the higher derivatives $\partial^n\omega/\partial k^n$ in the expansion.

Knowledge of the expression for the radiation field allows one to calculate its energy as well as the radiation power of a soliton at any time. Using the well-known formula for the energy density of a quasimonochromatic electromagnetic wave W,[7] we have

$$W=\frac{1}{4}\left(\frac{\partial\omega C_2}{\partial\omega}UU^*+\frac{\partial\omega L_2}{\partial\omega}II^*\right), \quad (20)$$

where U and I are the complex voltage and current amplitudes in the wave, C_2 and L_2 are the frequency-dependent capacitance and inductance of the electrodynamical systems (8),(9). With Eqs. (5), (6), and (7) in mind we find the energy E of the radiation field in the form

$$E(t)=\int_{-\infty}^{+\infty}W(t,x)dx=\frac{\epsilon^2}{4}\frac{|f(k_c)|^2k_c^4}{|\partial D/\partial\omega|^2|v-v_g(k_c)|}\frac{1-e^{-2\Gamma t}}{2\Gamma}\times\left(v^2\frac{\partial\omega C_2}{\partial\omega}+L_2^{-2}\frac{\partial\omega L_2}{\partial\omega}\right), \quad (21)$$

in which the values of all functions are taken in points $\omega(k_c)$ and k_c. The power P of the linear waves' generation by a soliton is easily found from the energy expression as

$$P=\left|\frac{dE}{dt}\right|_{t=0}=\frac{\epsilon^2}{4}\frac{|f(k_c)|^2k_c^4}{|\partial D/\partial\omega|^2|v-v_g|}\times\left(v^2\frac{\partial\omega C_2}{\partial\omega}+L_2^{-2}\frac{\partial\omega L_2}{\partial\omega}\right). \quad (22)$$

One important circumstance we would like to emphasize is that, despite the smallness of the coupling coefficient, the energy in the radiation field may be much higher than in a soliton,

$$E_{\rm sol}=\int_{-\infty}^{+\infty}\frac{\varphi_t^2}{2}+\frac{\varphi_x^2}{2}+(1-\cos\varphi)dx=\frac{8}{\sqrt{1-v^2}}.$$

Thus, we have shown that a soliton moving in a long Josephson junction coupled to a transmission line with dispersion may continuously radiate a linear wave as a result of the Cherenkov radiation effect. The energy of this wave is limited only by damping and by the finite length of the electrodynamical system.

If more than one vortex propagate in the junction, the total radiation field is a superposition of the radiation fields of individual vortices, and the collective radiation power of the vortex bunch depends on the relative position of vortices and on the difference in their velocities. The maximum radiation power which is proportional to the vortex number squared is achieved when all vortices radiate coherently, i.e., when all of them move with the same velocity, and the intervortex distance is a multiple of their radiation wavelength. In the following section we show that radiation from vortices becomes coherent automatically, due to interaction of vortices through the radiation field, which equalizes their velocities and promotes bunching of vortices in the optimal phase of the collectively radiated wave.

IV. BUNCHING OF SOLITONS IN THE FIELD OF A RADIATED WAVE

In this section we consider the effects arising by the action of linear waves on solitons. We assume that a field in a Josephson junction can be represented as a chain of solitons either of which is described by the distribution of phase $\varphi_{\rm sol}^{(n)}$ and the linear wave $F(x,t)$,

$$\varphi=\sum_n \varphi_{\rm sol}^{(n)}(x,t)+F(x,t).$$

Then, assuming the right-hand sides in the sets (10) and (11) relatively small we can use the perturbation method[19,20] to derive the expression for the motion of solitons and for the dynamics of linear waves in the junction and in the external electrodynamical system. Representing the field in the nth soliton as

$$\varphi_{\rm sol}^{(n)}(x)=4\arctan \exp Y_n(x-Z_n),$$

where Z_n is the coordinate of the center of the nth soliton, $Y_n^{-1}=\sqrt{1-\dot{Z}_n^2}$ is its dimensionless width, and introducing the soliton momentum P in the form $P_n=Y_n\dot{Z}_n$, we have the set

$$\dot{P}_n+\gamma P_n+\frac{\pi}{4}[\exp(Y_nZ_n-Y_{n+1}Z_{n+1})-\exp(Y_{n-1}Z_{n-1}-Y_nZ_n)]=\int_{-\infty}^{\infty}F(x,t)\frac{\partial\varphi_{\rm sol}^{(n)}(x,t)}{\partial x}dx, \tag{23}$$

$$\hat{D}\psi=-\sum_n (\alpha-\beta\dot{Z}_n^2)\frac{\partial^2}{\partial x^2}\varphi_{\rm sol}^{(n)}(x,t), \tag{24}$$

$$F_{tt}+\gamma F_t-F_{xx}+F=-\alpha\psi_{xx}+\beta\psi_{tt}+j. \tag{25}$$

This approximation is valued provided that the distance between the solitons is much greater than their width.

We shall use this set to analyze the dynamics of an equidistant chain of solitons moving at the velocity v, whose spatial period d_0 is incommensurable with the radiated wave length

$$\left|D\left(\omega=\frac{2\pi l}{d_0}v,k=\frac{2\pi l}{d_0}\right)\right|\gg 1,$$

so that a phase mismatch of the contributions by individual solitons eliminates the Cherenkov radiation. The coordinates of the solitons' centers in such a case are given by the formula

$$Z_n=d_0n+vt,$$

and the velocity v is defined from the equation

$$\frac{v}{\sqrt{1-v^2}}=\frac{\pi}{4}\frac{j}{\gamma}. \tag{26}$$

Further we show that this state is unstable with respect to soliton bunching and wave build-up in an electrodynamical system.

We shall study this instability assuming the deviations of solitons from the initial position

$$\xi_n=Z_n-d_0n-vt,$$

and the wave amplitude ψ small: $\xi_n\ll 1$, $\dot{\xi}_n\ll v$, $\psi\ll 1$, and linearizing the sets (23)–(25) near the state $\xi_n=0$, $\dot{\xi}_n=0$, $\psi=0$. We find the expression to describe the dynamics of perturbations:

$$\ddot{\xi}_n+\gamma\xi_n+\frac{c_s^2}{d_0^2}(2\xi_n-\xi_{n-1}-\xi_{n+1})=(1-v^2)^{3/2}\int_{-\infty}^{\infty}F(x,t)\frac{\partial\varphi_{\rm sol}^{(n)}(x,t)}{\partial x}dx, \tag{27}$$

$$\hat{D}\psi=(\alpha-\beta v^2)\xi_n\frac{\partial^3}{\partial x^3}\sum_n \varphi_{\rm sol}^{n}(x,t), \tag{28}$$

$$F_{tt}+\gamma F_t-F_{xx}+F=-\alpha\psi_{xx}+\beta\psi_{tt}, \tag{29}$$

where $c_s=(\sqrt{\pi}d_0/2)(1-v^2)^{3/2}\exp(-d_0/2\sqrt{1-v^2})$ is the velocity of acoustic waves in the soliton chain. This linear system can be solved in the following way. We seek its solution in the form

$$\psi(x,t)=\psi\exp(-i\omega t+ikx)+\text{c.c.},$$

$$F(x,t)=F\exp(-i\omega t+ikx)+\text{c.c.},$$

$$\xi_n(t)=\xi\,\exp(-i\omega t+iknd_0+ikvt)+\text{c.c.}$$

Then we substitute these expressions in the sets (27)–(29) and extract the terms with the same dependence on x and t to eventually obtain a set of algebraic equations for the complex amplitudes ψ, F, ξ

$$\left[-(\omega-kv)^2-i\gamma(\omega-kv)+\frac{4c_s^2}{d_0^2}\sin^2\left(\frac{kd_0}{2}\right)\right]\xi=\frac{1}{8}(1-v^2)^{3/2}f(k)F, \tag{30}$$

$$D(\omega,k)\psi=\frac{k^2\xi\epsilon f(k)}{d}, \tag{31}$$

$$(-\omega^2-i\gamma\omega+k^2+1)F=(\alpha k^2-\beta\omega^2)\psi, \tag{32}$$

here $\epsilon=(\alpha-\beta v^2)$, just as in the second part is the coupling coefficient, $f(k)$ is determined by Eq. (16) and is essentially a Fourier spectrum of the magnetic field in a soliton. The resolvability condition for this set of equations is the dispersion relation

$$\left[-(\omega-kv)^2-i\gamma(\omega-kv)+\frac{4c_s^2}{d^2}\sin^2\left(\frac{kd}{2}\right)\right](-\omega^2-i\gamma\omega+k^2+1)D(\omega,k)=\frac{1}{8}\frac{k^2\epsilon f^2(k)(\alpha k^2-\beta\omega^2)}{d}(1-v^2)^{3/2}, \tag{33}$$

which defines the spectrum of the eigenwaves in the set. Generally speaking, this equation should be solved numerically, but given a rather weak coupling of waves it is possible to find the solution by means of the perturbation theory.

In the absence of interaction ($\epsilon=0$) Eq. (33) describes the dispersion of uncoupled density waves in a soliton chain, the

Swihart waves in a Josephson junction, and waves in an electrodynamical system. If the coupling is weak, the interaction effect will be appreciable only near the crossing points of the dispersion curves of noninteracting waves, and Eq. (33) can be largely simplified.

Let us now consider the dispersion equation (33) in the vicinity of the Cherenkov synchronism point k_c defined by the relation

$$\omega(k_c)=k_c v,$$

where $\omega(k)$ is the wave dispersion in the electrodynamical system in which this condition can be met. We now introduce the new variables $\chi=k-k_c$ and $\delta=\omega-\omega(k_c)=\omega-k_c v$ characterizing deviation from the point k_c with respect to wave number and frequency, respectively. Then, expansion of the dispersion relation (33) near the synchronism point in the small damping limit $[\Gamma\ll\omega(k_c)]$ yields

$$[(\delta-\chi v)^2+i\gamma(\delta-\chi v)-\omega_s^2](\delta-\chi v_g+i\Gamma)=G, \quad (34)$$

in which $\omega_s^2=(c_s^2/d_0^2)\sin^2(k_c d_0/2)$ is the parameter for the splitting of the dispersion characteristics of the acoustic waves in the soliton chain in point k_c, the coupling constant G is defined by the relation

$$G=\frac{k_c^4\epsilon^2 f^2(k_c)}{8d[1+k_c^2(1-v^2)]|\partial D/\partial\omega|}(1-v^2)^{3/2}.$$

This expression is a complete analog of the dispersion relation for the waves in a TWT (for $v_g>0$) and in a BWT (for $v_g<0$),[21] G is analogous to the Pierce parameter, and ω_s^2 is similar to the parameter accounting for the spatial charge of a beam.

Equation (34) is still quite complicated at this stage, and a complete study of the dependence $\mathrm{Im}\,\delta(\chi)$ which determines the growth rate of the instability for different values of G, ω_s^2, γ, Γ, v, v_g requires numerical calculations. This equation is well studied; the details of its solution procedure can be found, for example, in Ref. 22. We shall consider some limiting cases that allow simple analytical solutions.

Without interaction, $G=0$, Eq. (34) breaks down into the product of factors and is easy to solve:

$$\delta_{1,2}=\chi v-\frac{i\gamma}{2}\pm\sqrt{\omega_s^2-\frac{\gamma^2}{4}},$$

$$\delta_3=\chi v_g-i\Gamma. \quad (35)$$

The above dependences — actually, the dispersion curves near the point k_c, are shown in Fig. 4(a) for $v_g>0$ and Fig. 4(b) for $v_g<0$. Let us first consider the case of low damping in the junction, $\gamma\ll\omega_s$, and a small coupling coefficient G, i.e., when the resulting growth rate satisfies the condition $\mathrm{Im}\,\delta\ll\omega_s$. Splitting of the dispersion curves due to $G\neq 0$ in this case will be inconsiderably small as compared to ω_s, and the points where the dispersion curve of eigenwaves of the electrodynamical system crosses the dispersive curves of fast and slow waves in a soliton chain can be examined separately.

Equation (34) then reduces to a quadratic equation,

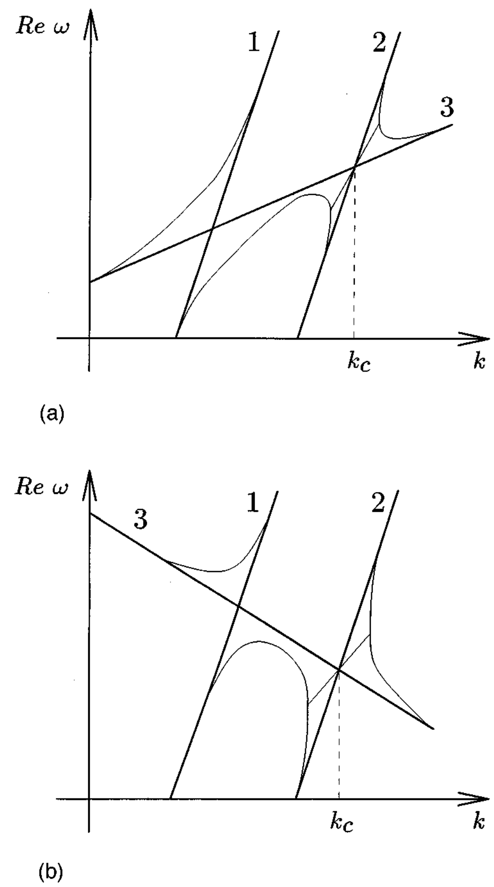

FIG. 4. (a) The dependence of the real part of ω on k for the system in question when $v_g>0$. Curves 1 and 2 correspond to the fast and slow acoustic waves in the soliton chain, curve 3 corresponds to the wave in transmission line. Thin lines show the perturbation of the dispersion characteristics when the coupling between the Josephson junction and the transmission line takes place. (b) The dependence of the real part of ω on k for the system in question when $v_g<0$. Curves 1 and 2 correspond to the fast and slow acoustic waves in the soliton chain, curve 3 corresponds to the wave in transmission line. Thin lines show the perturbation of the dispersion characteristics when the coupling between the Josephson junction and the transmission line takes place.

$$\left(\delta-\chi v+\frac{i\gamma}{2}\pm\omega_s\right)(\delta-\chi v_g+i\Gamma)=\mp\frac{G}{2\omega_s} \quad (36)$$

and can be studied in detail. The upper sign in Eq. (36) corresponds to the interaction between a wave in the electrodynamical system and a slow wave in the soliton chain, the lower one — to the same interaction but with a fast wave of the soliton chain. One can easily see that instability is possible only in the first case, when the right-hand side is negative. It is this situation that we are going to address.

By introducing the variables δ' and χ' in the form

$$\chi=\frac{\omega_s}{v-v_g}+\chi',$$

$$\delta=\frac{v_g}{v-v_g}\omega_s+\delta',$$

and dropping the primes we can rewrite the equation as

$$\left(\delta-\chi v+\frac{i\gamma}{2}\right)(\delta-\chi v_g+i\Gamma)=-\gamma_0^2,$$

where $\gamma_0^2=G/2\omega_s$ is the new coupling constant. From here follows a direct solution

$$\delta=\frac{v+v_g}{2}\chi-\frac{i\left(\Gamma+\frac{\gamma}{2}\right)}{2}\pm i\sqrt{\gamma_0^2+\frac{1}{4}\left(\Gamma-\frac{\gamma}{2}+i\chi(v-v_g)\right)^2}, \tag{37}$$

defining the complex frequency of the interacting waves for a set wave number. The maximum $\mathrm{Im}\,\delta$ is reached at the exact resonance $(\chi=0)$. From Eq. (37) it is readily found that the threshold of instability is

$$\gamma_0^2>\gamma_{\mathrm{th}}^2=\frac{\Gamma\gamma}{2} \tag{38}$$

or

$$G>\Gamma\gamma\omega_s. \tag{39}$$

This threshold can be easily exceeded for real microstrip circuits. For a small excess of the threshold

$$\gamma_0^2-\frac{\Gamma\gamma}{2}\ll\frac{1}{4}\left(\Gamma+\frac{\gamma}{2}\right)^2$$

the maximum growth rate of the instability is expressed as

$$\mathrm{Im}\,\delta=\frac{\gamma_0^2-\Gamma\gamma/2}{\Gamma+\gamma/2};$$

if the opposite condition is met,

$$\gamma_0^2-\frac{\Gamma\gamma}{2}\gg\frac{1}{4}\left(\Gamma+\frac{\gamma}{2}\right)^2,$$

then

$$\mathrm{Im}\,\delta=\gamma_0.$$

The wave number range in which instability $\mathrm{Im}\,\delta(\chi)>0$ occurs is defined by the relation $-\Delta\chi/2<\chi<\Delta\chi/2$, where $\Delta\chi$ is the instability domain

$$\Delta\chi=\frac{4\sqrt{\gamma_0^2-\Gamma\gamma/2}}{v-v_g}. \tag{40}$$

So we see that the soliton chain which was initially out of phase with the wave is unstable to growing perturbations in the positions of solitons ξ_n and the wave in the electrodynamical system ψ. The physical mechanism responsible for this instability is clearly understood as soliton bunching in the decelerating phase of the wave. The growth of the wave, in its turn, is related to bunching of solitons. At $v_g>0$ the instability exhibits a convective behavior.[23] The wave packet is increasing with a growth rate (37) in the reference frame moving at the velocity $v=(v_g+v)/2$, while in the laboratory frame perturbation tends to zero at any fixed point. It is a well-known fact that a system with a convective instability will feature a spatial exponential amplification of waves. The gain per unit length is determined from the dispersion equation (36) if that is resolved for χ. The maximum gain is expressed as $g=e^{\mathrm{Im}\chi x}$, where $\mathrm{Im}\chi$ is defined by the formula

$$\mathrm{Im}\chi=-\frac{\Gamma/v_g+\gamma/2v}{2}+\sqrt{\frac{\gamma_0^2}{vv_g}+\frac{1}{4}\left(\frac{\Gamma}{v_g}-\frac{\gamma}{2v}\right)^2},$$

which for a large supercriticality reduces to a simpler form:

$$\mathrm{Im}\chi=\frac{\gamma_0}{\sqrt{v_g v}}=\left(\frac{G}{\omega_s v_g v}\right)^{1/2}.$$

If the coupling coefficient G is large: $G\gg\omega_s^3$, such that $\mathrm{Im}\,\delta$ becomes comparable with ω_s, it is no longer possible to reduce Eq. (34) to Eq. (36), so we have to solve the cubic equation (34). For a large supercriticality it gives the following expression to describe the growth rate of the instability:

$$\mathrm{Im}\,\delta=\frac{\sqrt{3}}{2}G^{1/3},$$

and the imaginary part of the wave number responsible for amplification will be defined as

$$\mathrm{Im}\chi=\frac{\sqrt{3}}{2}\left(\frac{G}{v_g v^2}\right)^{1/3}.$$

This leads us to a conclusion that a Josephson junction coupled to an electrodynamical system with dispersion can be used for amplification of electromagnetic waves. Naturally, large gains will take a rather large length of the system to be achieved.

If $v_g(k_s)<0$ in the point of crossing, k_s, of the dispersion curves for a slow density wave in a soliton chain and for a wave in electrodynamical system, then the instability described by Eq. (34) will be absolute and the system in question will be able to operate as a generator. This case is a very close analogy with generation of waves by a backward-wave tube oscillator.[22] We look for the critical values of G and the system length that will make generation possible. Solution of this problem takes more than the dispersion equation (34): we also need to know the relations between ψ, χ, and F in the growing waves. Therefore, we now get back to the set (30)–(32).

Assuming $G\ll\omega_s^3$, we expand the left-hand sides in this set in a series in the vicinity of the Cherenkov synchronism point exactly as we did it in the attempt to simplify the dispersion equation. The result is a linear set of equations for the amplitudes ξ and ψ,

$$\left(\delta-\chi v+i\frac{\gamma}{2}\right)\xi=-\sigma_1\psi,$$

$$(\delta-\chi v_g+i\Gamma)\psi=\sigma_2\xi,$$

where

$$\sigma_1=\frac{1}{8}\frac{k_c^2\epsilon f(k_c)(1-v^2)^{3/2}}{1+k_c^2(1-v^2)}\quad\text{and}\quad\sigma_2=\frac{k_c^2\epsilon f(k_c)}{2\omega_s d(\partial D/\partial\omega)}$$

are the coupling coefficients of the slow wave in the soliton chain and of the wave in the electrodynamical system, respectively. From this set we directly derive the dispersion equation (33) considered earlier, since, as is easily seen, $\sigma_1\sigma_2=\gamma_0^2$. It follows from this set that the fields can be defined as

$$\xi=C_1(\delta+\chi_1|v_g|+i\Gamma)\exp(-i\delta t+i\chi_1 x)+C_2(\delta+\chi_2|v_g|+i\Gamma)\exp(-i\delta t+i\chi_2 x),$$

$$\psi=C_1\sigma_2\exp(-i\delta t+i\chi_1 x)+C_2\sigma_2\exp(-i\delta t+i\chi_2 x),$$

where C_1,C_2 are the arbitrary constants, χ_1,χ_2 are the solutions to Eq. (33),

$$\chi_{1,2}=\chi_0\pm\chi'$$

in which

$$\chi_0=\frac{1}{2}\left(\frac{\delta+i(\gamma/2)}{v}+\frac{\delta+i\Gamma}{|v_g|}\right)$$

and

$$\chi'=\sqrt{\frac{1}{4}\left(\frac{\delta+i(\gamma/2)}{v}-\frac{\delta+i\Gamma}{|v_g|}\right)^2+\frac{\gamma_0^2}{v|v_g|}}.$$

Satisfying the boundary conditions

$$\xi(0)=0,\quad \psi(L)=0,$$

by which waves do not enter the system (L is the length of the system), we derive the characteristic equation for the instability parameters versus a length of the interaction region and the system parameters, which can be represented in a convenient form as follows:

$$\frac{y}{\sqrt{a^2-y^2}}=-\tan y,$$

where $y=\chi' L$, $a=\gamma_0 L/\sqrt{v|v_g|}$. It is easily seen that nontrivial solutions of this equations $y_n\neq 0$, appear at $a>\pi/2(2n+1)$, where n is the integer. We find solutions to this equation and, further, expressing χ' via δ obtain the complex frequencies of the eigenmodes

$$\frac{\delta+i(\gamma/2)}{v}+\frac{\delta+i\Gamma}{|v_g|}=\frac{\sqrt{a^2-y_n^2(a)}}{L},$$

and the threshold value of the pump parameter a is determined from the equation

$$\frac{\gamma}{2v}+\frac{\Gamma}{|v_g|}=\frac{2}{L}\sqrt{a^2-y^2(a)}.$$

If the length of the system is large, such that $a\gg\pi/2$, $y(a)\to\pi$, the root can be expanded in series to yield an explicit expression for the dependence of the fundamental mode growth rate on damping and the system length:

$$\delta\left(\frac{1}{v}+\frac{1}{|v_g|}\right)=i\left(\frac{2\gamma_0}{\sqrt{vv_g}}-\frac{\Gamma}{|v_g|}+\frac{\gamma}{2v}\right)-\frac{\pi^2(a)}{aL}.$$

This is a much more strict threshold condition on the density and velocity of a soliton chain than Eqs. (38) and (39), which makes it by far more complicated to achieve generation in such a soliton analog of BWT than to realize the amplification mode at $v_g>0$.

Note, however, that generation may also be made possible at $v_g>0$ by introducing an auxiliary feedback loop providing transfer of the amplified wave energy from the output to the input of the system. The feedback can be effected either via reflection at the unmatched ends of the system, or by using a ring geometry. In the latter case, given instability, the system will always work as a generator regardless of its dispersion characteristics, and the growth rate of the instability will be defined by Eq. (37) with a range of possible values for χ determined by the size of the system, $\chi=2\pi n/L$, where n is the integer. Extraction of energy from such a system will take an additional transmission line.

It has been shown that interaction of a soliton chain with a wave in electrodynamical system may give rise to amplification and generation of an electromagnetic wave and to soliton bunching in the decelerating phase of the wave. In the following section we analyze the nonlinear stage of this bunching instability with a view to estimating the output power of the proposed device.

V. NONLINEAR STAGE OF SOLITON BUNCHING

At the nonlinear stage of bunching the solitons' shifts from their initial positions can no longer be assumed small as in derivation of the dispersion relations (30)–(32). To derive the equations describing the nonlinear mode of interaction between solitons and a synchronous wave in the electrodynamical system we consider the simplest case, when the amplification band defined by the relation $\mathrm{Im}\,\delta(\chi)>0$ from Eq. (37) includes only one wave whose field can be written in the form

$$\psi(x,t)=a(t)\exp(-i\omega t+ikx)+\text{c.c.},$$

where $a(t)=|a|e^{i\varphi}$ is the slow ($|\dot{a}|\ll\omega_c|a|$) complex amplitude independent of the coordinate. This situation may take place, for example, in an annular geometry of Josephson junction and electrodynamical system. The wave number k, due to periodicity of the boundary conditions, can have only a discrete series of values $k_n=2\pi n/L$, where L is the length of the system; so, if the intermode distance in the k space exceeds the amplification band $2\pi/L>\Delta\chi$ defined by Eq. (40), then only one wave will be amplified.

It is convenient to represent a soliton's coordinates in the chain as

$$Z_n=vt+\xi_n,$$

where the velocity $v=\omega_r/k_r$, ω_r is the frequency of the resonator eigenmode with the wave number k_r nearest to k_c. Here, unlike the case in Sec. III, we do not assume the soliton shifts ξ much smaller than the wavelength. Setting the vortex velocities close: $|\dot{\xi}_n|\ll v$ and the soliton chain rather rare, such that a soliton-soliton interaction can be neglected, we find from Eqs. (10) and (11) the expression

$$\ddot{\xi}_n+\gamma\dot{\xi}_n=\sigma_1(ae^{ik_r\xi_n}+\text{c.c.})+\Delta I, \tag{41}$$

where $\Delta I=(\pi/4)I-\gamma(v/\sqrt{1-v^2})$ is the correction to the bias current, σ_1 is the earlier introduced coefficient describing the force of the wave-vortex coupling.

We derive the expression for the slow amplitude $a(t)$ from Eq. (28) in the following way. First, we expand the function $D(\omega,k)$ just as above in Eq. (33), near the point ω_r, k_r:

$$D(\omega_r+\delta,k_r)=D(\omega_r,k_r)+\frac{\partial D}{\partial\omega}(\delta+i\Gamma).$$

Then, rewriting this expression in terms of t by replacing the multiplication operators $i\delta$ with the operators $-\partial/\partial t$ and integrating the right- and left-hand sides of Eq. (28) over the space coordinate we have

$$\frac{\partial a}{\partial t}+\Gamma a=-\frac{\sigma_3}{L}\sum_{n=0}^{N}\exp(-ik_r\xi_n), \tag{42}$$

where $\sigma_3=\epsilon k_r f(k_r)/|\partial D/\partial\omega|$, N is the number of solitons in the system, L is the system length. The resulting set of equations (41), (42) provides the basis for investigation on the nonlinear stage of soliton bunching in the field of a wave; however, despite the above simplifications it remains too complicated to be studied analytically. To estimate the radiation power we assume that all solitons are in phase with the excited wave, i.e., $\xi_n=\xi+2\pi l/k_r$, l is an integer. We seek a solution to Eqs. (41), (42) in the form

$$\xi=\xi_0+\Delta v t,$$

$$a=Ae^{-i\delta t+\phi_0},$$

where Δv is the perturbation of soliton velocity, $\delta=k_c\Delta v$ is the frequency mismatch, A is the real amplitude of the wave. Assuming further A and Δv to be stationary: $\partial/\partial t=0$, we find the equations

$$\gamma\Delta v=\sigma_1 A\cos\Theta+\Delta I,$$

$$\Gamma A=-\sigma_3 n\cos\Theta, \tag{43}$$

$$\delta A=-\sigma_3 n\sin\Theta,$$

which define the steady-state value of the amplitude and frequency of the electromagnetic wave; here $\Theta=k_r\xi_0$ is the soliton phase with respect to the wave, $n=N/L$ is the concentration of solitons. Since there is a straight relationship between the perturbation of solitons' velocity Δv and the voltage perturbation ΔU in the Josephson junction,

$$\Delta U=2\pi n\Delta v,$$

we can, knowing the dependence $\Delta v(\Delta I)$, also determine the I-V characteristic of the junction coupled to electrodynamical system. Solution (43) is easily found in the form

$$A=\frac{\sigma_3 n}{\sqrt{\Gamma^2+k_r^2\Delta v^2}},\quad \cos\Theta=-\frac{\Gamma}{\sqrt{\Gamma^2+k_r^2\Delta v^2}},$$

and the velocity perturbation is related to ΔI as $\gamma\Delta v+\sigma_3\sigma_4 n\Gamma/(\Gamma^2+k_r^2\Delta v^2)=\Delta I$, which defines the junction I-V characteristic. Its qualitative curve in the vicinity of the resonance point is shown in Fig. 5. Note that the resonance velocity of solitons is determined by the Cherenkov synchronism condition $k_c v=\omega(k_c)$, and, since the spectrum of the wave numbers $k_r=2\pi l/L$ and frequencies $\omega(k_r)$ in the ring geometry considered here is discrete, the voltage in the junction will be defined as $U=2\pi vn/L$, where N is the number of solitons in the system, which relates to the frequencies of excited modes as $U=(N/l)\omega$. The presence of the steps on the current-voltage characteristic is not the evidence of Cherenkov emission (similar steps may be caused by Fiske resonances, etc.). Moreover, if the Cherenkov radiation takes place and the spectrum of the resonator is very dense, then the C-V characteristic will not have any steps, it will only go lower than in the unloaded Josephson junction case. So, to distinguish Cherenkov emission, direct measurements of the radiation are necessary.

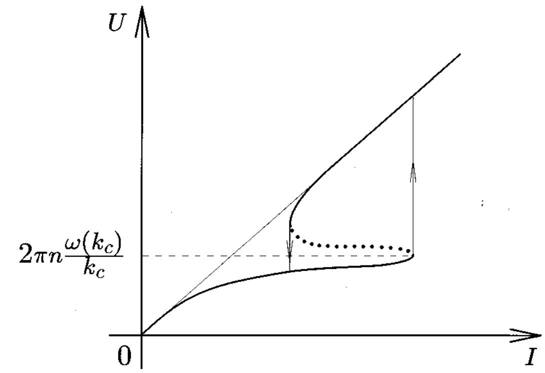

FIG. 5. The current-voltage characteristic of the system considered. Dotted line shows the unstable branch of the I-U characteristic. Thin line depicts the current-voltage characteristic of the long Josephson junction without coupling to an additional transmission line.

The maximum amplitude of the wave

$$A_{\max}=\frac{\sigma_3 n}{\Gamma}=\frac{\epsilon n k_r f(k_r)}{\Gamma|\partial D/\partial\omega|}$$

is reached at exact resonance $\Delta v=0$ which corresponds to $\Theta=\pi$ and occurs at $\Delta I=\sigma_1\sigma_3 n$. Knowledge of the wave amplitude and the expression for the energy density in a dispersive medium (20) provide a sufficient basis for estimation of the energy stored in an annular resonator:

$$E=\frac{\epsilon^2 n^2 L k_r^4|f(k_r)|^2}{4\Gamma^2|\partial D/\partial\omega|^2}\left(v^2\frac{\partial(\omega C_2(\omega))}{\partial\omega}+L_2^{-2}\frac{\partial(\omega L_2(\omega))}{\partial\omega}\right),$$

where $n=N/L$ is the density of a soliton chain. From here we derive the radiation power:

$$P=2\Gamma E=\frac{\epsilon^2 n^2 L k_r^4|f(k_r)|^2}{2\Gamma|\partial D/\partial\omega|^2}\times\left(v^2\frac{\partial(\omega C_2(\omega))}{\partial\omega}+L_2^{-2}\frac{\partial(\omega L_2(\omega))}{\partial\omega}\right).$$

It is clearly seen that its value increases with a higher Q factor of the resonator.

We shall now use the obtained formulas to estimate a practically feasible radiation emission power of solitons moving in a dispersive electrodynamical system. The power unit in the dimensionless system of units used here is the quantity $P_0 = \Phi_0 j_c W v_s / 2\pi$, where j_c is the critical current density per unit area, W is the width of a Josephson junction. Using the typical values for the Nb-AlO-Nb-based Josephson junctions (see, for example, Refs. 3–5), $j_c = 10^4$ A/cm^2, $W = 10^{-5}$ m, $v_s = 10^7$ m/s, we find a definition for the power unit in our system of units:

$$P_0 = 3 \times 10^{-6} W.$$

The dimensionless power can be estimated by putting in the characteristic parameter values attainable in the experiments on Josephson junctions and strip lines on the Nb basis. Assuming there is only one soliton on the length of an excited wave we have $n = k_c / 2\pi$ and taking, for example, $k_c = 1$, $\epsilon = 0.3$, $C_2 = 10$, $\Gamma = 10^{-3}$ we come to a very hopeful estimate for the radiation power:

$$P = 100 P_0 = 3 \times 10^{-4} W.$$

The mechanism underlying such a high radiation efficiency is the Cherenkov emission of solitons, which makes possible their interaction with electromagnetic waves in the entire junction.

VI. CONCLUSION

The main results achieved through this effort can be formulated as follows. We propose a method of using a Josephson junction for generation and amplification of electromagnetic waves, which involves the Cherenkov emission of linear waves by Josephson solitons. The device by which this principle can be realized is essentially a long Josephson junction electrodynamically coupled to a dispersive waveguide system that allows for resonance interaction of moving solitons with a wave. From the viewpoint of general physics the proposed Josephson oscillator is an analog of the traveling-wave and backward-wave tubes. The problem of interaction between the Josephson vortices moving in a long Josephson junction electrodynamically coupled to a transmission line with dispersion is considered. It is shown that the Cherenkov effect causes the Josephson vortices to effectively excite electromagnetic waves, the radiation intensity is estimated. We found out and studied the effect of soliton bunching in the field of excited wave. By bunching in the decelerating phase of the wave the solitons are shown to cause wave amplification due to coherence contributions of individual solitons. The coefficients of spatial amplification and the growth rate of bunching instability are calculated. Other results include derivation of the equations describing a nonlinear stage of soliton bunching and estimation of the Cherenkov radiation power. The Josephson flux-flow oscillator based on the Cherenkov effect is shown to hold the promise of good energy characteristics. It seems quite likely that the soliton bunching effect will largely upgrade the FFO noise properties.

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