# Dynamical vortices in superfluid films

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The coupling of superfluid film to a moving vortex is a gauge coupling entirely dictated by topology. From the definition of a linking number, one can define a gauge field $\mathcal{A}^\mu$, whose (2+1)-dimensional curl is the vortex three-current $J^\mu$, and to which the superfluid is minimally coupled. We compute the superfluid density and current response to a moving vortex. Exploiting the analogy to (2+1)-dimensional electrodynamics, we compute the effective vortex mass $M(\omega)$ and find that it is logarithmically divergent in the $\omega\to 0$ limit, with a constant imaginary part, yielding a super-Ohmic dissipation in the presence of an oscillating superflow. Numerical integration of the nonlinear Schrödinger equation supports these conclusions. The interaction of vortices with impurities coupling to the density also is discussed. [S0163-1829(97)08501-9]

## I. INTRODUCTION

In this paper we investigate the effective action and dynamics of vortices in compressible superfluid films at zero temperature. In an *incompressible* two-dimensional superfluid, vortices behave as massless charges in a uniform magnetic field—their motion is along an equipotential, the sum of logarithmic contributions from each of the point ''charges.'' [1] The Lagrangian for a charge-neutral system of vortices may be written

$$L=-\kappa\overline{\rho}\sum_i n_i X_i \dot{Y}_i+\frac{\overline{\rho}\kappa^2}{2\pi}\sum_{i<j} n_i n_j \ln|\mathbf{X}_i-\mathbf{X}_j|$$

where $\kappa=h/m$ is the rotational quantum, $\overline{\rho}$ is the bulk density (or superfluid density[2]), $n_i$ is the integer charge and $\mathbf{X}_i$ is the position of the $i$th vortex. The equations of motion,

$$\mathbf{X}_i=\frac{\kappa}{2\pi}\sum_{\substack{j\\(j\neq i)}} n_j \frac{\hat{z}\times(\mathbf{X}_i-\mathbf{X}_j)}{|\mathbf{X}_i-\mathbf{X}_j|^2},$$

preserve the total potential energy of the vortices, which of course is just the kinetic energy of the superfluid. These equations are first order in time—there is no inertial term $\Sigma_i(1/2)M_i\dot{\mathbf{X}}_i^2$ in $L$.

In a compressible superfluid, the speed of sound $c$ is finite. This leads to retardation effects in the vortex dynamics. Furthermore, accelerating vortices may radiate phonons, leading to dissipation. Both effects are described by a complex frequency-dependent mass term $M(\omega)$, derived below. This physics is present in granular films and Josephson-junction arrays as well.[3] The basic idea is to integrate out the phonons, which represent a bosonic bath to which vortices are coupled, in the spirit of Ref. 4, and thereby derive an effective action for the vortices alone.[4,5]

This paper is organized as follows: In Sec. II we derive the analog of backflow for moving vortices in superfluid films. In Sec. III we review the correspondence between superfluid dynamics and electrodynamics in two space dimensions and show how the results of Sec. II may be obtained by a Lorentz transformation of a static vortex solution. Self-interaction effects, vortex mass, and dissipation are discussed in Sec. IV. In Sec. V, we report on the results of numerical simulations of a vortex in an oscillating superflow, from which we can extract $M(\omega)$ and compare with theoretical predictions. Section VI discusses the interaction of vortices and dynamical impurities.

## II. ANALOG OF BACKFLOW FOR DYNAMICAL VORTICES

Consider a vortex moving in a (2+1)-dimensional Bose fluid. The only information we have about the vortex is that it is a point object which accrues a geometric phase of $2\pi$ in the many-body boson propagator each time it encircles a boson. We write the vortex current density as[6]

$$J^\mu=c\kappa\int d\tau\sum_l n_l \frac{dX_l^\mu}{d\tau}\,\delta^{(3)}(x-X_l(\tau)),$$

where $\tau$ parametrizes the vortex ''world lines'' $X_l^\mu(\tau)$, which are one-dimensional filaments running through (2+1)-dimensional spacetime. The many-boson Lagrangian is written

$$\mathcal{L}=\frac{1}{2}m\sum_i\left(\frac{d\mathbf{x}_i}{dt}\right)^2-\sum_{i<j}v(|\mathbf{x}_i-\mathbf{x}_j|)+\mathcal{L}_{\rm top},$$

where we assume a simple generic interacting Bose fluid (isotropic, single component). Here, $\mathcal{L}_{\rm top}$ is the topological term in the Lagrangian which counts the winding number of the vortices relative to the bosons. This is explicitly written in terms of the linking number of their trajectories,[7]

$$\mathcal{S}_{\rm top}=\int d^2x\,dt\,\mathcal{L}_{\rm top}=2\pi\hbar N_{\rm link}=\frac{1}{c}\int d^2x\,dt\,j^\mu\,\frac{\epsilon_{\mu\nu\lambda}\partial^\nu}{\partial^2}J^\lambda$$
$$\equiv\int d^2x\,dt\,j^\mu\mathcal{A}_\mu, \qquad (1)$$

where the boson mass current density is $j^\mu=(c\rho,\mathbf{j})$, and where $\partial^\nu/\partial^2$ is a formal expression for a nonlocal operator. Vortex current conservation, $\partial_\mu J^\mu=0$, allows one to construct a gauge potential $\mathcal{A}^\mu$ whose curl is $J^\mu$,[7]

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$$J_{\mu}=-c\epsilon_{\mu\nu\lambda}\partial^{\nu}\mathcal{A}^{\lambda}, \tag{2}$$

and thereby express the linking number as a local interaction between the boson current density $j^{\mu}$ and the vortex gauge potential $\mathcal{A}_{\mu}$.

The time-dependent Hamiltonian for the bosons in the presence of moving vortices is thus

$$\mathcal{H}(A^{\mu})=\mathcal{H}(0)-\int d^2x j^p_{\mu}(\mathbf{x})\mathcal{A}^{\mu}(\mathbf{x},t)+\frac{1}{2}\int d^2x\rho(\mathbf{x})\boldsymbol{\mathcal{A}}^2(\mathbf{x},t), \tag{3}$$

where $\rho$ is the boson density and $\mathbf{j}^p$ is given by

$$j^p_0(x)=c\rho(\mathbf{x})=mc\sum_i \delta(\mathbf{x}-\mathbf{x}_i),$$

$$\mathbf{j}^p(\mathbf{x})=\frac{1}{2}\sum_i [\mathbf{p}_i\delta(\mathbf{x}-\mathbf{x}_i)+\delta(\mathbf{x}-\mathbf{x}_i)\mathbf{p}_i].$$

The gauge-invariant boson current density $j_{\mu}$ is then written[8]

$$j_{\mu}=-\frac{\delta\mathcal{H}}{\delta\mathcal{A}^{\mu}}=j^p_{\mu}+\rho\mathcal{A}_{\mu}(1-\delta_{\mu 0}).$$

At this point we have reproduced the well-known analogy between a rotating superfluid and a superconductor in a magnetic field.[9] What is new here is the explicit gauge-covariant, time-*dependent* description, through Eqs. (2) and (3), of the coupling of a superfluid to vortices, which are quanta of rotation.

The linear response of the boson system is given by the Kubo formula,

$$\langle j^{(1)}_{\mu}(\mathbf{x},t)\rangle=\int d^2x'dt'K_{\mu\nu}(\mathbf{x},t;\mathbf{x}',t')\mathcal{A}^{\nu}(\mathbf{x}',t'),$$

where

$$K_{\mu\nu}(\mathbf{x},t;\mathbf{x}',t')=\frac{i}{\hbar}\langle[j^p_{\mu}(\mathbf{x},t),j^p_{\nu}(\mathbf{x}',t')]\rangle_0\Theta(t-t')-\langle\rho(\mathbf{x},t)\rangle_0\delta_{\mu\nu}(1-\delta_{\mu 0})\delta(\mathbf{x}-\mathbf{x}')\times\delta(t-t').$$

The spatial part of $K_{\mu\nu}$ may be written in terms of longitudinal and transverse components in Fourier space, viz.

$$K^{ij}(q)=-\hat{q}^i\hat{q}^jK_{\parallel}(q)-(g^{ij}+\hat{q}^i\hat{q}^j)K_{\perp}(q).$$

Gauge invariance, $K_{\mu\nu}(q)q^{\nu}=0$, may be used to relate the 00 and $0i$ components to $K_{\parallel}$:

$$K^{00}(q)=-\frac{c^2|\mathbf{q}|^2}{\omega^2}K_{\parallel}(q),\quad K^{i0}(q)=-\frac{cq^i}{\omega}K_{\parallel}(q).$$

At zero temperature, the single-mode approximation (SMA) gives for the density response[10]

$$K^{\mathrm{SMA}}_{00}(q)=\frac{mc^2\bar{\rho}}{\hbar}S(\mathbf{q})\left\{\frac{1}{\omega+c|\mathbf{q}|+i\epsilon}-\frac{1}{\omega-c|\mathbf{q}|+i\epsilon}\right\},$$

where $\bar{\rho}=\langle\rho\rangle$ is the average mass density and $S(\mathbf{q})$ is the ground-state static structure function. Note that $K^{\mathrm{SMA}}_{\perp}(q)=\bar{\rho}$, since the phonon is purely longitudinal. Recall that $\lim_{\mathbf{q}\to 0}\hbar|\mathbf{q}|/2mcS(\mathbf{q})=1$.[10]

*Static Vortex.* We choose a gauge in which $\mathcal{A}^0=0$ and $\nabla\times\boldsymbol{\mathcal{A}}=n\kappa\delta(\mathbf{x})\hat{z}$, which is satisfied by $\boldsymbol{\mathcal{A}}(\mathbf{x},t)=n\kappa\hat{z}\times\mathbf{x}/2\pi|\mathbf{x}|^2$. Now $\boldsymbol{\mathcal{A}}(\mathbf{q},\omega)=-in\kappa(\hat{z}\times\mathbf{q}/|\mathbf{q}|^2)\cdot 2\pi\delta(\omega)$ is purely transverse, so the density response vanishes and the current density response gives the usual $\langle\mathbf{j}^{(1)}(\mathbf{x})\rangle=n\kappa\bar{\rho}\hat{z}\times\mathbf{x}/2\pi|\mathbf{x}|^2$. The absence of a density variation in response to the vortex seems to contradict the result that $\delta\rho(r)/\bar{\rho}=-n^2K_S\bar{\rho}\hbar^2/2m^2|\mathbf{x}|^2$ far from a vortex of strength $n$, where $K_S$ is the adiabatic compressibility. However, the $n^2$ dependence tells us that this is a *nonlinear* response. The second-order response is formally written

$$\langle j^{(2)}_{\alpha}\rangle=\int d^2x'dt'\int d^2x''dt''R_{\alpha\beta\gamma}(x;x';x'')\mathcal{A}^{\beta}(x')\mathcal{A}^{\gamma}(x'').$$

$R_{\alpha\beta\gamma}\equiv R^{\mathrm{I}}_{\alpha\beta\gamma}+R^{\mathrm{II}}_{\alpha\beta\gamma}$ may be divided into two contributions. The first, $R^{\mathrm{I}}_{\alpha\beta\gamma}$, is the second-order nonlinear susceptibility arising from the linear $j^p_{\mu}\mathcal{A}^{\mu}$ coupling in $\mathcal{H}$. The second, $R^{\mathrm{II}}_{\alpha\beta\gamma}$, is the linear susceptibility arising from the $(1/2)\rho\boldsymbol{\mathcal{A}}^2$ term—this is given by the density-density correlation function, so

$$R^{\mathrm{II}}_{\alpha\beta\gamma}(x;x';x'')=-\frac{1}{2c^2}K_{00}(\mathbf{x},t;\mathbf{x}',t')\delta(\mathbf{x}'-\mathbf{x}'')\times\delta(t'-t'')\delta_{\beta\gamma}(1-\delta_{\beta 0}).$$

It is easy to check that the nonlinear response arising from $R^{\mathrm{II}}_{\alpha\beta\gamma}$ exactly reproduces the asymptotic density variation due to the vortex, and that $R^{\mathrm{I}}_{\alpha\beta\gamma}$ does not contribute to $\mathcal{O}(1/|\mathbf{x}|^2)$.

*Moving vortex.* Consider now a vortex moving with uniform velocity: $\mathbf{X}=\mathbf{v}t$. We choose the gauge $\mathcal{A}^0=0$ and

$$\boldsymbol{\mathcal{A}}(\mathbf{q},\omega)=-in\kappa\left[\frac{\hat{z}\times\mathbf{q}}{|\mathbf{q}|^2}+\frac{\mathbf{q}}{\omega|\mathbf{q}|^2}\hat{z}\cdot\mathbf{v}\times\mathbf{q}\right]\cdot 2\pi\delta(\omega-\mathbf{q}\cdot\mathbf{v}).$$

Note that this is no longer purely transverse, so there will be a linear response of the density to the moving vortex:

$$\langle\rho^{(1)}(\mathbf{q},\omega)\rangle=8in\pi^2\bar{\rho}S(\mathbf{q})\frac{c}{|\mathbf{q}|}\frac{\hat{z}\cdot\mathbf{v}\times\mathbf{q}}{(c\mathbf{q})^2-(\mathbf{v}\cdot\mathbf{q})^2}\delta(\omega-\mathbf{q}\cdot\mathbf{v}).$$

Further assuming $|\mathbf{v}/c|\ll 1$, we obtain, at large distances

$$\langle\rho^{(1)}(\mathbf{x},t)\rangle=\frac{n\kappa\bar{\rho}}{2\pi c^2}\frac{\hat{z}\cdot\mathbf{v}\times\mathbf{x}}{|\mathbf{x}-\mathbf{v}t|^2}, \tag{4}$$

which is identical to the result obtained by Duan.[12]

The current density in the presence of a moving vortex is similarly computed and found to be

$$\langle\mathbf{j}^{(1)}(\mathbf{x},t)\rangle=\frac{n\kappa\bar{\rho}}{2\pi}\frac{\hat{z}\times\mathbf{R}}{\mathbf{R}^2}\left\{1-\frac{\mathbf{v}^2}{2c^2}[(\hat{v}\cdot\hat{R})^2-(\hat{v}\times\hat{R})^2]\right\},$$

valid to order $\mathbf{v}^2/c^2$, with $\mathbf{R}=\mathbf{x}-\mathbf{v}t$.

### Contrast with impurity backflow

It is important to contrast this behavior with the standard picture of backflow in neutral systems. A local perturbation coupling to the superfluid as

$$\mathcal{H}'=\int d^2x\rho(\mathbf{x})U(\mathbf{x},t)$$

leads to a superfluid response[10]

$$\langle\rho^{(1)}(\mathbf{q},\omega)\rangle=-\frac{1}{c^2}K_{00}(\mathbf{q},\omega)U(\mathbf{q},\omega),$$

$$\langle\mathbf{j}^{(1)}(\mathbf{q},\omega)\rangle=-\frac{1}{c}K^{i0}(\mathbf{q},\omega)U(\mathbf{q},\omega)\hat{e}_i\,.$$

Using the SMA response functions, and assuming $U(\mathbf{x},t)=U_0\delta(\mathbf{x}-\mathbf{v}t)$, one obtains

$$\langle\rho^{(1)}(\mathbf{x},t)\rangle=-\frac{\overline{\rho}U_0}{c^2}\,\delta(\mathbf{R}),\tag{5}$$

$$\langle\mathbf{j}^{(1)}(\mathbf{x},t)\rangle=-\frac{\overline{\rho}U_0}{2\pi c^2}\frac{v}{\mathbf{R}^2}\{\hat{v}-2(\hat{v}\cdot\hat{R})\hat{R}\}.\tag{6}$$

The density response is purely local, in contrast to that of Eq. (4), and the current, which vanishes in the static case $v=0$, is dipolar and falls off as $1/R^2$. The superfluid-vortex gauge coupling leads to a much different linear response.

### III. ANALOGY TO QED$_{2+1}$

These results may be understood in terms of the well-known correspondence between (2+1)-dimensional superfluids and quantum electrodynamics,[13–17] which we now review. One starts with the standard Ginzburg-Landau Lagrangian density in the presence of an external gauge field $Z^\mu$,

$$\mathcal{L}[\Psi^*,\Psi]=\Psi^*(i\hbar\partial_t+eZ^0)\Psi-\frac{1}{2m}\left|\left(\frac{\hbar}{i}\boldsymbol{\nabla}+\frac{e}{c}\mathbf{Z}\right)\Psi\right|^2$$
$$-\lambda\left(|\Psi|^2-\frac{\overline{\rho}}{m}\right)^2.\tag{7}$$

For a superconductor, $Z^\mu$ would represent the electromagnetic gauge potential, while in our case it can be used to describe an externally imposed current. At this point, the ''charge'' $e$ and velocity $c$ are arbitrary parameters; we will take $c$ to be the speed of sound, defined below. One substitutes $\Psi\equiv\sqrt{\rho/m}e^{i\theta}e^{i\chi}$, where $\theta(\mathbf{x},t)$ is a smooth ''spin-wave'' field and $\chi(\mathbf{x},t)$ the singular vortex field, which satisfies $J^\mu(x)=(\hbar c/m)\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi$. This gives

$$\mathcal{L}'=-\frac{\hbar\rho}{m}\left(\partial_t\theta+\partial_t\chi-\frac{e}{\hbar}Z^0\right)-\frac{\hbar^2\rho}{2m^2}\left(\boldsymbol{\nabla}\theta+\boldsymbol{\nabla}\chi+\frac{e}{\hbar c}\mathbf{Z}\right)^2$$
$$-\frac{\hbar^2}{8m^2\rho}(\boldsymbol{\nabla}\rho)^2-\frac{\lambda}{m^2}(\rho-\overline{\rho})^2$$

after subtracting a time derivative term. Decoupling the $(\boldsymbol{\nabla}\theta+\boldsymbol{\nabla}\chi+e\mathbf{Z}/\hbar c)^2$ term, one arrives at

$$\mathcal{L}''=-\frac{\hbar\mathbf{Q}}{m}\cdot\left(\boldsymbol{\nabla}\theta+\boldsymbol{\nabla}\chi+\frac{e}{\hbar c}\mathbf{Z}\right)-\frac{\hbar\rho}{m}\left(\partial_t\theta+\partial_t\chi-\frac{e}{\hbar}Z^0\right)$$
$$+\frac{\mathbf{Q}^2}{2\rho}-\frac{\hbar^2}{8m^2}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho}-\frac{\lambda}{m^2}(\rho-\overline{\rho})^2.$$

Integrating over the spin-wave field $\theta(\mathbf{x},t)$ now generates the constraint $\boldsymbol{\nabla}\cdot\mathbf{Q}+\partial_t\rho=0-\mathbf{Q}$ is the mass current—which is satisfied by introducing the gauge field $A^\mu=(A^0,\mathbf{A})$, where

$$(\rho-\overline{\rho},\mathbf{Q})=-\frac{\overline{\rho}}{c}(\hat{z}\cdot\boldsymbol{\nabla}\times\mathbf{A},c\hat{z}\times\boldsymbol{\nabla}A^0+\hat{z}\times\partial_t\mathbf{A}).$$

The coupling of the vortex current to this gauge field is due to the term

$$\frac{\hbar}{m}(\mathbf{Q}\cdot\boldsymbol{\nabla}\chi+\rho\partial_t\chi)=\frac{\hbar\overline{\rho}}{m}\epsilon^{\mu\nu\lambda}(A_\mu+a_\mu)\partial_\nu\partial_\lambda\chi+\partial(\cdots)$$
$$=\frac{1}{c}\overline{\rho}J^\mu(A_\mu+a_\mu)+\partial(\cdots),$$

where the *nondynamical* gauge field $a^\mu$ generates a static magnetic field, i.e.,

$$\mathbf{e}=-\boldsymbol{\nabla}a^0-\frac{1}{c}\frac{\partial\mathbf{a}}{\partial t}=0,\quad b=\hat{z}\cdot\boldsymbol{\nabla}\times\mathbf{a}=-c,$$

which is satisfied by the gauge choice $\mathbf{a}=(1/2)c(y,-x)$, $a^0=0$, for example. The coupling between the gauge fields $A^\mu$ and $Z^\nu$ is then given by

$$\frac{e}{m}\left(\rho Z^0-\frac{1}{c}\mathbf{Q}\cdot\mathbf{Z}\right)=\frac{e\overline{\rho}}{mc}\epsilon^{\mu\nu\lambda}Z_\mu\partial_\nu(A_\lambda+a_\lambda).$$

Finally, one introduces the field strength tensor

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=\begin{pmatrix}0 & E_x & E_y\\ -E_x & 0 & -B\\ -E_y & B & 0\end{pmatrix},$$

and obtains

$$\mathcal{L}_{\rm eff}=-\frac{\overline{\rho}}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\overline{\rho}}{c}J^\mu(A_\mu+a_\mu)+\frac{e\overline{\rho}}{mc}\epsilon^{\mu\nu\lambda}Z_\mu\partial_\nu$$
$$\times(A_\lambda+a_\lambda)+\frac{\overline{\rho}}{2c}\frac{B\mathbf{E}^2}{(1-B/c)}-\frac{\overline{\rho}}{8}\frac{(\xi\boldsymbol{\nabla}B)^2}{(1-B/c)},\tag{8}$$

where $c=\sqrt{2\lambda\overline{\rho}/m^2}$ is the speed of sound and $\xi=\hbar/mc$ is the coherence length. Note $K_s^o=m^2/2\overline{\rho}^2\lambda$ is the bare compressibility.

The Lagrangian density $\mathcal{L}_{\rm eff}$ describes ''charged'' particles (vortices) moving in a background ''magnetic field'' $-c\hat{z}$, minimally coupled to a dynamical gauge field $A^\mu$ [note that this is not the gauge field defined in Eq. (2)]. That the background magnetic field is the average boson density is of course due to the fact that the vortices see the bosons as sources of geometric phase. This was recognized by Haldane and Wu,[18] who computed the Berry phase accrued by a vortex as it executes adiabatic transport in the superfluid film. If the vortex position is $\boldsymbol{\xi}$, and the adiabatic wave function is $|\Psi\rangle$, then

$$\gamma_C=i\oint_C d\boldsymbol{\xi}\cdot\langle\Psi|\boldsymbol{\nabla}_\xi|\Psi\rangle=-2\pi\frac{\overline{\rho}}{m}S_C,$$

where $S_C$ is the area enclosed by the path $C$ along which the vortex travels. Note that this immediately tells us that vortices experience a Lorentz force when moving through a superfluid. In the nonrelativistic limit, their dynamics can be described in terms of vortices being advected in each other's flow field, or as charged particles in a background magnetic field moving under the influence of each other's electric field. A vortex-antivortex pair, for example, behaves like an exciton in a magnetic field.

The linearized, long-wavelength Lagrangian density is obtained by dropping terms in $\mathcal{L}_{\rm eff}$ which are higher than second order in the field strength or which involve higher derivatives acting on the $A^\mu$ field. One is then left with (2+1)-dimensional quantum electrodynamics,

$$\mathcal{L}_{\rm QED}=-\frac{\bar{\rho}}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\bar{\rho}}{c}\left(\frac{e}{m}\epsilon^{\mu\nu\lambda}\partial_\nu Z_\lambda-J^\mu\right)(A_\mu+a_\mu),$$

in the presence of a uniform background magnetic field. Note that the gauge field $A^\mu$ couples to a sum of the (quantized) vortex current density $J^\mu$ and the (not quantized) externally imposed ''current'' $(e/m)\epsilon^{\mu\nu\lambda}\partial_\mu Z_\lambda$, which could represent a global rotation of the system.[9] When no vortices are present ($J^\mu=0$), one can integrate out the gauge field $A^\mu$ to obtain

$$S=\frac{\bar{\rho}}{2}\left(\frac{e}{mc}\right)^2\int\frac{d^2k\,d\omega}{(2\pi)^3}\left[Z_\mu(-k)\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)Z_\nu(k)+\frac{e\bar{\rho}}{m}Z^0(k)(2\pi)^3\delta^{(3)}(k)\right],$$

where $k^2=k^\mu k_\mu=c^{-2}\omega^2-\mathbf{k}^2$, and setting $e/mc\equiv 1$ one can read off the response tensor

$$K^{\mu\nu}(k)=\bar{\rho}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right).$$

When $Z^\mu=0$, one has a theory of vortices minimally coupled to the gauge field $A^\mu$, and the action extremizing equations for the fields are Maxwell's equations: $\partial_\mu F^{\mu\nu}=J^\nu/c$, or

$$\nabla\cdot\mathbf{E}=\frac{1}{c}J^0,$$

$$\hat{z}\cdot\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial B}{\partial t},$$

$$\nabla B\times\hat{z}=\frac{1}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}.$$

(Note $\nabla\cdot\mathbf{B}=\partial B/\partial z=0$, trivially.) For the vortices, one has the Lorentz force law,

$$\frac{d\mathbf{X}_l}{dX_l^0}=\frac{\hat{z}\times\mathbf{E}(\mathbf{X}_l)}{c-B(\mathbf{X}_l)},\tag{9}$$

which says that the vortices move perpendicular to the local electric field, with a magnetic-field strength of $c-B$ which is the sum of a uniform background contribution (the average boson density) and a dynamical contribution (due to fluctuations in the boson density).

### Lorentz transformations

To investigate the effects of moving vortices, it is useful to appeal to the Lorentz invariance of $\mathcal{L}_{\rm QED}$ and transform static solutions.[19] Recall that in (2+1) dimensions, the Lorentz group has three generators, corresponding to two boosts and one rotation. The general boost transformation is written

$$L^\mu_{\ \nu}=(gL)_{\mu\nu}=\begin{pmatrix}\gamma & \gamma\beta_x & \gamma\beta_y\\ \gamma\beta_x & \frac{\gamma-1}{\beta^2}\beta_x^2+1 & \frac{\gamma-1}{\beta^2}\beta_x\beta_y\\ \gamma\beta_y & \frac{\gamma-1}{\beta^2}\beta_x\beta_y & \frac{\gamma-1}{\beta^2}\beta_y^2+1\end{pmatrix}\tag{10}$$

with $\boldsymbol{\beta}=\mathbf{v}/c$ and $\gamma=1/\sqrt{1-\beta^2}$. Applying the Lorentz transformation $z'^\mu=L^\mu_{\ \nu}z^\nu$ to the coordinates and field strength tensor gives the familiar results

$$x'^0=\gamma x^0+\gamma\boldsymbol{\beta}\cdot\mathbf{x}$$

$$\mathbf{x}'=\gamma x^0\boldsymbol{\beta}+\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}+\mathbf{x}$$

and

$$\mathbf{E}'=\gamma\mathbf{E}-\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{E})\boldsymbol{\beta}+\gamma B\hat{z}\times\boldsymbol{\beta}$$

$$B'=\gamma\hat{z}\cdot\boldsymbol{\beta}\times\mathbf{E}+\gamma B.$$

We may now transform solutions

$$\{x^\mu,J^\mu,F^{\mu\nu}\}\to\{x'^\mu,J'^\mu,F'^{\mu\nu}\}.$$

A static charge 1 vortex generates an electric field $\mathbf{E}=\kappa\mathbf{x}/2\pi|\mathbf{x}|^2$ and a magnetic field $B=0$. Upon applying the boost of Eq. (10), we obtain (dropping primes),

$$\mathbf{E}=\frac{\gamma\kappa}{2\pi}\frac{x_\perp\hat{z}\times\hat{\beta}+(x_\parallel-\beta x^0)\hat{\beta}}{x_\perp^2+\gamma^2(x_\parallel-\beta x^0)^2},$$

$$B=\frac{\gamma\kappa}{2\pi}\frac{\beta x_\perp}{x_\perp^2+\gamma^2(x_\parallel-\beta x^0)^2},\tag{11}$$

where we have written $\mathbf{x}=x_\parallel\hat{\beta}+x_\perp\hat{z}\times\hat{\beta}$.

Now the rules for translating from $\mathbf{E}$ and $B$ to $\mathbf{j}$ and $\rho$ are as follows:

$$\mathbf{j}=\bar{\rho}\hat{z}\times\mathbf{E},\quad \rho=\bar{\rho}(1-B/c).$$

We now see that the linear-response formulas [Eq. (4) and accompanying discussion] exactly reproduce these results to lowest order in $\beta$. The vortex velocity is the the ratio $\mathbf{j}/\rho$, which is the content of the Lorentz force law, Eq. (9).

### A tale of two vortices

Consider now an elementary vortex-antivortex pair separated by a distance $a$. We choose $\mathbf{X}_\pm(t)=vt\hat{e}_1\pm(1/2)a\hat{e}_2$. Computing the electric and magnetic fields at one of the singularities due to the presence of the other is easily accomplished with the Lorentz transformation. One obtains

$$\mathbf{E}(\mathbf{X}_{\pm}) = -\frac{\gamma\kappa}{2\pi a}\,\hat{e}_2, \quad B(\mathbf{X}_{\pm}) = -\frac{\beta\gamma\kappa}{2\pi a}.$$

But if in the moving frame the vortices are stationary, we must have that $c\boldsymbol{\beta}=d\mathbf{X}_{\pm}/dt$, which leads to the result

$$\beta(a) = \left(\frac{\xi^2}{a^2+\xi^2}\right)^{1/2}.$$

Thus, at large separations the pair's velocity is $c\xi/a$, but at smaller separations the velocity asymptotically approaches the sound speed $c$. We stress that this is true for the model defined by $\mathcal{L}_{\rm QED}$, where the vortices have no core. The naive expression $v(a)=c\xi/a$ begins to break down at distances on the order of $\xi$, where a proper accounting of the terms neglected in the QED action must be taken in order to reproduce the correct core structure.

### Superflow and the magnus force

In the above examples, we derived results for a moving vortex in a stationary superfluid. In this section, we make a Galilean transformation (the original theory is Galilean invariant) in order to discuss what happens to vortices in the presence of a background superflow. Starting with the Galilean-transformed Lagrangian density

$$\mathcal{L} = i\hbar\Psi^*\partial_t\Psi + i\hbar\mathbf{v}\cdot\Psi^*\boldsymbol{\nabla}\Psi - \frac{\hbar^2}{2m}\,|\boldsymbol{\nabla}\Psi|^2 - \lambda\left(\left|\Psi\right|^2 - \frac{\bar{\rho}}{m}\right)^2,$$

and proceeding as before, one derives the effective Lagrangian density

$$\mathcal{L}_{\rm eff} = \frac{\bar{\rho}}{2}\left[\frac{(\mathbf{E}-B\hat{z}\times\boldsymbol{\beta})^2}{1-B/c} - B^2\right] - \frac{\bar{\rho}}{8}\,\frac{(\xi\boldsymbol{\nabla}B)^2}{1-B/c} - \frac{1}{c}\,\bar{\rho}J^{\mu}(A_{\mu}+a_{\mu}),$$

where $a^{\mu}$ now generates a static electric field as well as a static magnetic field:

$$\mathbf{e} = -c\hat{z}\times\boldsymbol{\beta}, \quad b = -c.$$

The Lorentz force due to $\mathbf{e}$ is the Magnus force. The vortex equation of motion is

$$\frac{d\mathbf{X}_l}{dX_l^0} = \boldsymbol{\beta} + \frac{\hat{z}\times\mathbf{E}(\mathbf{X}_l) - \boldsymbol{\beta}B(\mathbf{X}_l)}{c-B(\mathbf{X}_l)}.$$

When $\mathbf{E}(\mathbf{X}_l)+\mathbf{e}=0$, the forces on vortex $l$ cancel, and it is stationary.

## IV. SELF-INTERACTION, INERTIAL MASS, AND DISSIPATION

The analogy to electrodynamics suggests that there should be an electrodynamic contribution to the mass and retardation effects, as there are in (3+1)-dimensional classical electrodynamics.[20] In the superfluid, this is due to the phonon cloud carried by the vortex—a polaronic effect. However, as we have seen, the coupling of vorticity to superfluid density and current fluctuations is a gauge coupling which is rather different from the local density coupling used in conventional polaron theories. Still, this coupling is of the general form considered in Ref. 4, i.e., an external coordinate (the vortex position) coupled to a bath of oscillators (the phonons).

We wish to integrate out the dynamical field $A^{\mu}$ corresponding to the phonon degrees of freedom and obtain an effective action for the vortices. Working in Lorentz gauge ($\partial_{\mu}A^{\mu}=0$), we integrate out the Gaussian field $A^{\mu}$ in $\mathcal{L}_{\rm QED}$ by solving the equations of motion, yielding

$$A^{\mu} = \frac{1}{c}\,\Box^{-1}J^{\mu}$$

and an effective action for the vortices of

$$\mathcal{S}_{\rm eff} = -\frac{\bar{\rho}}{c^3}\int d^3x\int d^3x'\,J^{\mu}(x)\Box^{-1}(x,x')J_{\mu}(x') - \frac{\bar{\rho}}{c^2}\int d^3x J^{\mu}(x)a_{\mu}(x).$$

The inverse D'Alambertian has the form

$$\Box^{-1}(x,x') = \frac{1}{2\pi}\,\frac{\Theta(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)}{\sqrt{(x^0-x'^0)^2-|\mathbf{x}-\mathbf{x}'|^2}}$$

in (2+1) dimensions and

$$\Box^{-1}(x,x') = \frac{\delta(x^0-x'^0-|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

in (3+1) dimensions. Thus, in contrast to the case of three spatial dimensions, where $\Box^{-1}$ vanishes unless $x-x'$ is lightlike, in our (2+1)-dimensional case $\Box^{-1}$ is nonzero everywhere inside the light cone.[21] The finite sound speed $c$ leads to retardation effects. One might naïvely think that this would lead to the collapse of the vortex-antivortex pair, since the vortex should ''see'' the antivortex at earlier times and *vice versa*. However, although the *potentials* are retarded, the *fields* of a uniformly moving charge point to the *instantaneous* position of the charge (as we have derived above), and so for the special case of a uniformly moving vortex-antivortex pair, there is no apparent time delay.[22]

The self-interaction part of $\mathcal{S}_{\rm eff}$ for a vortex of strength $n$ is

$$\mathcal{S}_{\rm self} = -\frac{n^2\kappa^2\bar{\rho}}{4\pi c}\int_{-\infty}^{\infty}du\int_0^{\infty}d\sigma[1-\boldsymbol{\beta}(u)\cdot\boldsymbol{\beta}(u+\sigma)] \times\frac{\Theta(\sigma-|\mathbf{X}(u+\sigma)-\mathbf{X}(u)|)}{\sqrt{\sigma^2-|\mathbf{X}(u+\sigma)-\mathbf{X}(u)|^2}}, \tag{12}$$

where we have taken $x^0(u)=u$ as a parametrization of the vortex world line, and $\boldsymbol{\beta}(u)=d\mathbf{X}(u)/du$. If the integrand in $S_{\rm self}$ were well-behaved and allowed an expansion of the form

$$\mathcal{S}_{\rm self} = \int_{-\infty}^{\infty}dt\left\{-m_0c^2 + \frac{1}{2}\,m_1\mathbf{v}^2(t) + \cdots\right\},$$

then we would associate a rest mass with $m_0$ and an inertial mass with $m_1$. The remaining terms, involving higher derivatives and powers of the velocity, would be negligible in the Newtonian limit. However, the integral over $\sigma$ in Eq. (12) diverges logarithmically, both for large and small $\sigma$. The small $\sigma$ divergence is remedied by a proper treatment of the core structure, which lies beyond the QED approximation. The large $\sigma$ divergence, on the other hand, is real. In the case of the parameter $m_0$, this is to be expected, since we know the energy of a static vortex diverges logarithmically with the size of the system, owing to the slow falloff of the current density $|\mathbf{j}|\propto 1/r$. In the electrodynamic language, the energy density, $\mathcal{E}=(1/2)\overline{\rho}(\mathbf{E}^2+B^2)$, dies off as $1/r^2$ in the vicinity of a static vortex, yielding a logarithmic divergence when integrated over the system. In (3+1)-dimensional classical electrodynamics, by contrast, the energy density dies off as $1/r^4$, and there is no infrared divergence. Now we ask whether $m_1$ is finite. The answer again is no. Since $\mathcal{S}_{\rm self}$ is a Lorentz scalar, for constant $\boldsymbol{\beta}$ one has

$$\mathcal{S}_{\rm self}(\boldsymbol{\beta})=-\sqrt{1-\beta^2}\int_{-\infty}^{\infty}dt\ m_0c^2,$$

which says that $m_1=m_0$. As recently emphasized by Duan,[11,12] this may be understood in terms of the density variation $\langle\rho^{(1)}\rangle\propto\hat{z}\cdot\mathbf{v}\times\mathbf{x}/|\mathbf{x}-\mathbf{v}t|^2$, which produces a logarithmically infinite energy shift

$$\Delta E=\frac{1}{2K_S\overline{\rho}^2}\int d^2x[\delta\rho(\mathbf{x})]^2.$$

Similarly, the total momentum $\mathbf{P}$ of the moving vortex,

$$\mathbf{P}^{(1)}(t)=m\int d^2x\ \frac{\mathbf{j}(\mathbf{x},t)}{\rho(\mathbf{x},t)},$$

diverges logarithmically.[12]

### Frequency-dependent inertial mass

In this section we compute the low-frequency inertial mass of a single vortex and find that it is frequency dependent and logarithmically divergent as $\omega\to 0$. We start by expanding the self-interaction contribution $\mathcal{S}_{\rm self}$ for a single vortex:[23]

$$\begin{aligned}\mathcal{S}_{\rm self}=&-\frac{n^2\kappa^2\overline{\rho}}{4\pi c}\int_{-\infty}^{\infty}du\int_0^{\infty}d\sigma[1-\boldsymbol{\beta}(u)\cdot\boldsymbol{\beta}(u+\sigma)]\\&\times\left[\frac{1}{\sigma}+\frac{|\mathbf{X}(u+\sigma)-\mathbf{X}(u)|^2}{2\sigma^3}+\cdots\right]\\&=\mathcal{S}_{\rm static}+\frac{1}{2}n^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}M'(\omega)\omega^2|\mathbf{X}(\omega)|^2+\cdots,\end{aligned}$$

where $\mathcal{S}_{\rm static}$ is action for a static vortex of strength $n$. The quantity $M'(\omega)$ is found to be

$$\begin{aligned}M'(\omega)&=2\mu\int_{\delta}^{\infty}\frac{ds}{s}\left[\cos\omega s-\frac{1-\cos\omega s}{\omega^2s^2}\right]\\&=-\mu\left\{\mathrm{ci}(|\omega|\delta)+\frac{1-\cos\omega\delta}{\omega^2\delta^2}+\frac{\sin\omega\delta}{\omega\delta}\right\},\end{aligned}\tag{13}$$

where $\mu\equiv\pi\xi^2\overline{\rho}=\pi\hbar^2/2\lambda$ is the ''core mass'' of the vortex,[12] and $\mathrm{ci}(z)$ is the cosine integral.[24] We have introduced an ultraviolet temporal cutoff $\delta\approx\xi/c$ to regularize the $s$ integrals. This crudely accounts for the core structure of the vortex which lies beyond the approximation afforded by $\mathcal{L}_{\rm QED}$. The important point is that the *infrared* divergence of $(\mathcal{S}-\mathcal{S}_{\rm static})$ is suppressed by the finite frequency $\omega$, leading to a low-frequency mass $n^2M'(\omega)$ which diverges logarithmically as $\omega\to 0$:

$$M'(\omega)=\mu\{-\ln(|\omega|\delta)-(C+\tfrac{3}{2})+\tfrac{11}{24}\omega^2\delta^2+\cdots\},$$

where $C=0.577215...$ is Euler's constant.

The effective Lagrangian $\mathcal{L}_{\rm eff}$ does contain a term $-(1/8)\overline{\rho}(\xi\nabla B)^2$ which is dropped in the long-wavelength effective theory but is still quadratic in the field strengths. (It also breaks Lorentz invariance.) Retaining this term, there are no ultraviolet divergences, and the mass is

$$\begin{aligned}M(\omega)&=\frac{\mu}{2\sqrt{\Delta(\omega)}}\ln\left(\frac{\sqrt{\Delta(\omega)}+1}{\sqrt{\Delta(\omega)}-1}\right)+\frac{i\pi\mu\ \mathrm{sgn}\omega}{2\sqrt{\Delta(\omega)}}\\&\equiv M'(\omega)+iM''(\omega),\end{aligned}\tag{14}$$

$$\Delta(\omega)\equiv 1+\frac{\omega^2\xi^2}{c^2},\tag{15}$$

where we have now included the imaginary part $M''(\omega)$. The logarithmic divergence at small $\omega$ is still present, but in the large-$\omega$ limit we find that $M'(\omega)$ vanishes as $\omega^{-2}$ and $M''(\omega)$ as $\omega^{-1}$. The Fourier transform of $M(\omega)$ is then causal:

$$\begin{aligned}M(t)&=\frac{\pi\mu c}{2\xi}[I_0(ct/\xi)-L_0(ct/\xi)]\Theta(t)\\&=\frac{\pi\mu c}{2\xi}\left[1-\frac{2}{\pi}(ct/\xi)+\frac{1}{4}(ct/\xi)^2+\cdots\right]\quad(t\to 0)\\&=\frac{\mu}{t}[1-(\xi/ct)^2+\cdots]\quad(t\to\infty),\end{aligned}$$

where $I_0(z)$ and $L_0(z)$ are modified Bessel and Struve functions, respectively.[24] The logarithmic frequency dependence has previously been obtained by Eckern and Schmid,[3] who investigated vortices in granular films, and by Stamp, Chudnovsky, and Barbara[25] in the context of magnetic domain walls.

When several vortices are present, the effective vortex action for low frequencies becomes (see the Appendix)

$$\begin{aligned}\mathcal{S}_{\rm eff}=&\frac{1}{2}\int\frac{d\omega}{2\pi}M(\omega)\omega^2\left|\sum_i n_i\mathbf{X}_i(\omega)\right|^2+\cdots\\&+\frac{\overline{\rho}\kappa^2}{4\pi}\int dt{\sum_{i\neq j}}'n_in_j\ln|\mathbf{X}_i(t)-\mathbf{X}_j(t)|\\&-\kappa\overline{\rho}\int dt\sum_i n_iX_i(t)\dot{Y}_i(t),\end{aligned}\tag{16}$$

where the first term arises from an expansion of $J^{\mu}\square^{-1}J_{\mu}$ in terms of the vortex coordinates themselves,[26] and where the prime on the sum is a zero total vorticity restriction: $\Sigma_in_i=0$. Notice that the first (''kinetic'') term, discussed in the Ap-

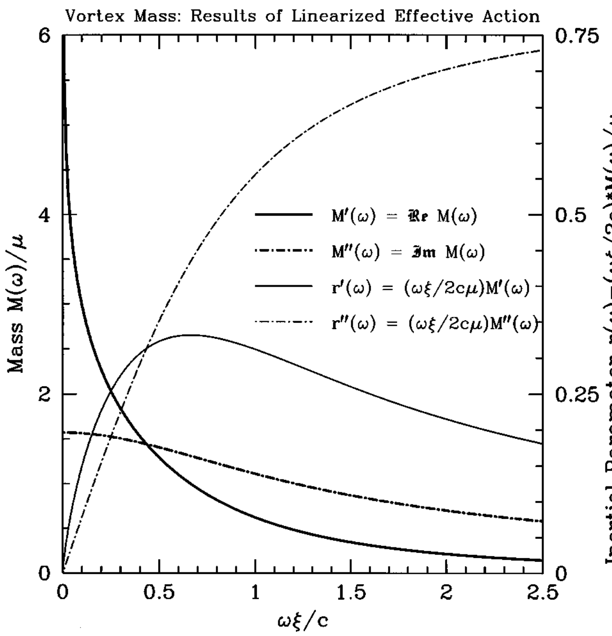


FIG. 1. Complex frequency-dependent vortex mass $M(\omega)$ from Eq. (14). Also shown is the complex dimensionless inertial parameter $r(\omega)$.

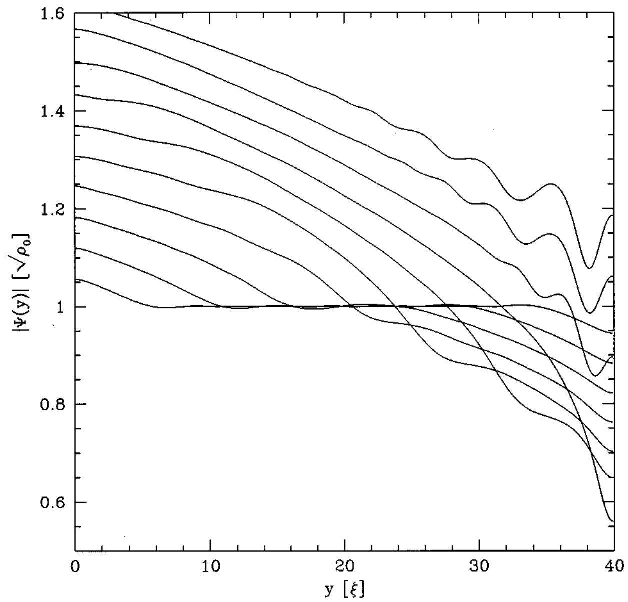


FIG. 2. Amplitude of $\phi(\mathbf{x},t)$ for a uniformly accelerated condensate, plotted as a function of length along the direction of flow, at ten equally spaced time intervals separated by $\Delta t = 5\xi/c$.

pendix, involves only the total dipole moment operator $\mathbf{D}(t) = \Sigma_i n_i \mathbf{X}_i(t)$; this fact is intimately connected with Galilean invariance and the stability of superflow in the absence of disorder. Consider, for example, an elementary $(n=\pm 1)$ vortex-antivortex pair. Let $\mathbf{X} \equiv (1/2)(\mathbf{X}_+ + \mathbf{X}_-)$ be the "center of mass" (CM) coordinate and $\mathbf{x} \equiv (\mathbf{X}_+ - \mathbf{X}_{-1})$ be the dipole moment. The CM coordinate appears only in the Berry phase term of the Lagrangian, which is

$$L_B = -\kappa\overline{\rho}(X_+\dot{Y}_+ - X_-\dot{Y}_-) = -\kappa\overline{\rho}(X\dot{y} - Y\dot{x})$$

up to a total time derivative. Thus, a path integral over the CM coordinates generates a $\delta$ function at each time step, enforcing $d\mathbf{x}(t)/dt = 0$ always.

### Relative importance of inertial terms

To investigate the importance of inertial terms relative to those arising from the Lorentz force, we consider the response of an isolated $n=1$ vortex to a time-dependent field $\mathbf{e}(t)$, which might represent a sudden switching on of a superflow which will accelerate the vortex[3] or an oscillating superflow. We find that the velocity $\mathbf{V}(\omega) = -i\omega\mathbf{X}(\omega)$ satisfies

$$\mathbf{V}(\omega) = \left[\frac{1}{1-r^2(\omega)}\right]\hat{z}\times\mathbf{e}(\omega) + \left[\frac{ir(\omega)}{1-r^2(\omega)}\right]\mathbf{e}(\omega), \quad (17)$$

where the dimensionless function

$$r(\omega) \equiv \frac{\omega M(\omega)}{\kappa\overline{\rho}} = \frac{\omega\xi}{2c}\,\frac{M(\omega)}{\mu},$$

shown in Fig. 1, describes the inertial and frictional aspects of the vortex's motion.

We now see that inertial effects will be relatively unimportant at frequencies where the "inertial parameter" $r(\omega)$ is small. Note that $r'(\omega)$ vanishes both for very low and very high frequencies, as shown in Fig. 1. Taking $c \sim 100$ m/s and $\xi \sim 5$ Å, one obtains a characteristic frequency $\omega_0 \equiv c/\xi \approx 10^{11}$ Hz. At low frequencies, both real and imaginary components of $r(\omega)$ are small, and inertial effects are relatively unimportant. From Eq. (17), we find that an elementary vortex in an oscillating superflow will move at a Hall angle $\theta_H(\omega) = \tan^{-1}|r(\omega)|$ relative to the $\hat{z}\times\mathbf{e}$ direction. The power dissipation per unit frequency is given by

$$P(\omega) = \kappa\overline{\rho}\,\mathrm{Re}[\mathbf{V}(\omega)\cdot\mathbf{e}^*(\omega)] \underset{\omega\to 0}{=} \omega M''(\omega)|\mathbf{e}(\omega)|^2, \quad (18)$$

which means $P(\omega) = (\pi/2)\mu v_s^2|\omega|$ for a superfluid velocity oscillation of amplitude $v_s$.

## V. NUMERICAL SIMULATION

Since the predictions of the linearized theory are essentially classical, we should expect to see the aforementioned effects of phonon radiation by solving the non-linear Schrödinger equation (NLSE),

$$i\dot{\psi} = -\frac{1}{2}\nabla^2\psi + (|\psi|^2 - 1)\psi,$$

where we now measure all distances in units of $\xi$ and times in units of $\xi/c$, and $\psi$ itself in units of $\sqrt{\rho/m}$. The NLSE was numerically integrated on a two-dimensional grid using an operator splitting method.[27] To impose the background oscillating superflow the condensate was defined as

$$\psi(\mathbf{x},t) = e^{-i\mathbf{v}_s(t)\cdot\mathbf{x}}\varphi(\mathbf{x},t),$$

where $\mathbf{v}_s(t)$ is the chosen time-dependent superflow (in units of $c$) uniformly defined over the whole region, and $\varphi(\mathbf{x},t)$ is, initially, a static vortex solution of the NLSE. With the initial state representing a condensate with a vortex *plus* a super-

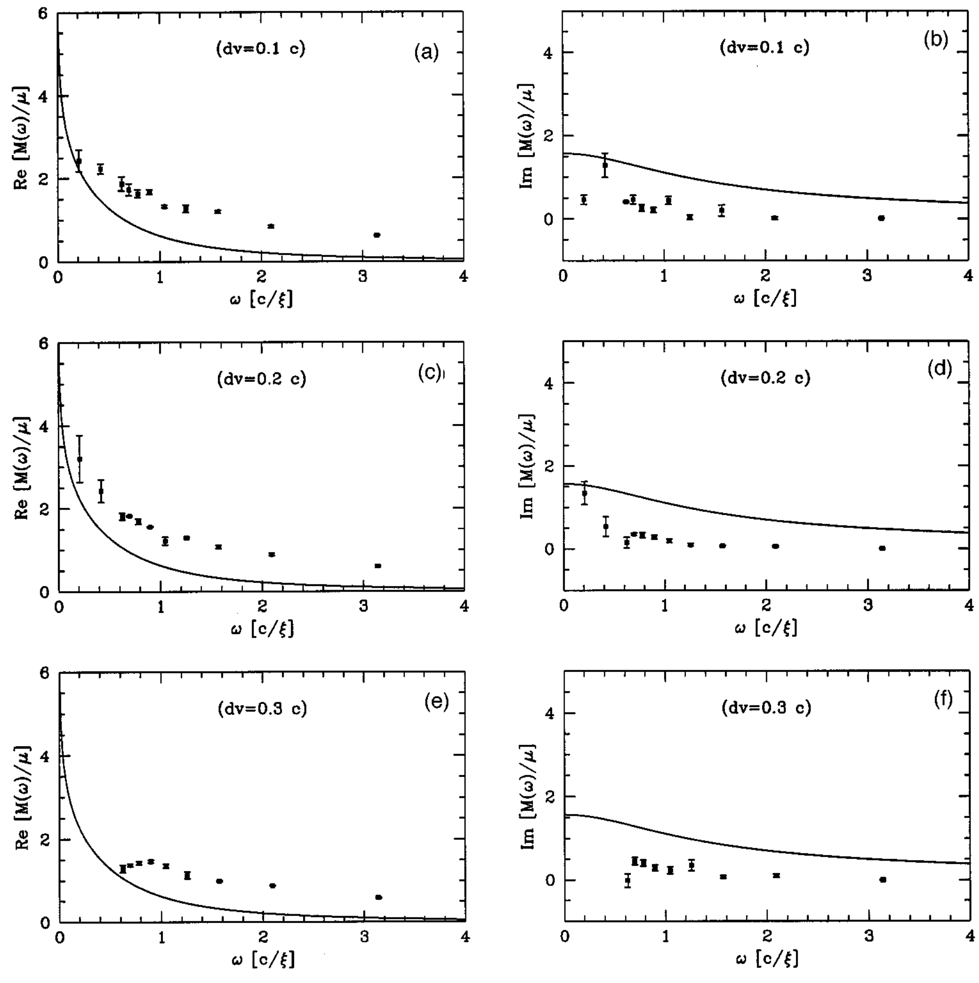


FIG. 3. Complex mass $M(\omega)$ inferred from Eq. (20). Solid curves are $M'(\omega)$ and $M''(\omega)$ from the linearized electrodynamic theory. (a) $M'(\omega)$ for driving amplitude $0.1c$. (b) $M''(\omega)$ for driving amplitude $0.1c$. (c) $M'(\omega)$ for driving amplitude $0.2c$. (d) $M''(\omega)$ for driving amplitude $0.2c$. (e) $M'(\omega)$ for driving amplitude $0.3c$. (f) $M''(\omega)$ for driving amplitude $0.3c$.

flow given by $\mathbf{v}_s(t=0)$, $\psi(\mathbf{x},t)$ was evolved according to the NLSE. The resulting equation for $\varphi(\mathbf{x},t)$ is

$$i\dot{\varphi}=(\dot{\mathbf{v}}_s\cdot\mathbf{x})\varphi-\tfrac{1}{2}(\boldsymbol{\nabla}+i\mathbf{v}_s)^2\varphi+(|\varphi|^2-1)\varphi. \tag{19}$$

When $\mathbf{v}_s$ is constant in time, Galilean invariance means that a solution to Eq. (19) is given by

$$\varphi(\mathbf{x},t)=\exp(-i\mathbf{v}_s^2t/2)f(\mathbf{x}-\mathbf{v}_st)$$

$$0=-\tfrac{1}{2}\boldsymbol{\nabla}^2f+(|f|^2-1)f.$$

In particular, if initially $f(\mathbf{x})$ is a static vortex solution of the NLSE, the time evolution of $\psi(\mathbf{x},t)$ represents a vortex being rigidly translated with velocity $\mathbf{v}_s$. The deviations from this ''massless'' behavior will become evident as $\mathbf{v}_s$ becomes time dependent.

In the method used the right-hand side of (19) is split into two parts which are successively integrated, making the algorithm first-order accurate in time:[27]

$$i\dot{\varphi}=-\tfrac{1}{2}\boldsymbol{\nabla}^2\varphi-i\mathbf{v}_s\cdot\boldsymbol{\nabla}_\varphi \quad (\text{1st step}),$$

$$i\dot{\varphi}=\dot{\mathbf{v}}_s\cdot\mathbf{x}\varphi+\tfrac{1}{2}v_s^2\varphi+(|\varphi|^2-1)\varphi \quad (\text{2nd step}).$$

The first step was integrated using the Crank-Nicholson method,[27] which is unconditionally stable and second-order accurate in time; the second step was integrated exactly. The time step was 0.01 $\xi/c$ and the time-dependent velocity was along the longest dimension of the grid, a channel whose spatial dimensions were $256\times400$ points, with a spacing equal to 0.1 or 0.05 $\xi$. Along the edges we adopted von Neumann boundary conditions for $\varphi(\mathbf{x},t)$, which means that the superfluid velocity computed from $\psi(\mathbf{x},t)$ was $\mathbf{v}_s(t)$ at the beginning and at the end of the channel (this also implies that the initial circular vortex flow field had to be slightly distorted).

There are other ways of imposing the superflow, for instance using the boundary condition $\hat{n}\cdot\boldsymbol{\nabla}\psi=i\mathbf{v}_s\psi$ for the full condensate. The influence of the edges however takes some time to reach the center where the vortex is located, which can be specially inconvenient if one wants $\mathbf{v}_s$ to vary rapidly with time, another problem is that the velocity field one obtains is not spacially uniform and does not lend itself so easely to a comparison with the electrodynamical theory.

Figure 2 shows the effect of a constant acceleration on an initially uniform condensate. Here one sees a wake in the superfluid density propagating at the speed of sound from the beginning of the channel towards the end, and a counterwake propagating in the opposite direction. This consequence of the imposed accelerated flow takes a finite amount of time to reach the channel center, where the vortex was placed in the subsequent simulations, and is responsible for an observed delay in the vortex response. This effect is related to the

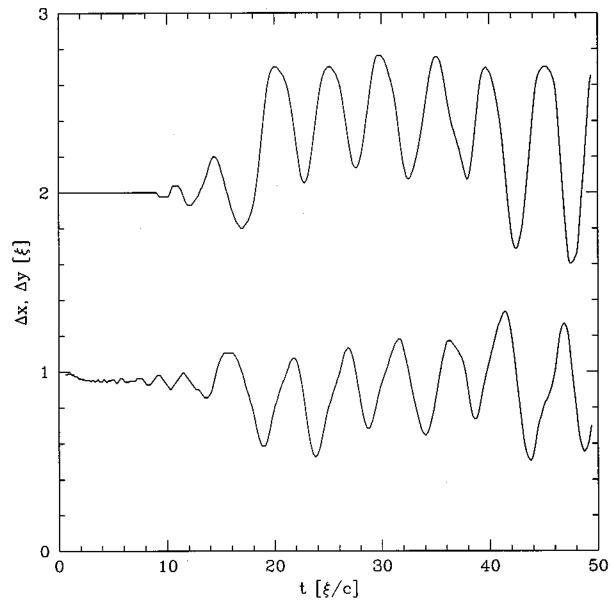


FIG. 4. Typical vortex trajectory $\mathbf{X}(t)$ with harmonic forcing of period $T=5\xi/c$.

finite compressibility of the superfluid and does no harm to the observation of vortex oscillations, it is in fact what causes them.

A more dangerous effect is the reflection of the wake off the end of the channel. To avoid it, one has to restrict the observation time to about 40 $\xi/c$, for a channel length of 40 $\xi$. This reflection probably explains problems related to vortex shedding that were observed in some cases at the end of the observation period.

### Results of the simulations

The equation of motion one gets for the vortex position in the presence of a time-dependent background superflow is $(n=\pm 1)$

$$-i\omega M(\omega)\mathbf{V}(\omega)=\pm\kappa\overline{\rho}\hat{z}\times[\mathbf{v}_s(\omega)-\mathbf{V}(\omega)].$$

Thus, in the absence of the inertial term, the vortex drifts with the superflow. $M(\omega)$ is the Fourier transform of the causal kernel, as above. These dynamics imply that for a monochromatic flow one should get a response only at the driving frequency or at the resonance where $\omega^2 M^2(\omega)=\kappa^2\overline{\rho}^2$, which for a frequency-independent mass corresponds to cyclotron oscillations.

We considered different forms of time-dependent flow and compared the observed trajectories with the equations of motion above. To simulate an oscillating flow we took $v_s(t)$ to be

$$v_s(t)=v_s^o\,\sin(2\pi t/T),$$

where $v_s^o$ ranged from 0.1 to $0.3c$ and $T$ ranged from 2 to 30 $\xi/c$. The trajectories we have obtained display a nearly periodic structure with a characteristic frequency equal to the driving frequency. We obtained the frequency dependence of $M(\omega)$, as implied by the model's equation of motion, by taking the Fourier transform of the trajectory and reading its amplitude at the driving frequency according to

$$M(\omega)=\mp i\kappa\overline{\rho}\,\frac{X(\omega)}{\omega Y(\omega)}. \tag{20}$$

The results are shown in Fig. 3 together with the functional form obtained from the linearized effective QED theory.

Since we did not get a perfect steady-state response, as can be seen in a typical trajectory as in Fig. 4, it was not very clear which region of the data array to use in taking the Fourier transform. We chose to ignore an initial structureless region, corresponding to the delay mentioned above, and to take several Fourier transforms using time intervals equal to an integer number of periods, all starting at the same point. Hence, with the same trajectory, we obtained several values of $M(\omega)$, which were averaged. The error bars in the figures correspond to this averaging process.

We obtained $M(\omega)$ for three amplitudes of the oscillatory velocity field, namely 0.3, 0.2, and 0.1 $c$. The last two cases gave very similar values for the mass, except at lower frequencies, where the vortex response was further from a steady state, and we could not observe several periods of oscillations. Nonetheless, the expected qualitative behavior was observed.

The higher amplitude case, $v_s^o=0.3c$, posed more difficulties at low frequencies. Up to the lowest frequency we were able to get, one sees a qualitatively different behavior, which may be due to the onset of nonlinearities not visible in the other cases.

Note that in the QED theory the real and imaginary parts of $M(\omega)$ obey a Kramers-Kronig relation. We could not check if such a relation existed in the measured $M(\omega)$ because the points obtained were too scattered to allow for a reliable fit.

We also tried a pulse form for $v_s(t)$,

$$v_s(t)=-\sqrt{2e}\,v_{\max}\left(\frac{t}{T}\right)e^{-t^2/T^2}. \tag{21}$$

This flow would produce a Gaussian displacement along the channel for a ''massless'' vortex. What is observed in Fig. 5 is a delayed main peak in the parallel direction with an accompanying structure in the perpendicular direction. The real part $M'(\omega)$ is displayed in Fig. 6. Since the channel is finite in both width and length, the zero-frequency limit of $M(\omega)$ should cross over to a finite value given by the system size. In our system, with length 40 $\xi$, values of $M'(\omega)$ greater than $\ln 40\approx 3.7$ are difficult to interpret.

## VI. IMPURITIES AND VORTICES

We consider as a simple model of an impurity a point object which couples linearly to the boson density, $\rho=\overline{\rho}(1-B/c)$. The Lagrangian describing the impurity is taken to be

$$L_{\text{imp}}=\sum_a\ \frac{1}{2}\,m_a\dot{\mathbf{R}}_a^2+\frac{\overline{\rho}}{mc}\int d^2x\sum_a\ U_a(|\mathbf{x}-\mathbf{R}_a|)B(\mathbf{x}),$$

where $\mathbf{R}_a(t)$ is the position of the $a^{\text{th}}$ impurity. Upon integrating out the gauge field $A^\mu(\mathbf{x},t)$, we obtain the effective action

$$\mathcal{S}_{\rm eff}[\{\mathbf{X}_i(t)\},\{\mathbf{R}_a(t')\}]=\int dt\left\{-\frac{1}{2}\kappa\overline{\rho}\sum_i n_i\epsilon^{\alpha\beta}X_i^\alpha\dot{X}_i^\beta+\sum_a\frac{1}{2}m_a\dot{\mathbf{R}}_a^2\right\}$$
$$+\frac{\overline{\rho}\kappa^2}{4\pi}\int dt\sum_{i\neq j}{}' n_in_j\ln|\mathbf{X}_i(t)-\mathbf{X}_j(t)|-\int dt\int dt'\int\frac{d^2k}{(2\pi)^2}\int\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{\omega^2-\omega^2(k)}$$
$$\times\left\{\frac{1}{2}\overline{\rho}\kappa^2\sum_{i,j}n_in_je^{-i\mathbf{k}\cdot[\mathbf{X}_i(t)-\mathbf{X}_j(t')]}\hat{z}\times\hat{k}\cdot\dot{\mathbf{X}}_i(t)\hat{z}\times\hat{k}\cdot\dot{\mathbf{X}}_j(t')\right.$$
$$-\frac{i\overline{\rho}\kappa}{m}\sum_{i,a}n_i|\mathbf{k}|U_a(k)e^{-i\mathbf{k}\cdot[\mathbf{X}_i(t)-\mathbf{R}_a(t')]}\hat{z}\times\hat{k}\cdot\dot{\mathbf{X}}_i(t)$$
$$\left.+\frac{\overline{\rho}}{2m^2}\sum_{a\neq b}\mathbf{k}^2U_a(k)U_b(-k)e^{-i\mathbf{k}\cdot[\mathbf{R}_a(t)-\mathbf{R}_b(t')]}\right\}, \tag{22}$$

where $\omega^2(k)=c^2k^2[1+(1/4)\xi^2k^2]$ is the dispersion relation derived from the nonlinear Schrödinger equation. [Superscripts in Eq. (22) refer only to spatial indices.] We now focus on the final two terms, which involve the impurity coordinates.

The last contribution in Eq. (22) describes the interaction between impurities mediated by phonons, including a self-interaction term analogous to the self-interaction of vortices already discussed. Ignoring retardation effects, the purely local superfluid response [see Eq. (6)] means that the impurity-impurity interaction will be short-ranged provided the $U_a(k)$ are nonsingular in the infrared. The self-interaction term contributes a mass shift

$$\Delta m_a(\omega)=\frac{\overline{\rho}}{m^2}\int\frac{d^2k}{(2\pi)^2}\frac{k^4|U_a(k)|^2}{\omega^2(k)(\omega^2(k)-\omega^2)}, \tag{23}$$

whose imaginary part [taking $\omega(k)=ck$],

$$\Delta m_a''(\omega)=\frac{\overline{\rho}}{4m^2c^6}|U_a(\omega/c)|^2\omega^2$$

is super-Ohmic provided $U(k\to 0)\propto k^{-\sigma}$ with $\sigma<3/2$.[4]

Consider now the vortex-impurity interaction term. We will ignore retardation effects, and further assume a point interaction between impurities and superfluid, so that $U_a(k)=U_a$ is a constant. Without loss of generality, we may consider a single vortex-impurity pair. The contribution to the effective action is then

$$\mathcal{S}_{v-i}=-\frac{nU\overline{\rho}\kappa}{2\pi mc^2}\int dt\,\frac{\hat{z}\times(\mathbf{X}-\mathbf{R})}{(\mathbf{X}-\mathbf{R})^2}\cdot\dot{\mathbf{X}}$$
$$=-\frac{nU\overline{\rho}\kappa}{2\pi mc^2}\int dt\dot{\mathbf{X}}\cdot\frac{\partial}{\partial\mathbf{X}}\Theta(\mathbf{X}-\mathbf{R})$$

where $n$ is the integer charge of the vortex, $U$ is the impurity-boson density coupling, and $\Theta(\mathbf{x})=\tan^{-1}(y/x)$ is the angle function. Note that if $\mathbf{R}(t)$ is time independent, then, for closed paths, $\mathcal{S}_{v-i}$ is a topological quantity equal to $-(nU\overline{\rho}\kappa/mc^2)W_{v-i}$, where $W_{v-i}$ is the winding number of the vortex about the impurity. This makes excellent sense: the vortex effectively counts the number of bosons it encircles, and the density response of the superfluid generates a point accumulation of $\Delta N=-U\overline{\rho}/m^2c^2$ bosons to "screen" the impurity.

Now let the impurity move throughout the superfluid. Varying $\mathcal{S}_{v-i}$ with respect to $X^\alpha(t)$, we find the force on the vortex to be

$$F^\alpha=\frac{\delta\mathcal{S}_{v-i}}{\delta X^\alpha}=-\frac{nU\overline{\rho}\kappa}{2\pi mc^2}\left\{\frac{\delta^{\alpha\beta}-2\hat{\Delta}^\alpha\hat{\Delta}^\beta}{\mathbf{\Delta}^2}\right\}\epsilon^{\beta\gamma}\dot{R}^\gamma$$

with $\mathbf{\Delta}\equiv\mathbf{X}-\mathbf{R}$. This is simply related to the backflow current. If we ignore the self-interaction term for the vortices, which is appropriate at very low frequencies, then the vortex equation of motion,

$$n\overline{\rho}\kappa\epsilon^{\alpha\beta}\dot{X}^\beta=F^\alpha$$

just says that the vortex moves in the dipolar backflow field of the moving impurity.

Finally, consider the force on the impurity due to a moving vortex. We find, neglecting the impurity mass renormalization term,

$$m_{\rm imp}\ddot{R}^\alpha=\frac{nU\overline{\rho}\kappa}{2\pi mc^2}\left\{\frac{\delta^{\alpha\beta}-2\hat{\Delta}^\alpha\hat{\Delta}^\beta}{\mathbf{\Delta}^2}\right\}\epsilon^{\beta\gamma}\dot{X}^\gamma.$$

This too has a simple interpretation. The quantity $U\rho(\mathbf{R})/m$ is the local potential in which the impurity moves, and hence the force on the impurity is $\mathbf{F}=-(U/m)\boldsymbol{\nabla}\rho$. Now recall Duan's result [Eq. (4)] for the density response to a moving vortex. Taking the gradient gives us the appropriate force.

### Polaron model of a quantum vortex

Niu, Ao, and Thouless (NAT) have investigated a model of a quantum vortex coupled to superfluid density fluctuations.[5] They describe the vortex as a nonrelativistic particle of mass $M_e$ in a background uniform magnetic field (corresponding to the average superfluid density) and assume a *scalar* coupling to the phonons, i.e.,

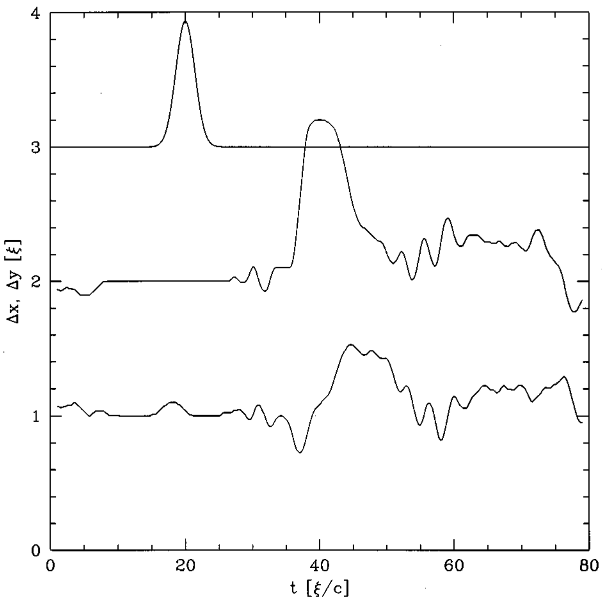


FIG. 5. Vortex trajectory $\mathbf{X}(t)$ for a pulsed superflow given by Eq. (21). In this case $T=2\xi/c$ and $v_{\max}=0.4c$. The smooth curve is the pulse shape.

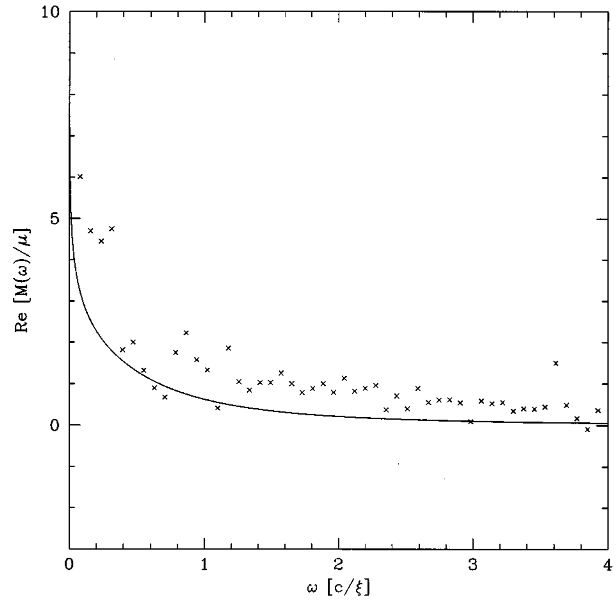


FIG. 6. Real part of the mass $M'(\omega)$ obtained from analysis of response to the pulse flow. Solid curve is the prediction of the linearized electrodynamic theory.

$$\mathcal{H}_{\mathrm{NAT}}=\frac{1}{2M_e}(\mathbf{p}+n\kappa\overline{\rho}\mathbf{a}/c)^2+\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(d_{\mathbf{k}}^{\dagger}d_{\mathbf{k}}+\frac{1}{2}\right)+\sum_{\mathbf{k}}W(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}(d_{\mathbf{k}}+d_{-\mathbf{k}}^{\dagger}),$$

where $n$ is the integer vorticity, $\boldsymbol{\nabla}\times\mathbf{a}=-c\hat{z}$ accounts for the geometric phase due to the background bosons, and $\omega_{\mathbf{k}}$ is the phonon frequency at wave vector $\mathbf{k}$. NAT go on to investigate a simple polaronic wave function which accounts for the phonon cloud around a vortex and conclude that an infinite vortex mass would shrink the quantum uncertainty in the vortex position to zero, a situation in conflict with explicit calculations using Feynman's trial vortex wave function. At this level, the mass $M_e$ is phenomenological and does not include the effects of phonons. It may be perhaps more appropriate to consider $M_e$ as the mass of an external particle trapped in the vortex core, as considered by Demircan, Po, and Niu.[28]

In keeping with the general philosophy that the vortex is a topological object which couples to the boson density as in Eq. (3), we propose a variant of the NAT model:

$$\mathcal{H}=\frac{1}{2M_e}[\mathbf{p}+n\kappa\overline{\rho}(\mathbf{a}+\mathbf{A})/c]^2+\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(d_{\mathbf{k}}^{\dagger}d_{\mathbf{k}}+\frac{1}{2}\right),$$

where $\omega_{\mathbf{k}}=\hbar ck$ and

$$\mathbf{A}(\mathbf{r})=\frac{i}{\sqrt{\Omega}}\sum_{\mathbf{k}}\sqrt{\frac{\hbar c}{2\overline{\rho}|\mathbf{k}|}}\,\hat{z}\times\hat{k}e^{i\mathbf{k}\cdot\mathbf{r}}(d_{\mathbf{k}}+d_{-\mathbf{k}}^{\dagger})$$

is the quantized radiation field corresponding to the superfluid density and current fluctuations[29] ($\Omega$ is the area of the system). We rewrite $\mathcal{H}$ as

$$\mathcal{H}=\frac{\mathbf{\Pi}^2}{2M_e}+\frac{n\kappa\overline{\rho}}{M_ec}\mathbf{A}\cdot\mathbf{\Pi}+\frac{n^2\kappa^2\overline{\rho^2}}{2M_ec^2}\mathbf{A}^2,$$

where $\mathbf{\Pi}=\mathbf{p}-(1/2)n\kappa\overline{\rho\hat{z}}\times\mathbf{r}$ is the cyclotron momentum operator for the vortex. Note that $[\Pi_x,\Pi_y]=i\hbar^2/l_o^2$, where $l_o\equiv\sqrt{m/2\pi n\overline{\rho}}$ is the "magnetic length" for a vortex of strength $n$. We now work to lowest order in $\mathbf{A}$, and follow NAT by assuming a trial state

$$|\Psi[\mathbf{R},\{n_{\mathbf{k}}\}]\rangle=|\chi_{\mathbf{R}}\rangle\otimes|\Psi_{\mathrm{ph}}\rangle$$

with

$$\chi_{\mathbf{R}}(\mathbf{r})=\frac{1}{\sqrt{2\pi l^2}}e^{-(\mathbf{r}-\mathbf{R})^2/4l^2}e^{-i\hat{z}\cdot\mathbf{r}\times\mathbf{R}/2l_o^2},$$

treating $l$ as a variational parameter. Taking the expectation value of $\mathcal{H}$ is the state $\chi_{\mathbf{R}}(\mathbf{r})$, we obtain the effective phonon Hamiltonian

$$\mathcal{H}_{\mathrm{ph}}=\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(d_{\mathbf{k}}^{\dagger}d_{\mathbf{k}}+\frac{1}{2}\right)+\sum_{\mathbf{k}}W(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{R}}(d_{\mathbf{k}}+d_{-\mathbf{k}}^{\dagger}),$$

$$W(\mathbf{k})=\frac{n\kappa^2}{4\pi c}\frac{l^2}{l_o^2}\frac{m}{M_e}\sqrt{\frac{\overline{\rho}\hbar\omega_{\mathbf{k}}}{2\Omega}}e^{-(1/2)\mathbf{k}^2l^2}.$$

The phonon ground state is a coherent state,

$$|\Psi_{\mathrm{ph}}\rangle=\exp\left\{\sum_{\mathbf{k}}\frac{W(\mathbf{k})}{\hbar\omega_{\mathbf{k}}}(d_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{R}}-d_{\mathbf{k}}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{R}})\right\}|0\rangle,$$

and the total energy is

$$E=\int d^2r\,\chi^*_{\mathbf{R}}(\mathbf{r})\left(\frac{\mathbf{\Pi}^2}{2M_e}\right)\chi_{\mathbf{R}}(\mathbf{r})-\sum_{\mathbf{k}}\frac{|W(\mathbf{k})|^2}{\hbar\omega_{\mathbf{k}}}$$
$$=\frac{\hbar^2}{4M_e l_o^2}\left[\frac{l_o^2}{l^2}+(1-n^2\mu/M_e)\,\frac{l^2}{l_o^2}\right],$$

where $\mu=\pi\overline{\rho}\xi^2=\kappa^2\overline{\rho}/4\pi c^2$ as before. Setting $\partial E/\partial l=0$ gives

$$l=l_o/\sqrt[4]{1-n^2\mu/M_e}.$$

Note that no solution exists for $M_e<n^2\mu$, which we interpret in the following manner. A cyclotron mode will show up as a pole in the denominator of Eq. (17), which means $\omega M_e(\omega)=\pm\kappa\overline{\rho}$. Now no such pole exists in the absence of the external mass $M_e$, but if $M_e$ is added to our $M(\omega)$, then a damped cyclotron resonance does exist at the cyclotron frequency $\omega\approx\kappa\overline{\rho}/M_e$, provided that $M_e\gtrsim n^2\mu$.

NAT compute a renormalized magnetic length $\tilde{l}$ according to the relation

$$|\langle\Psi(\mathbf{R})|\Psi(\mathbf{R}+\boldsymbol{\eta})\rangle|^2\equiv\exp\left[-\frac{|\boldsymbol{\eta}|^2}{2\tilde{l}^2}+\mathcal{O}(|\boldsymbol{\eta}|^4)\right],$$

which is obtained from the overlaps

$$\langle\chi_{\mathbf{R}}|\chi_{\mathbf{R}+\boldsymbol{\eta}}\rangle=\exp\left[-\frac{i\hat{z}\cdot\mathbf{R}\times\boldsymbol{\eta}}{2l_o^2}-\frac{|\boldsymbol{\eta}|^2}{8l^2}-\frac{|\boldsymbol{\eta}|^2}{8l_o^2}\right]$$

and

$$\langle\Psi_{\rm ph}(\mathbf{R})|\Psi_{\rm ph}(\mathbf{R}+\boldsymbol{\eta})\rangle$$
$$=\exp\left(-\sum_{\mathbf{k}}\frac{|W(\mathbf{k})|^2}{[\hbar\omega_{\mathbf{k}}]^2}\,(1-\cos\mathbf{k}\cdot\boldsymbol{\eta})\right).$$

We find

$$\frac{1}{\tilde{l}^2}=\frac{1}{2l^2}+\left[1+\frac{\pi n}{2\sqrt{2}}\left(\frac{m}{M_e}\right)^2\frac{\xi^3 l}{l_o^4}\right]\frac{1}{2l_o^2}.$$

### VII. CONCLUSION

In this paper we have explored the theory of dynamical vortices in superfluid films, deriving a frequency-dependent vortex mass which enters into the vortex equations of motion, as well as describing dissipation by radiation of phonons. Numerical simulations corroborating the predicted behavior of $M(\omega)$ were presented as well. These calculations may be extended to $(3+1)$-dimensional superfluids as well.[30] These results will be presented in a future publication.

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### APPENDIX: INERTIAL TERM FOR MANY VORTICES

The velocity-dependent part of the effective action induced by integrating out the phonon field is

$$\Delta\mathcal{S}=-\frac{1}{2}\,\overline{\rho}\kappa^2\int dt\int dt'\int\frac{d^2k}{(2\pi)^2}\int\frac{d\omega}{2\pi}\,\frac{e^{i\omega(t-t')}}{\omega^2-\omega^2(k)}$$
$$\times\sum_{i,j}n_in_je^{-i\mathbf{k}\cdot[\mathbf{X}_i(t)-\mathbf{X}_j(t')]}\hat{z}\times\hat{k}\cdot\dot{\mathbf{X}}_i(t)\hat{z}\times\hat{k}\cdot\dot{\mathbf{X}}_j(t').$$

We define $\mathbf{\Delta}_{ij}(t,t')=\mathbf{X}_i(t)-\mathbf{X}_j(t')$. Using

$$\int\frac{d\hat{k}}{2\pi}\,e^{-i\mathbf{k}\cdot\mathbf{\Delta}}\hat{k}^\alpha\hat{k}^\beta=\frac{1}{2}\,J_0(k\Delta)\,\delta^{\alpha\beta}$$
$$+\frac{1}{2}\,J_2(k\Delta)(\delta^{\alpha\beta}-2\hat{\Delta}^\alpha\hat{\Delta}^\beta),$$

where $J_n(z)$ is the Bessel function of order $n$. Taking $\omega(k)=ck$, we have

$$\Delta\mathcal{S}=-\frac{\overline{\rho}\kappa^2}{8\pi c^2}\int dt\int dt'\int\frac{d\omega}{2\pi}\,e^{i\omega(t-t')}$$
$$\times\sum_{i,j}n_in_j\dot{X}_i^\alpha(t)\dot{X}_j^\beta(t')\left[\left(K_2(-i\omega\Delta_{ij}/c)+\frac{2c^2}{\omega^2\Delta_{ij}^2}\right)\right.$$
$$\left.\times(\delta^{\mu\nu}-2\hat{\Delta}^\mu\hat{\Delta}^\nu)-K_0(-i\omega\Delta_{ij}/c)\,\delta^{\mu\nu}\right]\epsilon^{\alpha\mu}\epsilon^{\beta\nu},$$

where $K_n(z)$ is a modified Bessel function, and where $\omega\to\omega+i0^+$ is understood. Since the above integrand is already quadratic in velocities, which we assume are small compared with $c$, we may approximate $\Delta_{ij}$ as a constant. Expansions of $K_n(z)$ for small $z$ (Ref. 24) yield

$$K_0(-iz)=-C-\ln(-iz/2)+\cdots,$$

$$K_2(-iz)+\frac{2}{z^2}=\frac{1}{8}\,z^2\,\ln(-iz/2)+\cdots,$$

and at low frequencies the first of these terms dominates, so provided $\omega\ll c/\Delta_{\rm rms}$ we recover the action of Eq. (16).

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