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Energy oscillations of scattered waves for disorder-induced crossing resonance

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We investigate the energy dynamics of compounded states, which arise in a system of waves linearly coupled via a random interaction. Randomness of the interaction causes energy flow from the coherent wave to scattered states of the second participating wave. If disorder-induced crossing resonance occurs, we discovered that there is also a current from scattered waves to the coherent component. The interference of these flows causes the distinctive temporal oscillation of the energy of scattered waves. [S0163-1829(97)02417-X]

I. INTRODUCTION

The term ''crossing resonance'' describes a wide class of phenomena occurring when the dispersion curves of two excitations intersect each other in the absence of an interaction. Even a weak interaction between these excitations removes the degeneracy giving rise to new compounded excitations. Such crossing resonances as magnetoelastic resonance,[1] polaritons,[2] and others play an important role in the solid state physics of ordered materials and have been studied in detail.

Recently, interest in the behavior of such compounded states in *disordered* materials has arisen. The influence of disorder on polaritons has been considered in a number of papers.[3–5] It was implicitly assumed in all of these papers that only electromagnetic parameters of the medium were random. In this case peculiarities of the influence of disorder on phenomena related to crossing resonances cannot fully manifest themselves. The only difference between this situation and the case of pure electromagnetic waves propagating through a random medium is a more complicated dispersion law.

Since the interaction plays a crucial role in crossing resonances, one can assume that randomness of the coupling parameter should significantly change the manifestation of this phenomenon in disordered materials. In order to emphasize this point, a model with a random zero-mean interaction has been considered in Ref. 6. In this model the coupling of waves is provided by fluctuations only. This causes a *stochastic* or *disorder-induced crossing resonance* (DICR). The main result of Ref. 6 is the discovery that a random coupling can generate well-defined compounded excitations. This fact is not surprising in the geometric optical limit. In this case waves have the opportunity to interact due to an approximately homogeneous coupling parameter while traveling a distance much greater than their wavelengths. As a result, compounded states with dispersion laws dependent on position arise.[7] The surprising thing is that the impact of a random interaction can extend far beyond the geometric optical limit. In Ref. 6 well-defined compounded states with wavelengths much longer than the correlation radius have been discovered. The main peculiarity of these excitations is that they consist of a coherent component of one wave and a scattered component of the second participating wave. These states were shown to exhibit unusual dispersion and relaxation properties.

This phenomenon opens a direction in the field of wave propagation through random media because the additional kind of excitation and materials become involved in the study. Though a zero-mean random interaction may seem to be an exotic one, it may exist in various physical systems. A magnetoelastic resonance in zero-mean magnetostrictive alloys[8] gives one opportunity for application of the model.[9,10] Another example may be the interaction between light and excitations that are normally inactive in an ideal crystal. A random distortion of the ideal structure results in the appearance of a local random coupling parameter with a zero-mean value.[11]

In the present paper we discuss another effect that can exist in systems with DICR. We are interested here in the process of energy exchange between a coherent part of the compounded state and its scattered partner. We consider an initially excited coherent wave with an acousticlike dispersion law and with a fixed wave number. The dependence of the scattering energy of the second participating wave upon time is the main subject for the study. In the ordinary case of a single wave excited in a random media the solution of this problem would be trivial. Only an exponential decay of the coherent component accompanied by a corresponding increase of the scattered energy would be found in this case. In the model considered in this paper the situation is more interesting and unusual. Alongside the routine exponential relaxation we find a distinctive dynamical stage of a temporal evolution of the system. We show that it appears in the form of an oscillatory exchange of the energy between coherent and scattered components of the compounded states. This

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behavior is effective for a time less than the relaxation time of the coherent component and it exists only if conditions for DICR are fulfilled.

Similar energy oscillations are typical phenomena in deterministic systems with cross resonances. In polariton physics they are known, for example, as nutation or Rabi oscillations. They reflect the coherent nature of the interaction between different waves. A similar behavior in random systems brings the idea that a random interaction can keep the coherency of scattered waves over longer distances than in systems with other kinds of disorder. In other words, we deal here with an interesting phenomenon where the interaction destroys the coherency due to its random nature and, at the same time, stabilizes the scattered waves.

II. MODEL AND GENERAL EXPRESSION FOR SCATTERED ENERGY

We consider the model, which may be described by a Lagrangian of the form

$$\mathcal{L}=\int\left[\frac{1}{2}\rho_1\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2}\rho_2\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}d_1\left(\frac{\partial u}{\partial\mathbf{r}}\right)^2-\frac{1}{2}d_2\left(\frac{\partial\phi}{\partial\mathbf{r}}\right)^2-\frac{1}{2}m\phi^2-\mathbf{P}\nabla u\,\phi\right]d^3r\,dt. \tag{1}$$

This Lagrangian describes two scalar waves, one of which has an acoustic dispersion law

$$\omega_k=v_1k, \tag{2}$$

and the second wave, which has an optical spectrum

$$\epsilon^2=\omega_0^2+v_2^2k^2, \tag{3}$$

where

$$v_1^2=\frac{d_1}{\rho_1},\quad \omega_0^2=\frac{m}{\rho_2},\quad v_2^2=\frac{d_2}{\rho_2}. \tag{4}$$

The parameters ρ_1,ρ_2 and d_1,d_2,m represent densities and force parameters, respectively.

These waves are linearly coupled due to the last term in Eq. (1). Unlike the model considered in Ref. 6 we introduce the gradient form of the interaction with a vector coupling parameter **P**. This form of interaction corresponds to magnetoelastic coupling between elastic and spin waves.[9,10,12] The direction of the coupling parameter should be assigned to the equilibrium direction of the magnetization, and the modulus of **P** is a magnetostriction parameter. To apply the model to a magnetoelastic resonance one should also substitute squares of frequencies by the corresponding frequencies themselves.

Lagrangian (1) can describe the polariton problem as well, if one replaces the spatial gradient in the interaction term by a corresponding temporal derivative. In this case **P** may be interpreted as a random local dipole moment coupling an electromagnetic wave and optical phonons.[11] In ω-k space this alteration can be done by means of a simple replacement of a corresponding k^2 term by ω^2. The results of the calculations shall not be affected considerably due to this replacement.

The spatial components P_i of the vector **P** are considered as random functions with zero mean and with given correlation properties

$$\langle P_i\rangle=0,$$

$$\langle P_i(\mathbf{r}_1)P_j(\mathbf{r}_2)\rangle=\lambda^2\Phi_{ij}K(\mathbf{r}_1-\mathbf{r}_2). \tag{5}$$

Here λ is the rms fluctuation of the coupling parameter, Φ_{ij} reflects tensor properties of the correlations, and $K(\mathbf{r}_1-\mathbf{r}_2)$ is a normalized $[K(0)=1]$ scalar function that introduces a characteristic scale to the inhomogeneities by means of correlation radius r_c.

The total energy of the system with Lagrangian (1) has the form

$$E=\int\left[\frac{1}{2}\rho_1\dot{u}^2+\frac{1}{2}\rho_2\dot{\phi}^2+\frac{1}{2}d_1\left(\frac{\partial u}{\partial\mathbf{r}}\right)^2+\frac{1}{2}d_2\left(\frac{\partial\phi}{\partial\mathbf{r}}\right)^2+\frac{1}{2}m\phi^2+\mathbf{P}\frac{\partial\mathbf{u}}{\partial\mathbf{r}}\phi\right]dV. \tag{6}$$

The terms with the spatial gradient after integrating and neglecting surface parts become

$$\int\left(\frac{\partial u}{\partial\mathbf{r}}\right)^2dV=-\int u\frac{\partial^2u}{\partial\mathbf{r}^2}dV, \tag{7}$$

$$\int\left(\frac{\partial\phi}{\partial\mathbf{r}}\right)^2dV=-\int\phi\frac{\partial^2\phi}{\partial\mathbf{r}^2}dV. \tag{8}$$

Using the equations of motion

$$\rho_1\frac{\partial^2u}{\partial t^2}-d_1\frac{\partial^2u}{\partial\mathbf{r}^2}=\frac{\partial}{\partial r_i}(P_i\phi), \tag{9}$$

$$\rho_2\frac{\partial^2\phi}{\partial t^2}+m\phi-d_2\frac{\partial^2\phi}{\partial\mathbf{r}^2}=-P_i\frac{\partial u}{\partial r_i}, \tag{10}$$

following from Lagrangian (1), these terms can be turned into the form

$$d_1\int\left(\frac{\partial u}{\partial\mathbf{r}}\right)^2dV=-\int u\left[\rho_1\ddot{u}-\frac{\partial}{\partial\mathbf{r}}(\mathbf{P}\phi)\right]dV, \tag{11}$$

$$d_2\int\left(\frac{\partial\phi}{\partial\mathbf{r}}\right)^2dV=-\int\phi\left[\rho_2\ddot{\phi}+m\phi+\mathbf{P}\frac{\partial\mathbf{u}}{\partial\mathbf{r}}\right]dV. \tag{12}$$

Inserting these expressions into the original equation for the energy one can obtain

$$E=\frac{1}{2}\int dV\left\{\rho_1\left[\left(\frac{\partial u}{\partial t}\right)^2-u\frac{\partial^2u}{\partial t^2}\right]+\rho_2\left[\left(\frac{\partial\phi}{\partial t}\right)^2-\phi\frac{\partial^2\phi}{\partial t^2}\right]\right\}, \tag{13}$$

where we have omitted the terms

$$E_{\text{surf}}=\frac{1}{2}\int\left[\mathbf{P}\phi\frac{\partial u}{\partial\mathbf{r}}+u\frac{\partial}{\partial\mathbf{r}}(\mathbf{P}\phi)\right]dV, \tag{14}$$

which can obviously be reduced into the total derivative of $\mathbf{P}\phi\mathbf{u}$ and neglected. Equation (13) is an exact representation for the energy, which is more convenient for further use than the original equation (6). The absence of the interaction term in this expression is a consequence of the linear nature of coupling.

The Fourier-transformed expression for the energy takes the form

$$E=-\frac{1}{2}\int d\omega_1 d\omega_2 d^3k[\rho_1\hat{u}(\omega_1,\mathbf{k})\hat{u}(\omega_2,-\mathbf{k})$$

$$+\rho_2\hat{\phi}(\omega_1,\mathbf{k})\hat{\phi}(\omega_2,-\mathbf{k})]\omega_2(\omega_1-\omega_2)\exp it(\omega_1+\omega_2), \tag{15}$$

where the caret indicates the Fourier transform of the corresponding value and ω_1,ω_2 and **k** are frequencies and a wave number, respectively. This expression contains two terms, one of which relates to the energy of the acoustic wave and the second which represents the energy of the optical wave. Each of these terms depends upon time, but their sum must be a constant according to the energy conservation law. To analyze the evolution of the optical scattered energy one should calculate the value

$$Z_\phi=\langle\hat{\phi}(\omega_1,\mathbf{k})\hat{\phi}(\omega_2,-\mathbf{k})\rangle, \tag{16}$$

where brackets $\langle\cdots\rangle$ indicate averaging over realizations of the random function representing the coupling parameter. Equations of motion of the model (9) with initial conditions read as

$$u(\mathbf{r},t)|_{t=0}=u_{\rm in}(\mathbf{r}),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0,$$

$$\phi(\mathbf{r},t)=0|_{t=0},\quad \left.\frac{\partial\phi}{\partial t}\right|_{t=0}=0, \tag{17}$$

lead to the following expression for Z_ϕ:

$$Z_\phi=-\frac{1}{\rho_2^2}G_\phi(\omega_1,\mathbf{k})G_\phi(\omega_2,-\mathbf{k})\int\langle[\mathbf{k}_1\hat{\mathbf{P}}(\mathbf{k}-\mathbf{k}_1)]$$

$$\times[\mathbf{k}_2\hat{\mathbf{P}}(\mathbf{k}-\mathbf{k}_2)]u(\omega_1,\mathbf{k}_1)u(\omega_2,\mathbf{k}_2)\rangle d^3k_1d^3k_2, \tag{18}$$

where $G_\phi(\omega,\mathbf{k})$ denotes the initial Green's function for the optical waves which has the form

$$G_\phi(\omega,\mathbf{k})=\frac{1}{\omega^2-\epsilon_k^2}. \tag{19}$$

In order to evaluate Eq. (18) we replace the acoustic wave amplitude $u(\omega,k)$ by its average value $\langle u(\omega,k)\rangle$. This approximation takes into account all processes of second order in terms of the effective coupling parameter. The procedure is valid for $t<\tau$, where τ is the relaxation time of the coherent acoustic wave. When the time becomes greater than τ the coherent component almost disappears and an equilibrium mix of the incoherent participating waves occurs. In this case the approximation breaks down.

The average amplitude $\langle u(\omega,\mathbf{k})\rangle$ can be expressed in terms of the acoustical averaged Green's function $\langle G_u(\omega,\mathbf{k})\rangle$:

$$\langle\hat{u}(\omega,\mathbf{k})\rangle=i\omega\hat{u}_{\rm in}(\mathbf{k})\langle G_u(\omega,\mathbf{k})\rangle, \tag{20}$$

where the Green's function itself can be expressed in terms of the mass operator $Q_u(\omega,\mathbf{k})$ in the usual way:

$$\langle G_u(\omega,\mathbf{k})\rangle=\frac{1}{\omega^2-\omega_k^2-Q_u(\omega,\mathbf{k})}. \tag{21}$$

We approximate the mass operator by the first term in its expansion in terms of the effective coupling parameter (the Bourret approximation[13])

$$Q_u(\omega,\mathbf{k})=\frac{\Phi_{ij}k^ik^j}{\rho_1\rho_2}\int d^3k_1G_\phi(\omega,\mathbf{k}_1)S(\mathbf{k}-\mathbf{k}_1). \tag{22}$$

It is important to note that two approximations made for the correlation function Z_ϕ, Eq. (18), and for the Green's function, Eq. (22), are consistent in a sense that used together they preserve the energy of the system.

If the initially excited state, $u_{\rm in}(\mathbf{k})$, has the form

$$u_{\rm in}(\mathbf{k})=u_0\cos(\mathbf{k}_0\mathbf{r}), \tag{23}$$

we can deduce the following expression for the scattered opticlike part of the energy:

$$E_\phi=V\frac{\pi}{\rho_2}\frac{u_0^2}{2}\lambda^2\Phi_{ij}k_0^ik_0^j\int d\omega_1d\omega_2\omega_1\omega_2^2(\omega_1-\omega_2)$$

$$\times\langle G_u(\omega_1,\mathbf{k}_0)\rangle\langle G_u(\omega_2,-\mathbf{k}_0)\rangle e^{it(\omega_1+\omega_2)}$$

$$\times\int d^3kG_\phi(\omega_1,\mathbf{k})G_\phi(\omega_2,-\mathbf{k})S(\mathbf{k}-\mathbf{k}_0), \tag{24}$$

where the amplitude u_0 determines the total energy of the system,

$$E_0=\tfrac{1}{4}u_0^2\rho_1\omega_k^2V, \tag{25}$$

and k_0 fixes the wavelength of the originally excited wave. The notations V and $S(\mathbf{k})$ are used in Eqs. (24) and (25) for the total volume of the system and the spectral density of the inhomogeneities, respectively (the spectral density is defined as the Fourier transform of the correlation function).

Using Eq. (22), we can transform Eq. (24) for the energy into the form

$$E_\phi=\rho_1V\frac{u_0^2}{4}\int d\omega_1d\omega_2\frac{\omega_1\omega_2^2}{\omega_1+\omega_2}\langle G_u(\omega_1,k_0)\rangle\langle G_u(\omega_2,-k_0)\rangle$$

$$\times[Q_u(\omega_1,k_0)-Q_u(\omega_2,k_0)]\exp[it(\omega_1+\omega_2)]. \tag{26}$$

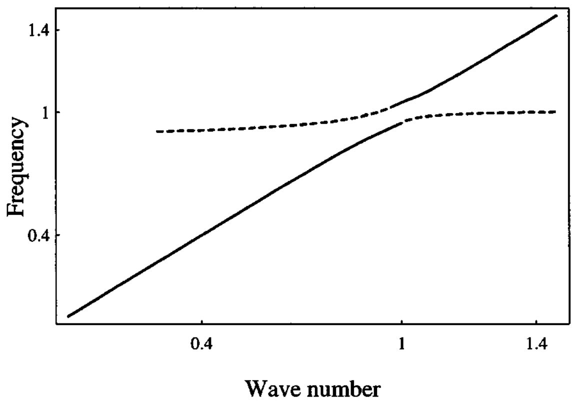

FIG. 1. Dispersion curves of compounded states in the case of DICR. The solid lines show parts of the branches, which tend to the initial acoustic dispersion curve in the off-resonance region. The dashed lines depict the states that arise due to DICR. These states exist in the vicinity of the resonance point only. Wave numbers and frequencies are normalized by the resonance wave number k_r and the resonance frequency ω_r, respectively.

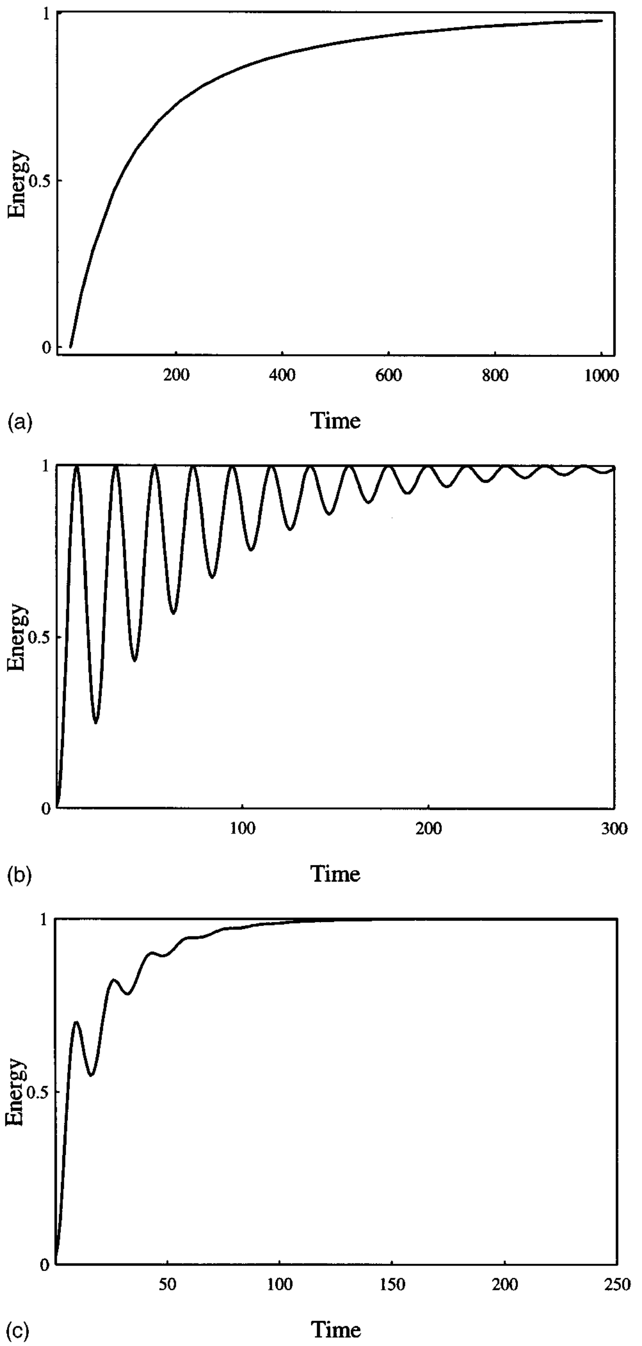

FIG. 2. Dependence of the energy of the scattered waves upon time when DICR is suppressed (a) and when DICR is in effect (b), (c). Here (b) and (c) correspond to resonant and off-resonant regions, respectively. The parameters $v_2/v_1 = 5\times 10^{-4}$ and $k_c/k_r = 250$ are the same for all figures.

The dynamics of the energy is determined by the poles of the integrand in Eq. (26). These are poles of the averaged Green's functions and a pole at $\omega_1 = -\omega_2$. Evaluation of integrals in Eq. (26) requires calculation of residues at these poles. The pole of the averaged Green's function determines the modification of the initial dispersion law and damping of the excitations. The dispersion equations determining the pole have been analyzed in detail in Ref. 6. It was found that the behavior of the system depends on the relation between the effective interaction and relaxation parameters Λ and Γ, respectively, at the resonance point. These parameters are given according to the expressions

$$\Lambda = \frac{\lambda}{\omega_r}, \quad \Gamma = \frac{v_2}{v_1}\frac{k_c}{k_r} v_2 \sqrt{k_c^2 + k_r^2}, \tag{27}$$

where ω_r is the resonance frequency and k_r is the corresponding wave number, and k_c is the correlation wave number that is reciprocal to the correlation radius, $k_c \sim 1/r_c$.

If $\Lambda < \Gamma$, the dispersion equation has only one well-defined solution and DICR does not occur. The dispersion curve in this case only slightly differs from the original one. In the case of opposite inequality one can find two solutions to the dispersion equations. Both of them are well defined within the resonance region. Consequently, a modified dispersion curve forms two branches separated by a gap Δ at the resonance point (Fig. 1). The gap is given by

$$\Delta = \sqrt{\Lambda^2 - \Gamma^2}. \tag{28}$$

Two parts of these branches turn into the original acoustic dispersion curve far from the resonance point and are well defined everywhere. The other two parts exist only in the vicinity of the resonance region and represent the above-mentioned additional solutions. They resemble part of the initial dispersion curve of the second participating wave. Far from the resonance point these parts become poorly defined. It should be emphasized that the nonzero average amplitude of the acoustic component is the same on both of these branches and the average amplitude of the optical partner on both branches is equal to zero. It allows one to conclude that both these branches relate to the same excitation rather than two different excitations. We call this excitation a quasiacoustic wave.

This situation differs from the case of a resonance caused by a deterministic interaction, where two different branches correspond to two different excitations with different ratios of component amplitudes. The second excitation in the case of DICR also exists and has a dispersion curve with a routine single mode shape, which is only slightly altered from the original optical curve.

The analysis of the dispersion curves conducted in Ref. 6 gives us a base for calculating the integral (26). The general expression for the energy, however, is rather cumbersome and we do not present it here. Since we are mainly interested in the frequency region near the resonance point, we discuss only simple formulas revealing the energy dynamics at resonance. Nevertheless, we use the general expression to plot the temporal dependence of the energy in Fig. 2.

If DICR does not occur, the dynamics of the energy follows the usual exponential law, which may be described by the expression

$$E_{\phi}=E_0\left[1-\exp(-\Gamma t)\frac{\Lambda^2}{\Gamma^2-\Lambda^2}\cosh^2\left(\frac{\sqrt{\Gamma^2-\Lambda^2}}{2}t+\tilde{\Psi}\right)\right], \tag{29}$$

and $\cosh\tilde{\Psi}=(\Gamma^2-\Lambda^2)/\Lambda$.

A graph in Fig. 2(a), showing the behavior of the energy in this situation, demonstrates a routine relaxation only.

DICR appears when $\Lambda>\Gamma$. The appearance of the gap changes the dynamics of the energy drastically. The expression describing this dynamics takes the form

$$E_{\phi}=E_0\left[1-\exp(-\Gamma t)\frac{\Lambda^2}{\Delta^2}\cos^2\left(\frac{\Delta}{2}t+\Psi\right)\right], \tag{30}$$

where $\cos\Psi=\Delta/\Lambda$. From Eq. (30) one can see that the relaxation controlled by the parameter Γ is accompanied by oscillations with frequency Δ. If $\Delta>\Gamma$, the interval $T<t<\tau$ of the distinct dynamic behavior of the energy exists where $\tau=1/\Gamma$ is the relaxation time and $T=2\pi/\Delta$. Within this interval the usual flow of the energy from the coherent component to the incoherent one is accompanied by the opposite current from the scattered waves to the coherent component. The interference of these flows gives rise to the oscillations.

The shape of the oscillations is shown in Fig. 2(b). This graph has been obtained from the general formula describing the dynamics of the energy for any value of the wave vector. Parameters of the system were chosen, however, to be very close to the resonance point. One can see that relaxation increases the minima of the energy with each oscillation, but does not change the maxima, which are always equal to the initial energy injected into the system. This means that at resonance the energy of the coherent wave transfers fully into the energy of the scattered waves *during each period of oscillation*. A part of this energy recycles back to the coherent wave, and the remainder accumulates in the scattered waves.

In the large-t limit both Eqs. (29) and (30), as well as corresponding graphs, show that the function E_{ϕ} saturates to one. This is an artifact of the approximations. We have neglected the scattered part of the acoustic energy that implies transformation of the coherent acoustic energy entirely into the scattered opticlike energy. However, this fact is important since it proves the consistency of the approximations with the energy conservation law.

The dynamics of energy is drastically changed in the off-resonance region [Fig. 2(c)]. In this case, not only does the number of oscillations decrease, but the curve's shape is also affected. Within each oscillation the energy of the coherent wave is no longer fully converted to the energy of the scattered waves. As a result the total picture more closely resembles the usual exponential relaxation.

The most astonishing aspect of these calculations is that the energy oscillations survive when the resonance wavelength is two orders of magnitude greater than the correlation radius. We use the value of the parameter $k_c/k_r=250$ to obtain these graphs. For this situation one might expect that the total averaging and elimination of the influence of inhomogeneities occur on the scale of a wavelength. It does not happen because of the factor v_2/v_1 in Eq. (27) for the relaxation parameter Γ. This factor is normally very small (for polaritons, for example, it is of order of 10^{-5}). It reduces the influence of the second factor k_c/k_r, which reflects a tendency to suppress a random interaction in the case of long waves. In the limit $k\to 0$ the last tendency of course prevails, as it should.

III. CONCLUSION

A model of two waves linearly coupled by a random zero-mean interaction has been considered. The main result of this paper is the discovery of an oscillatory exchange of energy between coherent and scattered components of mixed quasiacoustic states. These states appear in the case of disorder-induced crossing resonance. Such a behavior is effective only for times less than the relaxation time of the coherent component and reflects the shape of the dispersion law of the coherent quasiacoustic wave. This dispersion law consists of two branches separated by a gap. The period of the energy oscillations is determined by the gap and is usually much greater than the period of the waves.

The oscillations survive even if the resonance wavelength is much greater than the correlation radius. Although a random interaction becomes insufficient to cause DICR and energy oscillations in the case of extremely long waves, there still exists a wide interval of wavelengths where such oscillations do occur, even when these wavelengths are much greater than the correlation radius.

The energy oscillations discussed in this paper closely resemble well-known similar processes in deterministic systems. We should emphasize, however, that in our case one is dealing with energy exchange between deterministic (coherent wave) and randomized (scattered waves) via a zero-mean interaction. Establishing a well-defined quasicoherent behavior in such a system supports the general idea that fluctuations of a coupling parameter play a special role among other possible random parameters of the system. Scattered states caused by a random interaction possess long-lasting correlations which would not be expected for such systems.

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