Scaling behavior and universality near the quantum Hall transition

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A two-dimensional lattice system of noninteracting electrons in a homogeneous magnetic field with half a flux quantum per plaquette and a random potential is considered. For the large-scale behavior a supersymmetric theory with collective fields is constructed and studied within saddle-point approximation and fluctuations. The model is characterized by a broken supersymmetry indicating that only the fermion collective field becomes delocalized whereas the boson field is exponentially localized. Power counting for the fluctuation terms suggests that the interactions between delocalized fluctuations are irrelevant. Several quasiscaling regimes, separated by large crossover lengths, are found with effective exponents ν for the localization length ξ_l. In the asymptotic regime there is $\nu=1/2$ in agreement with an earlier calculation of Affleck and one by Ludwig *et al.* for a finite density of states. The effective exponent, relevant for physical system, is $\nu=1$ where the coefficient of ξ_l is growing with randomness. This is in agreement with recent high-precision measurements on Si metal-oxide-semiconductor field-effect transistor and $Al_xGa_{1-x}As/GaAs$ samples. [S0163-1829(97)01408-2]

I. INTRODUCTION

The transition between quantum Hall plateaus in a two-dimensional electron gas is characterized by a divergent localization length ξ_l with a critical exponent ν and a nonzero longitudinal conductivity σ_{xx}. ξ_l is finite and σ_{xx} is zero inside the Hall plateaus whereas σ_{xy} is a constant.

A direct measurement of the localization length exponent in an $Al_xGa_{1-x}As/GaAs$ sample by Koch *et al.*[1] gave a value for ν very close to 7/3. Recent high precision measurements on Si metal-oxide-semiconductor field-effect transistor[2,3] and $Al_xGa_{1-x}As/GaAs$ samples,[4] however, indicate that ξ_l diverges with the electron density n like $\approx b_n(n_c-n)^{-1}$ or with the magnetic field H like $\approx b_H(H_c-H)^{-1}$, where the quantum Hall transition (QHT) is at $n=n_c$ or $H=H_c$, respectively. The exponent $\nu\approx 1$ appears to be almost independent of the material or the Hall plateaus. On the other hand, the coefficients b_n, b_H are sensitive to disorder: they *increase* with increasing disorder.[3,4] This is a remarkable observation because in the scaling theory of the Anderson localization[5] the coefficient is related to the mean free path. That means it would *decrease* with increasing disorder. The observation of $\nu\approx 1$ is in sharp contrast to the experiment by Koch *et al.* The disagreement was explained in Ref. 2 by insufficient sample size in the earlier experiment.

The localization length scale near the QHT was also studied intensively in a number of numerical simulations using the network model[6,7] and the lowest Landau level approximation.[9,10] These calculations agree on the result that the critical exponent is $\nu\approx 7/3$. Concerning the exponent ν there is a calculation by Affleck[11] based on the $U(2n)/U(n)\times U(n)$ nonlinear σ model with a topological term in the replica limit $n\to 0$.[12] He obtains $\nu=1/2$. The same value was found for Dirac fermions with the random vector potential if the average density is finite.[13]

A reason for the impressive agreement of the numerical calculations, on the one hand, and the disagreement between experiments, numerical and analytic calculations, on the other hand, could be the sensitivity of the QHT to the size of the system and to the type of disorder. In particular, it may be related to the existence of a large characteristic length scale depending on the disorder. The existence of such a typical scale is also indicated by the numerical results due to the fact that there is a crossover from the pure network model ($\nu=1$) to the random network model ($\nu\approx 7/3$).

The purpose of this paper is to investigate the role of disorder induced length scales in a tight-binding model with a strong magnetic field. The work is based on an effective supersymmetric field theory for Dirac fermions with a random mass which enables us to study large scale properties. The main results are as follows.

(1) Spontaneous breaking of the supersymmetry. That is, the effective field theory for the QHT is *not* a nonlinear sigma model.

(2) The scaling behavior of the localization length depends on the characteristic scale $\exp(\pi/g)$, where g is the strength of disorder: If $\xi_l \ll \exp(\pi/g)$ the effective exponent is $\nu=1$, whereas for $\xi_l \gg \exp(\pi/g)$ the exponent is $\nu=1/2$.

(3) There is a universal value for the conductivity $\sigma_{xx}=e^2/h\pi$.

The paper is organized as follows: After the definition of the model in Sec. II an effective field theory is constructed for the averaged Green's functions (Sec. III). This field theory includes the description of the conductivity according to Kubo's formula. Then a collective field representation is introduced in order to cover symmetry breaking effects (Sec. III A). The latter are discussed using a saddle-point approximation for the collective field (Sec. III B). Gaussian fluctuations around the saddle points and corrections to Gaussian fluctuations are studied in Secs. III C and III D, respectively. Finally, the localization length (Sec. IV) and the conductivity (Sec. V) are evaluated.

II. THE MODEL

A lattice model is considered in this paper, stressing the universality in terms of the electron density and the magnetic

field observed in the experiment, to study the asymptotic behavior of the localization length near the QHT. A starting point is a microscopic model for noninteracting electrons on a regular lattice in a homogeneous magnetic field. Disorder enters only through a random potential on the lattice. This choice guarantees that the disorder does not affect the homogeneity of the magnetic field. The model is defined by the tight-binding Hamiltonian on a square lattice with magnetic flux $\phi = Ba^2$, where a is the lattice constant and B the homogeneous external magnetic field. There is nearest neighbor hopping with rate t and next nearest neighbor hopping with rate t'. The Hamiltonian reads in Landau gauge

$$H = -\sum_r \left[te^{2i\pi Bay/\phi_0} c(r)c^\dagger(r+e_x) + tc(r)c^\dagger(r+e_y) + t'e^{2i\pi Ba[y\pm(1/2)]\phi_0} c(r)c^\dagger(r+e_x\pm e_y) + \text{H.c.}\right] + \sum_r V(r)c(r)c^\dagger(r). \tag{2.1}$$

$e_{x,y}$ are lattice unit vectors, and $c^\dagger$ and c are fermion creation and annihilation operators, respectively. $V(r)$ is a random potential representing disorder on the lattice. Without disorder, i.e., for $V(r)=0$, this model was discussed extensively in the literature.[14–16] A central result is the occurrence of electron bands with a quantized Hall conductivity in each band. Gaps can be created in the model, for instance, by choosing a staggered potential $V(r)=(-1)^{r_1+r_2}\mu$ in the Hamiltonian (2.1).[13] By varying the staggered chemical potential one varies the concentration of electrons in the system. There are other methods to create gaps in a tight-binding model. For example, one could vary the magnetic field. This, however, would lead to a more complicated situation because the corresponding vector potential depends on space. In general, the relevant parameter for the quantum Hall transition is the filling factor $n\Phi_0/B$. This is essentially determined by the ratio of the concentration of electrons n and the magnetic field B. Therefore, the variation of the concentration of electrons (i.e., the chemical potential) is equivalent with the variation of the magnetic field in the quantum Hall system.

In general, the creation of new bands (''gap opening'') can be described by Dirac fermions.[16–18] Starting from the tight-binding Hamiltonian the Dirac fermions can be derived in a large-scale approximation. The simplest case is that with half a flux quantum per lattice plaquette ($\phi=\phi_0/2$).[13,19] (Such a strong magnetic flux is unrealistic in real crystals but typical for arrays of quantum dots in moderate magnetic fields.[20]) For half a flux quantum per plaquette it is easy to derive the Dirac theory from a sublattice representation which takes into account the phase factor $e^{i\pi y/a}$ of the tight-binding Hamiltonian and the staggered potential. The Fourier components of the nonrandom part $H(k)$ read

$$\begin{pmatrix} \mu & 1+e^{-ik_x} & \zeta(1-e^{-ik_y})(1-e^{-ik_x}) & 1+e^{-ik_y} \\ 1+e^{ik_x} & -\mu & 1+e^{-ik_y} & -\zeta(1-e^{-ik_y})(1-e^{ik_x}) \\ -\zeta(1-e^{ik_y})(1-e^{ik_x}) & 1+e^{ik_y} & \mu & -1-e^{ik_x} \\ 1+e^{ik_y} & \zeta(1-e^{ik_y})(1-e^{-ik_x}) & -1-e^{-ik_x} & -\mu \end{pmatrix}, \tag{2.2}$$

with $\zeta = it'/4$. All elements of the matrix are measured in units of the nearest neighbor hopping rate t. After expansion of $k=(\pm\pi,\pm\pi)+ap$ for small p vectors around the four nodes and a global orthogonal transformation $H(k)\to OH(k)O$ with

$$O = \begin{pmatrix} \sigma_0 & -i\sigma_0 \\ i\sigma_0 & -\sigma_0 \end{pmatrix}, \tag{2.3}$$

the Hamiltonian becomes

$$H'(p) = 2\begin{pmatrix} \mu+4i\zeta & ip_x-p_y & 2i\zeta(p_x+p_y) & 0 \\ -ip_x-p_y & -\mu-4i\zeta & 0 & 2i\zeta(p_x-p_y) \\ 2i\zeta(p_x+p_y) & 0 & \mu-4i\zeta & p_y+ip_x \\ 0 & 2i\zeta(p_x-p_y) & p_y-ip_x & -\mu+4i\zeta \end{pmatrix} \equiv \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \tag{2.4}$$

The corresponding Green's function,

$$\hat{G} = \begin{pmatrix} H_{11}+i\omega & H_{12} \\ H_{21} & H_{22}+i\omega \end{pmatrix}^{-1} \sim \begin{pmatrix} (H_{11}+i\omega)^{-1} & (H_{11}+i\omega)^{-1}H_{12}(H_{22}+i\omega)^{-1} \\ (H_{22}+i\omega)^{-1}H_{21}(H_{11}+i\omega)^{-1} & (H_{22}+i\omega)^{-1} \end{pmatrix}, \tag{2.5}$$

decays asymptotically into two diagonal blocks

$$\sim\begin{pmatrix}(H_{11}+i\omega)^{-1} & 0\\ 0 & (H_{22}+i\omega)^{-1}\end{pmatrix}. \tag{2.6}$$

The lattice constant a is implicitly scaled out in t', μ', and p_j. $\sim$ means asymptotics with respect to $\mu+4i\zeta\sim 0$ and $p_x\sim p_y\sim 0$. Thus the approximation breaks up the Hamiltonian (2.1) into two independent Dirac Hamiltonians $H_{11/22}=\sigma p+\sigma_3(\mu\mp t')$ with Pauli matrices σ_j. The two Dirac theories describe particles with different masses $\mu\mp t'$, respectively. The next nearest neighbor hopping term lifts the degeneracy of the two Dirac particles. Therefore, it plays an important role in this model and must be taken into account. A variation of the chemical potential implies a variation of the Dirac mass. In particular, Dirac fermions undergo a Hall transition if the mass vanishes.[13,21] This is a consequence of the fact that the mass breaks the time-reversal symmetry: depending on the sign of m there is a clockwise or counterclockwise Hall current. If the light Dirac particle undergoes a Hall transition at $\mu-t'=0$ its contribution to the Hall conductivity changes from $\sigma_{xy}=-1/2$ for $\mu-t'<0$ to $\sigma_{xy}=1/2$ for $\mu-t'>0$. (The conductivity is in units of e^2/h.) The heavy Dirac particle contributes $\sigma_{xy}=1/2$ because its mass is positive. Thus the combined effect is a Hall step from $\sigma_{xy}=0$ to $\sigma_{xy}=1$. This picture is particularly simple for $\Phi=\Phi_0/2$ but should also hold for other values of the flux as long as the low energy excitations are linear and can be described by Dirac fermions.

For large scale properties like for the critical behavior near the Hall transition it is sufficient to consider only the light particle with $\mu-t'$,

$$\begin{aligned}(H'+i\omega)^{-1}&=\begin{pmatrix}i\omega+\mu-t' & i\nabla_1+\nabla_2\\ i\nabla_1-\nabla_2 & i\omega-\mu+t'\end{pmatrix}^{-1}\\ &\equiv G(i\omega)\\ &\equiv\begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix},\end{aligned} \tag{2.7}$$

where ∇ is the lattice gradient operator. Disorder, originally introduced in H by the random potential V, appears in $H_{11/22}$ as a diagonal matrix V' with independent random elements V_1, V_2. The appearance of two random variables per site is a consequence of the sublattice representation required by the phase factor $e^{i\pi y/a}$ of the tight-binding Hamiltonian. The random matrix V' is equivalent to a random mass $\delta\mu\sigma_3$ and a random energy $\delta E\sigma_0$. For technical reasons the random energy term will be neglected in the following.

It should be noticed that the random mass is marginally irrelevant on a perturbative level.[22] However, going beyond perturbation theory, it turns out that the random mass leads to spontaneous symmetry breaking which changes the properties significantly.[23] This effect has not been included in previous studies of the localization properties near the Hall transition. It will be important for the considerations in this paper.

Dirac fermions can also be derived as the large scale approximation of the network model.[24] It was recently pointed out by Ho and Chalker[25] that the network implies a random Dirac mass (due to fluctuations in the tunneling rates), a random energy (due to fluctuations in the flux per plaquette), and a random vector potential (due to fluctuations in the phase of the hopping elements). That is, in terms of the network model the random Dirac mass requires a fixed flux per plaquette and a fixed phase for the current between the vertices of the network. This is probably the simplest situation for the realization of a QHT.

After averaging with respect to the random mass the localization length ξ_l, measured in lattice units $a=\sqrt{\phi_0/2B}$ for electrons in a magnetic field B, is defined as the decay length of the function $C_{jj'}(r,\omega)\equiv\langle|G_{jj'}(r,0;i\omega)|^2\rangle$. The relation $|G_{jj'}(r,0;i\omega)|^2=G_{jj'}(r,0;i\omega)G_{j'j}(0,r;-i\omega)$ means that ξ_l is given by the product of two Green's function at frequencies with *opposite sign* (retarded and advanced Green's functions). Due to the 2×2 block structure of G there exists a relation between Green's functions at $i\omega$ and Green's functions at $-i\omega$:

$$\begin{aligned}G_{jj}(r,r';-i\omega)&=-G_{j'j'}(r',r;i\omega),\\ G_{jj'}(r,r';-i\omega)&=-G_{jj'}(r',r;i\omega)\quad(j\neq j').\end{aligned} \tag{2.8}$$

This identity reflects the Lorentz covariance of the Dirac theory. It implies

$$|G_{jj}(r,r';i\omega)|^2=-G_{jj}(r,r';i\omega)G_{j'j'}(r,r';i\omega) \tag{2.9}$$

and

$$|G_{jj'}(r,r';i\omega)|^2=-G_{jj'}(r,r';i\omega)G_{j'j}(r,r';i\omega). \tag{2.10}$$

This means that only the Green's functions with one frequency is required for the evaluation of localization properties in the relativistic model. The averaged quantity $C_{jj'}(r,\omega)$ is translational invariant. Therefore, it can be expressed by its Fourier components $\tilde{C}_{jj'}(k,\omega)$. This can be used to calculate the localization length ξ_l. (The following discussion holds for any choice of j, j'. Therefore, these labels are not written explicitly.) The correlation function $C(r,\omega)$ for large r is proportional to $r^{-\alpha}\exp(-r/\xi_l)$ with some exponent α for which we assume that it is fixed for the model and does not depend on the parameters. This implies that

$$\frac{\Sigma_r r^2C(r,\omega)}{\Sigma_r C(r,\omega)}=\frac{\Sigma_r r^{2-\alpha}\exp(-r/\xi_l)}{\Sigma_r r^{-\alpha}\exp(-r/\xi_l)}=\xi_l^2\frac{\Sigma_x x^{2-\alpha}\exp(-x)}{\Sigma_x x^{-\alpha}\exp(-x)}. \tag{2.11}$$

Dropping the constant term from the ratio of sums on the right hand side of (2.11), the localization length can be defined in terms of the Fourier components as

$$\xi_l=\left|\sqrt{-\frac{\nabla_k^2\tilde{C}(k,\omega)}{\tilde{C}(k,\omega)}}\right|_{k=0}. \tag{2.12}$$

The localization length is finite for $\omega\neq 0$ but diverges in the regime of delocalized states with $\omega\to 0$.

The localization length for massless Dirac fermions without disorder diverges like $|\omega|^{-1}$ if $\omega=0$ is approached. This

behavior is probably unstable against arbitrarily weak randomness, as it will be shown in this paper. However, it has been shown in a previous paper[26] that the localization length of the averaged correlation function has a lower bound which is the energy-energy correlation length of the 2D random bond Ising model. Since the latter diverges at the critical points, this implies a divergent localization length at the QHT.

The calculation for the network model of Chalker and Coddington,[6] indicates that the critical exponent $\nu=1$ of the pure model may change to $\nu=7/3$ due to disorder. This was discussed as a possible appearance of a new random fixed point of the random model.[13] However, a new fixed point with such a behavior has not been found so far in terms of renormalization group calculations.

III. FUNCTIONAL INTEGRAL REPRESENTATION

It is convenient to introduce a functional integral representation for $C(r,\omega)$, because this provides a basis to apply an approximation using a saddle-point integration. The product of Green's functions on the right hand side of (2.9) and (2.10) can formally be written as

$$(i\omega+H)^{-1}_{rj,r'j'}(i\omega+H^T)^{-1}_{r'k',rk} = \int \chi_{r'j'}\overline{\chi_{rj}}\Psi_{rk}\overline{\Psi}_{r'k'}\exp(-S_0)\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}\chi\mathcal{D}\overline{\chi}, \tag{3.1}$$

with the quadratic form of the superfield (χ_r,Ψ_r),

$$S_0=-i\ \mathrm{sign}(\omega)\sum_{r,r'}\begin{pmatrix}\chi_r\\ \Psi_r\end{pmatrix}\begin{pmatrix}i\omega+H & 0\\ 0 & i\omega+H^T\end{pmatrix}_{r,r'}\begin{pmatrix}\overline{\chi_{r'}}\\ \overline{\Psi_{r'}}\end{pmatrix}. \tag{3.2}$$

χ is a complex field and Ψ a Grassmann field, respectively. It is important to notice that H appears in the quadratic form for the complex field, whereas H^T appears for the Grassmann field. This difference will turn out to be crucial for the localization properties of the Dirac fermions. In particular, it will give all the delocalized states expected near the Hall transition. In contrast, the quadratic form where H is used in the Grassmann sector instead of H^T does not give these critical properties.[26,21]

Averaging with respect to disorder leads to

$$\langle(i\omega+H)^{-1}_{rj,r'j'}(i\omega+H^T)^{-1}_{r'k',rk}\rangle = \int \chi_{r'j'}\overline{\chi_{rj}}\Psi_{rk}\overline{\Psi}_{r'k'}\langle\exp(-S_0)\rangle\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}\chi\mathcal{D}\overline{\chi}. \tag{3.3}$$

A Gaussian distribution of the random Dirac mass is assumed in the following with mean m and variance g. Then the average can be performed exactly giving an additional quartic interaction term in S_0 with coupling constant g. For weak disorder, i.e., small g, one could apply perturbation theory. Unfortunately, this does not lead to interesting results because it cannot catch spontaneous symmetry breaking. In order to deal with the latter one must construct a representation which describes the field which is the conjugate to the symmetry breaking terms of the Dirac theory, the mass m and the frequency ω. The appearance of the symmetry breaking terms in S_0 dictates the choice of $\chi_r\overline{\chi_r}$ and $\Psi_r\overline{\Psi}_r$ as the collective fields.

A. Collective field representation

In general, products of the fields χ, Ψ, in S_0 can be replaced by the collective fields as $\chi_r\overline{\chi_r}\to Q_r$, $\Psi_r\overline{\Psi}_r\to P_r$, $\chi_r\overline{\Psi}_r\to\Theta_r$, and $\Psi_r\overline{\chi_r}\to\Theta_r$. (Some care is necessary to choose the paths of integration correctly.[27]) One obtains for (3.3) in the collective field representation (for details see Ref. 27)

$$\frac{1}{g^2}\int \Theta_{r,kj}\overline{\Theta}_{r',j'k'}\exp(-S')\mathcal{D}P\mathcal{D}Q\mathcal{D}\Theta\mathcal{D}\overline{\Theta} \quad (r\neq r'), \tag{3.4}$$

with the supersymmetric effective action

$$\begin{aligned}S'=\frac{1}{g}\sum_r\ &(\mathrm{Tr}_2Q_r^2+\mathrm{Tr}_2P_r^2+2\,\mathrm{Tr}_2\overline{\Theta}_r\Theta_r)+\ln\det[(H_0+i\omega\sigma_0\\ &-2\tau Q\tau)(H_0^T+i\omega\sigma_0+2i\tau P\tau)^{-1}]\\ &+\ln\det[\mathbf{1}-4\tau\overline{\Theta}\tau(H_0^T+i\omega\sigma_0+2i\tau P\tau)^{-1}\\ &\times\tau\Theta\tau(H_0+i\omega\sigma_0-2\tau Q\tau)^{-1}].\end{aligned} \tag{3.5}$$

$H_0=i\sigma\nabla+m\sigma_3$ is the average Dirac Hamiltonian and τ the diagonal matrix $(1,i)$. The introduction of the collective fields is important to discover the finite length scale $e^{\pi/g}$, created by disorder, which is crucial for the properties of the random Dirac mass model.[27]

B. Saddle-point approximation

A saddle-point (SP) approximation is a crude approach for a two-dimensional system because it usually gives the wrong results for low dimensional systems due to strong effects of fluctuations. In the model under consideration it will be used as a starting point to study also the fluctuations. An argument in favor of a SP approach is the fact that some features of the model can be described which are not available from pertubation theory. An example is the creation of states in the massive Dirac theory due to randomness.[23,28] The hope is that the fluctuations around the SP are controlled by Gaussian fluctuations, at least if randomness is weak. This will be supported by the discussion presented below. Another argument for the SP approximation is its equivalence with the $N\to\infty$-limit, where N is a formal extension of the model to one for electrons with N states per lattice site.[21]

The SP of the functional integral is given by the equations $\delta_Q S'=\delta_P S'=0$. The two SP equations are identical if one substitutes $P=iQ$:

$$\sigma_3\tau Q_r\tau\sigma_3=g(H_0+i\omega\sigma_0-2\tau Q\tau)^{-1}_{r,r}. \tag{3.6}$$

An ansatz for a uniform SP solution reads $\tau Q_0\tau=-(i\eta\sigma_0+m_s\sigma_3)/2$. The SP equations imply a shift of the frequency $\omega\to\eta'\equiv\eta+\omega$ with

$$\eta'-\omega=\eta' gI, \tag{3.7}$$

and a shift of the average mass $m \to m' \equiv m + m_s$ with

$$m_s = -mgI/(1+gI), \tag{3.8}$$

$$\begin{aligned} I &\sim \frac{1}{\pi}\int_0^1 [\eta'^2 + (m+m_s)^2 + k^2]^{-1} k dk \\ &\sim -\frac{1}{2\pi}\ln[\eta'^2 + (m+m_s)^2] \\ &= -\frac{1}{\pi}\ln|\mu|, \end{aligned} \tag{3.9}$$

with $\mu = m + m_s + i\eta'$. In the pure limit $g \to 0$ the SP equations lead to $\eta' = \omega$ and $m_s = 0$. For a given $g > 0$ the SP depends on two parameters, m and ω. For $\omega = 0$ and large $|m|$ there is only a trivial solution of (3.7) with $\eta' = \eta = 0$ because $gI < 1$. As one varies $|m|$ there is a critical point $m_c = 2\mu_c = 2e^{-\pi/g}$, where $gI < 1$ approaches $gI = 1$. As a consequence, the SP solution of (3.7) bifurcates from $\eta = 0$ to $\eta \neq 0$ at $\mu = \mu_c$, and $\eta = 0$ becomes unstable.[27] In the following only the region with $\eta \neq 0$ is considered, where $\mathrm{sign}(\eta) = \mathrm{sign}(\omega)$. This has a nonzero density of states

$$\begin{aligned} \rho(m) &\approx (1/2\pi g) \lim_{\omega \to 0} \eta' \\ &= (1/2\pi g)\,\eta = (1/4\pi g)\sqrt{(m_c^2 - m^2)}\,\Theta(m_c^2 - m^2), \end{aligned} \tag{3.10}$$

which describes a semicircular behavior. Of course, this must be normalized with an energy cutoff dependent constant.

As an ansatz for the SP approximation with $\omega \neq 0$ one can write

$$\eta'(\omega)^2 = \eta'(\omega = 0)^2 + \delta = \frac{m_c^2 - m^2}{4} + \delta, \tag{3.11}$$

where the last equation follows from the SP equation for $\omega = 0$. This implies

$$|\mu|^2 = m_c^2/4 + \delta \tag{3.12}$$

and for Eq. (3.7)

$$(g/2\pi)\,\eta' \ln(1 + 4\delta/m_c^2) = \omega. \tag{3.13}$$

The expansion of the logarithm for $\delta \ll m_c^2/4$ yields a cubic equation for δ

$$\left(\frac{m_c^2 - m^2}{4} + \delta\right)\delta^2 \approx \omega^2 \left(\frac{\pi m_c^2}{2g}\right)^2. \tag{3.14}$$

Although this equation could be solved directly it is simpler to distinguish two different asymptotic regimes:

(i) $(m_c^2 - m^2)/4 \ll \delta \ll m_c^2/4$,

$$\delta \sim \omega^{2/3}\left(\frac{\pi m_c^2}{2g}\right)^{2/3} \tag{3.15}$$

and (ii) $(m_c^2 - m^2)/4 \gg \delta$,

$$\delta \sim \omega \frac{\pi m_c^2}{g}\frac{1}{\sqrt{m_c^2 - m^2}}. \tag{3.16}$$

C. Gaussian fluctuations

In order to evaluate the localization length ξ_l the Gaussian fluctuations around the SP must be calculated (semiclassical approximation). Since Q, P, and Θ are 2×2 matrices, the fluctuations can also be written as four-component vector fields: $q_1 = \delta Q_{11}$, $q_2 = (\delta Q_{12} + \delta Q_{21})/2$, $q_3 = -i(\delta Q_{12} - \delta Q_{21})/2$, $q_4 = \delta Q_{22}$ with analogous definitions for $p_1, \ldots, p_4$ and for the Grassmann field $\psi_1, \ldots, \psi_4$. The action of the Gaussian fluctuations reads in the Fourier representation

$$\begin{aligned} S' \approx \int \sum_{\mu,\mu'=1}^{4} [&(\mathbf{I}_k)_{\mu,\mu'}(q_{k,\mu}q_{-k,\mu'} + p_{k,\mu}p_{-k,\mu'}) \\ &+ 2(\mathbf{I}'_k)_{\mu,\mu'}\overline{\psi}_{k,\mu}\psi_{-k,\mu'}]d^2k, \end{aligned} \tag{3.17}$$

with the fluctuation matrices $\mathbf{I}_k$ and $\mathbf{I}'_k$. For the large-scale properties one needs only the asymptotic behavior for small wave vectors k. In particular, for a vanishing wave vector there is

$$\mathbf{I}_0 = \begin{pmatrix} 1/g - 2\alpha\mu^{*2} & 0 & 0 & 2\beta \\ 0 & 2/g - 4\alpha|\mu|^2 & 0 & 0 \\ 0 & 0 & 2/g - 4\alpha|\mu|^2 & 0 \\ 2\beta & 0 & 0 & 1/g - 2\alpha\mu^2 \end{pmatrix}, \tag{3.18}$$

$$\mathbf{I'}_0 = \begin{pmatrix} 1/g - 2\alpha\mu^{*2} & 0 & 0 & 0 \\ 0 & 2(1/g - 2\alpha|\mu|^2 - 2\beta) & 0 & 0 \\ 0 & 0 & 2(1/g - 2\alpha|\mu|^2 + 2\beta) & 0 \\ 0 & 0 & 0 & 1/g - 2\alpha\mu^2 \end{pmatrix}, \tag{3.19}$$

with

$$\alpha=\int (|\mu|^2+k^2)^{-2}d^2k/4\pi^2\sim|\mu|^{-2}/4\pi=1/\pi m_c^2 \tag{3.20}$$

and

$$\beta=\int k^2(|\mu|^2+k^2)^{-2}d^2k/4\pi^2$$
$$=I/2-1/4\pi\sim-\ln|\mu|^2/4\pi\sim 1/2g. \tag{3.21}$$

These quantities become quite large for small $|\mu|$ indicating a short range behavior of the related modes. However, the second diagonal element of $\mathbf{I}'$ vanishes for vanishing ω because of $1/g-2\alpha|\mu|^2-2\beta=\omega/g\eta'$. This is a direct consequence of the SP condition (3.7) and indicates a critical mode $\psi_{k,2}$ for all $|m|<m_c$. Moreover, it implies a divergent behavior of the localization length if $\omega\to 0$. The corresponding correlation function of the critical mode can be calculated in the large-scale limit by expanding $\mathbf{I}'_k$ in powers of k as

$$\tilde{C}(k,\omega)=g^{-2}(2\omega/g\eta'+Dk^2)^{-1}, \tag{3.22}$$

where

$$D=4\alpha\left[1+\alpha\left(\frac{\mu^2}{1/g-2\alpha\mu^2}+\frac{\mu^{*2}}{1/g-2\alpha\mu^{*2}}\right)\right]. \tag{3.23}$$

$g\eta' D/2$ is like a diffusion coefficient. It is real and it never becomes zero. The critical behavior describes a phase with a divergent sum $\Sigma_{r,j}\langle|G_{jj}(r,r')|^2\rangle$; i.e., the correlation function decays nonexponentially. It implies that $\tilde{C}(k=0,\omega)=\eta'/2g\omega=\pi\langle\rho(\omega)\rangle/\omega$. This holds not only on the SP level but, in general, due to the identity $\Sigma_{r',j,j'}|G_{jj'}(r,r')|^2=\pi\rho/\omega$.

$\tilde{C}$ in (3.22) is an approximation based on $Dk^2\ll 1$. Since D diverges like $1/\pi m_c^2$ it is not possible to use this expression for the pure limit. This also reflects the nonperturbative character of the SP approximation.

Apart from the critical (delocalized) fermion mode there is a boson mode which becomes critical at $m=\pm m_c$. This is due to a vanishing eigenvalue of $\mathbf{I}_0$ because $\eta=0$ at $m=\pm m_c$. Thus there are delocalized states for $|m|<m_c$ due to massless fermions, whereas a combination of critical fermions *and* critical bosons controls the QHT. The band of delocalized fermions simplifies the study of transport properties of the model away from the critical points $m=\pm m_c$.

An analogous calculation for the Gaussian fluctuations can be performed for Anderson localization [i.e., for $\phi=0$ in Eq. (2.1)]. In that case the Green's function of Eq. (2.7) must be replaced by

$$G(i\omega)\equiv\begin{pmatrix} i\omega+M+\nabla^2 & 0 \\ 0 & i\omega-M-\nabla^2\end{pmatrix}^{-1}. \tag{3.24}$$

Now the matrix $H=M+\nabla^2$ is symmetric in contrast to $H=i\sigma\nabla+(\mu-t')\sigma_3$. As a consequence the corresponding fluctuation matrix is degenerated for the fermion and the boson sector (i.e., $\mathbf{I}'_k=\mathbf{I}_k$) with

$$\mathbf{I}_0=\begin{pmatrix} 1/g-\alpha & 0 & 0 & 0 \\ 0 & 2(1/g-\beta) & 0 & 0 \\ 0 & 0 & 2(1/g-\beta) & 0 \\ 0 & 0 & 0 & 1/g-\alpha^*\end{pmatrix}, \tag{3.25}$$

where

$$\alpha=\frac{1}{(2\pi)^d}\int\frac{(k^2-E+i\eta)^2}{[(k^2-E)^2+\eta^2]^2}d^dk \tag{3.26}$$

and

$$\beta=\frac{1}{(2\pi)^d}\int\frac{1}{(k^2-E)^2+\eta^2}d^dk. \tag{3.27}$$

There are two vanishing eigenvalues of $\mathbf{I}_0$ for vanishing ω in the Grassmann as well as in the complex sector, each of them is $1/g-\beta=\omega/g\eta'$. Therefore, the large-scale behavior of this model is very different, and the critical properties are described by a nonlinear σ model including fermion as well as boson degrees of freedom.[29] This is a consequence of the fact that $\phi=0$ preserves the supersymmetry, whereas the model with $\phi=\phi_0/2$ breaks the supersymmetry implying $\mathbf{I}\neq\mathbf{I}'$. The latter implies that there is only one massless (delocalized) fermion field, and all other fields are massive (localized).

D. Corrections to Gaussian fluctuations

Gaussian fluctuations are usually not sufficient to describe the properties of a critical system, especially at low dimensionality, because the interaction between the fluctuations are a relevant perturbation. For Anderson localization the interaction is marginal in $d=2$ as one finds from power counting of the scaling behavior. This method can also be applied to the effective field theory of random Dirac fermions. It provides a first check for the effect of the interaction among the fluctuations. In the following we will see that the perturbation term for random Dirac fermions has dimensionality -2. Therefore, the interaction of the fluctuations is irrelevant. (It seems that the dimensionality is reduced by 2 in comparison to Anderson localization. This is similar to the dimensional reduction by 2 in supersymmetric theories, applied to the average density of states for a particle in a strong magnetic field and a random potential.[30,31])

Away from the critical points $|m|=m_c$ it is sufficient to study the Grassmann fluctuations because the complex fluctuations are massive. Their action reads for terms up to the fourth order in the fluctuations,

$$S'\approx 2\int(2\omega/g\eta'+Dk^2)\overline{\psi_{k,2}}\psi_{-k,2}d^2k$$
$$-8\sum_{r_1,\ldots,r_4}Tr_2[\overline{\Theta}_{r_1}\overline{G}_{r_1r_2}\Theta_{r_2}\overline{G^T}_{r_2r_3}\overline{\Theta}_{r_3}\overline{G}_{r_3r_4}\Theta_{r_4}\overline{G^T}_{r_4r_1}], \tag{3.28}$$

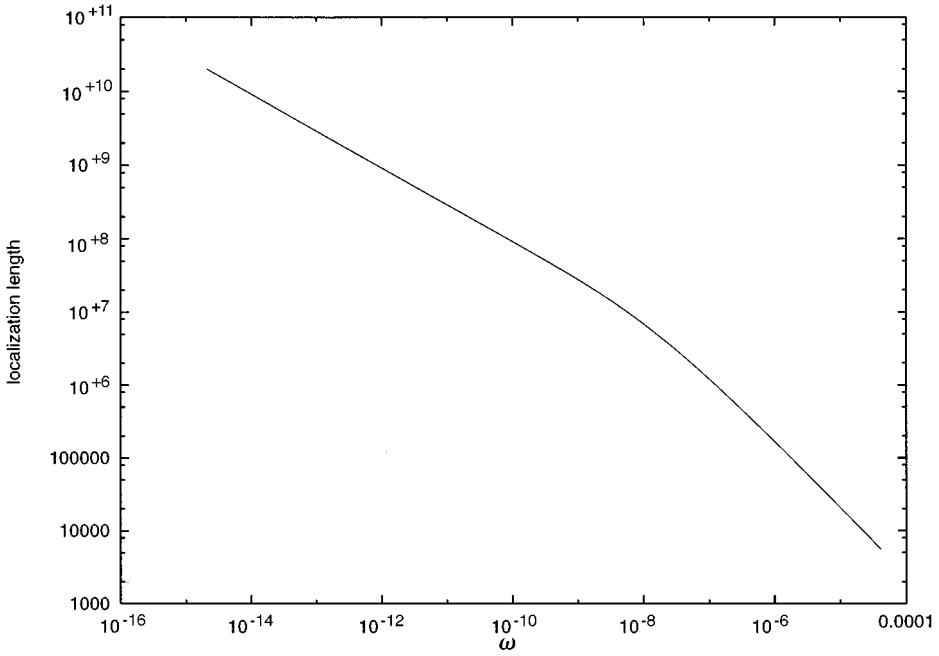

FIG. 1. Scaling of the localization length as a function of the frequency for disorder strength $g=0.2$ and $\mu-t'=0$.

where $\overline{G}=\tau G_0 \tau$. $\psi_2\sigma_1$ is the only critical mode of the collective Grassmann field Θ_r. Thus the interaction term can also be written as

$$-8\sum_{r_1,\ldots,r_4} Tr_2[\sigma_1\overline{G}_{r_1r_2}\sigma_1\overline{G^T}_{r_2r_3}\sigma_1\overline{G}_{r_3r_4}\sigma_1\overline{G^T}_{r_4r_1}] \times\overline{\psi}_{r_1,2}\psi_{r_2,2}\overline{\psi}_{r_3,2}\psi_{r_4,2}. \tag{3.29}$$

The trace term can be evaluated and yields after some straightforward calculation together with the approximation $G_{r,r'}\approx|\mu|^{-2}[(i\eta'\sigma_0-m\sigma_3)\delta_{r,r'}+i\nabla_{1;r,r'}\sigma_1+i\nabla_{2;r,r'}\sigma_2]$ the following expression

$$-8|\mu|^{-8}\sum_{r_1,\ldots,r_4}[\overline{\psi}_{r_1}d_{r_1,r_2}\psi_{r_2}d_{r_2,r_3}\overline{\psi}_{r_3}d_{r_3,r_4}\psi_{r_4}d_{r_4,r_1} + \overline{\psi}_{r_1}d^*_{r_1,r_2}\psi_{r_2}d^*_{r_2,r_3}\overline{\psi}_{r_3}d^*_{r_3,r_4}\psi_{r_4}d_{r_4,r_1}], \tag{3.30}$$

where the index 2 of the Grassmann field has been dropped and $d_{r,r'}=(\nabla_{1;r,r'}+i\nabla_{2;r,r'})$. This result reflects the fact that terms with an odd number of ∇ operators cancels in (3.29) and terms quadratic in ∇ cancel each other because of the anticommutation rule of the Grassmann field: $\psi_{r_1}\psi_{r_1}\psi_{r_3}\psi_{r_3}+\psi_{r_1}\psi_{r_3}\psi_{r_3}\psi_{r_1}=0$. Higher order terms with at most second order gradients disappear individually because they contain Grassmann fields at the same site. Simple power counting indicates that this interaction term has dimensionality -2. Therefore, the interaction is irrelevant in comparison with Gaussian fluctuations considered in the previous section, and it scales quickly to zero under renormalization group transformations.

The situation is different if we approach $m=\pm m_c$ because the complex field also becomes critical. As a consequence, the corrections to Gaussian fluctuations are marginal rather than irrelevant then. It is possible that the localization length becomes finite at $m=\pm m_c$ due to renormalization effects. This phenomenon near $\pm m_c$ requires a separate treatment which will not be considered in this paper.

The irrelevance of the interaction terms is very special for the model under consideration. In similar two-dimensional models, like the Gross-Neveu model or the tight-binding model without magnetic field (orthogonal nonlinear σ model) or with weak magnetic field (unitary nonlinear σ model), the interaction is always marginal (i.e., dimensionality of the interaction term is zero).

IV. LOCALIZATION LENGTH

According to the discussion of the Gaussian fluctuations in Sec. III C, the leading large scale behavior is given by the correlation function of (3.22). From the latter the localization length can be calculated, using Eq. (2.12), as $\xi_l=(2Dg\eta'/\omega)^{1/2}$. Together with the SP results (3.11) and (3.23), ξ_l reads

$$\xi_l\approx\sqrt{\frac{2g}{\pi}}\left(\frac{m_c^2-m^2}{4}+\delta\right)^{1/4}(m_c^2/4+\delta)^{-1/2}\omega^{-1/2}. \tag{4.1}$$

The localization length diverges like $\omega^{-1/2}$ because D and η' remain nonzero for $|m|<m_c$ according to the results of the SP approximation. To compare ξ_l with numerical or experimental results it is important that D can be large ($\sim|\mu|^{-2}$) and η' can be small. Therefore, several quasiscaling regimes exist as indicated by the graph in Fig. 1 which is simply the plot of (4.1) together with (3.13). Only in the asymptotic regime $\omega\sim 0$, i.e., for $(m_c^2-m^2)/4\gg\delta$, the localization length diverges like $\omega^{-1/2}$,

$$\xi_l\sim\sqrt{\frac{2g}{\pi}}\left(\frac{m_c^2-m^2}{4}\right)^{1/4}\frac{2}{m_c}\omega^{-1/2}. \tag{4.2}$$

Surprisingly, the exponent $\nu=1/2$ agrees with that of a completely different approach to the QHT by Affleck.[11] More-

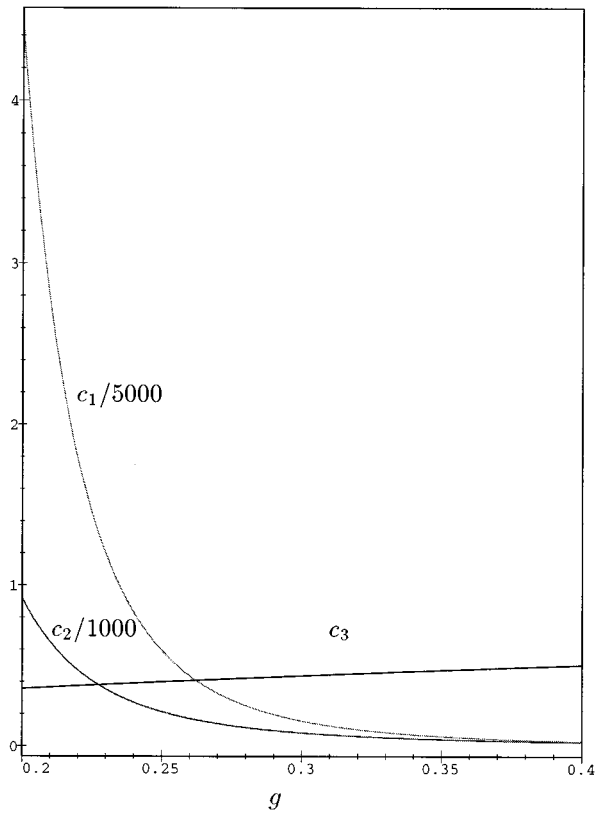

FIG. 2. Coefficients c_j of the power laws of ξ_l for $\mu - t' = 0$.

over, $\nu = 1/2$ was also found by Ludwig *et al.* for Dirac fermions with a random vector potential at that special point on their fixed point line where the average density of states is finite.[13]

A quasiscaling regime occurs for $(m_c^2 - m^2)/4 \ll \delta \ll m_c^2/4$, where we have

$$\xi_l \sim \sqrt{\frac{2g}{\pi}} (m_c^2/4 + \delta)^{-1/2} \left(\frac{\pi m_c^2}{2g} \right)^{1/6} \omega^{-1/3} \sim \sqrt{\frac{2g}{\pi}} \frac{2}{m_c} \left(\frac{\pi m_c^2}{2g} \right)^{1/6} \omega^{-1/3}, \tag{4.3}$$

i.e., the effective exponent is $\nu = 1/3$.

Going back to the SP equation (3.13) one could also assume $m_c^2/4 \ll \delta$. It gives

$$-\delta^{1/2} (g/2\pi) \ln(m_c^2) = \delta^{1/2} \sim \omega. \tag{4.4}$$

This result implies for the localization length

$$\xi_l \sim \sqrt{\frac{2g}{\pi}} \delta^{-1/4} \omega^{-1/2} \sim \sqrt{\frac{2g}{\pi}} \omega^{-1}, \tag{4.5}$$

i.e., the effective exponent is $\nu = 1$. The coefficient in (4.3) *decays* for small g like $g^{1/3} e^{2\pi/3g}$ whereas the corresponding coefficient in (4.5) *grows* like $g^{1/2}$ with the strength of disorder. The coefficients of the power law for $\nu = 1/3$ (c_1), for $\nu = 1/2$ (c_2) and for $\nu = 1$ (c_3) are plotted in Fig. 2. $\nu = 1$ and the behavior of the corresponding coefficient c_3 agree with the observation in experiments.[2–4]

V. CONDUCTIVITY

The longitudinal conductivity can be calculated via Kubo's formula,

$$\sigma_{xx}(\omega) = \frac{e^2}{h} \omega^2 \sum_r r^2 \langle G_{jj'}(r,0;i\omega) G_{j'j}(0,r;-i\omega) \rangle. \tag{5.1}$$

The correlation function is again the expression we have considered in the effective field theory. Since for small ω only the large-scale part of the correlation contributes significantly, we can use $\tilde{C}(k,\omega)$ of (3.22) to write

$$\sigma_{xx}(\omega) = -\frac{e^2}{h} \omega^2 \nabla_k^2 \tilde{C}(k,\omega)|_{k=0} = \frac{e^2}{h} \omega^2 g^{-2} D(\omega/g\eta')^{-2} = \frac{e^2}{h} D \eta'^2. \tag{5.2}$$

For weak disorder we use the approximation $D \sim 4\alpha$. Furthermore, for $\omega \sim 0$ we get $\alpha = 1/4\pi |\mu|^2 \sim 1/\pi m_c^2$. Therefore, the conductivity reads in the dc limit

$$\sigma_{xx}(\omega = 0) \sim \frac{e^2}{\pi h} \frac{m_c^2 - m^2}{m_c^2} \Theta(m_c^2 - m^2). \tag{5.3}$$

For $m = 0$ the dc conductivity is independent of disorder,

$$\sigma_{xx}^c(\omega = 0) \sim e^2/\pi h. \tag{5.4}$$

This result is in agreement with a calculation for Dirac fermions with a random vector potential.[13] Thus there is a universal conductivity in $d = 2$ at the center of the band of extended states $-m_c < m < m_c$.

The Hall conductivity can be calculated using a simpler field theory which does not break the supersymmetry.[21] In units of e^2/h it was found

$$\sigma_{xy} \approx 1/2 + \mathrm{sign}(m)[1/2 - (1/\pi) \arctan(\sqrt{m_c^2/m^2 - 1}) \times \Theta(m_c^2 - m^2)]. \tag{5.5}$$

For $m = 0$ the Hall conductivity is always $e^2/2h$, whereas for $m \neq 0$ it depends on the disorder via m_c. The resistivity, calculated from the averaged conductivities (which is an additional approximation because one should actually evaluate the averaged resistivity) yields

$$\rho_{xx}^c = \sigma_{xx}^c / ({\sigma_{xx}^c}^2 + {\sigma_{xy}^c}^2) \approx 0.9 h/e^2. \tag{5.6}$$

This value agrees with an error of about $\pm 10\%$ with various experimental results.[32]

VI. DISCUSSION OF THE RESULTS

For a given strength of randomness g there are three different regimes of the behavior of the localization length, depending on the value of δ. δ is directly related to the frequency ω according to the equations (3.15), (3.16), and (4.4), respectively. The corresponding power laws $\xi_l \sim c_j \omega^{-\nu_j}$ appear with effective exponents $\nu_1 = 1/3$ and $\nu_3 = 1$ and with the asymptotic exponent $\nu_2 = 1/2$. The exponents and the corresponding coefficients c_j are shown in Table I, and the g

TABLE I. Exponents for the localization length and the corresponding coefficients.

Regime	$\delta \ll (m_c^2 - m^2)/4$	$(m_c^2 - m^2)/4 \ll \delta \ll m_c^2/4$	$\delta \gg m_c^2/4$
ν	1/2	1/3	1
Coefficient	c_2	c_1	c_3

dependence of the coefficients is plotted in Fig. 2. Ludwig *et al.* evaluated the exponent ν for two-dimensional Dirac fermions with a random vector potential with variance g_a. In that case the exponent of the asymptotic localization length depends smoothly on randomness as $\nu = 1/(1 + g_a/\pi)$.[13] It should be noticed that the inequality $\nu \geqslant 1$ of Chayes *et al.*[33] for two-dimensional random systems does not apply to the problem under consideration (cf. also the discussion in Ref. 13). It is remarkable though that there is agreement of the asymptotic result $\nu = 1/2$ between the present calculation for the random Dirac mass, the random vector potential with finite average density of states (where $g_a/\pi = 1$) (Ref. 13), and the nonlinear σ model with the topological term.[11]

The result $\nu = 1/3$ is not reliable because it appears close to the critical points $|m| = m_c$, where the bosonic degrees of freedom become critical. Moreover, the asymptotic regime with $\nu = 1/2$ is not realistic: The typical width of the fluctuations $\sqrt{g}$ in a sample is about 10% of the hopping rate. I.e., $m_c = 2e^{-\pi/g}$ is immeasurably small. Therefore, only the regime $\delta \gg m_c^2/4$ would be accessible, and only the effective exponent $\nu = 1$, together with the coefficient $\sqrt{2g/\pi}$, is of practical relevance. Consequently, only the coefficient of the localization length is affected by randomness. This is in agreement with the observations in Refs. 2–4. However, the divergency of the localization length is controversial among the various experiments. In this context it would be interesting if the crossover length, evaluated in the paper as $\sim \exp(\pi/g)$, can be observed experimentally.

The density of states is nonzero near the QHT. This result is nonperturbative because $\rho \propto \exp(-\pi/g)$, and it reflects spontaneous symmetry breaking.[23] It agrees with a Monte Carlo simulation for the network model by Lee and Wang,[34] with an exact diagonalization of a finite system,[35] with a rigorous estimation,[23] and with an exact calculation for Lorentzian disorder.[28]

The longitudinal conductivity $\sigma_{xx}(\omega)$ is nonzero between $m = -m_c$ and $m = m_c$, the transition region between two Hall plateaus. In an experiment it may not be possible to resolve the width of this band of delocalized states since m_c is too small. Therefore, the width of the transition between the Hall plateaus will always be dominated by thermal broadening. Thus a power law for the width Δ at temperature T like $\Delta \sim T^{\kappa}$ (Ref. 36) is a realistic ansatz. The conductivity at the QHT should agree with the value of σ_{xx} at $m = 0$, i.e., $\sigma_{xx}^c = e^2/h\pi$. Converting this value into the corresponding value of ρ_{xx}^c gives $\approx 0.9h/e^2$ which is in good agreement with experiments. The universal value $\sigma_{xx}^c = e^2/h\pi$ agrees with the value found for random vector potential,[13] but not with the result found from the numerical simulation of the network model, where $\sigma_{xx}^c = e^2/2h$ was found.[7] This value was also obtained for the lowest Landau level projection.[8]

It seems that our results for the 2D Dirac fermions with a random mass are in good agreement with the corresponding results for 2D Dirac fermions with a random vector potential and with recent experiments. But there is disagreement with the results of the numerical simulation of the network model. This may be related to different types of randomness (e.g., strong randomness in the magnetic field) or to different geometries. It is also possible that there is a strong renormalization of the localization length near $m = \pm m_c$, due to extended boson fields leading to an exponent $\nu \approx 7/3$. This, however, would raise the question about the origin of the experimental value $\nu \approx 1$. Moreover, the disagreement of the values of σ_{xx}^c for the network model on the one side and for Dirac fermions with $m = 0$ on the other side cannot be explained by renormalization effects.

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