

Weak versus strong universality in the two-dimensional random-bond Ising ferromagnet

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We address the issue of universality in two-dimensional disordered Ising systems, by considering long, finite-width strips of ferromagnetic Ising spins with randomly distributed couplings. We calculate the free energy and spin-spin correlation functions (from which averaged correlation lengths, ξ^{av} , are computed) by transfer-matrix methods. An ansatz for the size dependence of logarithmic corrections to ξ^{av} is proposed. Data for both random-bond and site-diluted systems show that pure system behavior (with $\nu=1$) is recovered if these corrections are incorporated, discarding the weak-universality scenario. [S0163-1829(96)52938-3]

It is well known that the Harris criterion,¹ for the relevance or irrelevance of weak disorder upon critical behavior at a phase transition, is inconclusive for the two-dimensional Ising model where the specific heat of the pure system diverges logarithmically at the critical point. A great deal of effort has been dedicated to elucidating the properties of disordered versions of this model.² Both weak and strong disorder have been considered. Early proposals implying strong deviations from pure system behavior, such as a peculiar exponential decay of critical correlations with distance and a magnetization exponent $\beta=0$ (Ref. 3) have been ruled out by extensive numerical simulations.² Though nowadays there seems to be general agreement that no such drastic changes are expected to arise from disorder in this case, two main pictures have taken hold recently, which seem to be mutually excludent. The first, typically represented by the work of Heuer⁴ and of Talapov and Shchur⁵ maintains that the critical behavior is unaffected by disorder (apart from possible logarithmic corrections, which though not explicitly considered in Refs. 4 and 5, fit in with similar overall conclusions²); this view agrees with early numerical work on magnetization moments.⁶ According to the second view,^{7,8} critical quantities such as the zero-field susceptibility and correlation length display power-law singularities, with the corresponding exponents γ and ν changing continuously with disorder; however, this variation is such that the ratio γ/ν is kept constant at the pure system's value (the so-called *weak universality* scenario⁹).

The present work aims at shedding light into this controversy, by means of strip calculations which, since the work of Nightingale^{10,11} connecting finite-size scaling^{12,13} and renormalization group ideas, have proved to be among the most accurate techniques to extract critical points and exponents for nonrandom systems in two dimensions. Extensions of this approach to random systems require an appreciation of the subtleties involved in the corresponding averaging process;^{14,15} early efforts in this direction¹⁶ have since been extended and put into a wider perspective.¹⁷⁻¹⁹ We consider a two-dimensional, square-lattice, random-bond Ising model with a binary distribution of ferromagnetic interaction strengths, each occurring with equal probability. For this specific model, the transition temperature is exactly known from duality,^{20,21} so one can be sure that numerical errors due to imprecise knowledge of the critical point (such as

what may happen, e.g., for site-diluted cases) are absent. The only sources of such errors will then be the finite strip width and those arising from the averaging process. However, the former can be controlled by finite-size scaling theory,^{11,13} while the effects of the latter are reduced by studying large enough samples (though this is a subtle point when correlation functions are concerned, as seen below). A previous strip calculation for this model¹⁹ concentrated on testing for random systems, the well-known relation²² between the critical exponent η and the correlation length on a strip at the critical temperature of the two-dimensional system, as well as on extracting the conformal anomaly c (proportional to the leading finite-width correction to the bulk free energy^{23,24}). The conclusion was that, within error bars, $\eta=1/4$ and $c=1/2$ (the pure Ising values) for wide ranges of disorder. Since $\eta=2-\gamma/\nu$, those results could not be used to test any disorder dependence of γ and ν separately. Here, instead, we resort to numerical derivatives to obtain ν .

We have used long strips of a square lattice, of width $4 \leq L \leq 12$ sites with periodic boundary conditions. In order to provide samples that are sufficiently representative of disorder, we iterated the transfer matrix¹¹ typically along 10^7 lattice spacings, meaning much longer strips than those used in Ref. 19.

At each step, the respective vertical and horizontal bonds between first-neighbor spins i and j were drawn from a probability distribution

$$P(J_{ij}) = \frac{1}{2} (\delta(J_{ij} - J_0) + \delta(J_{ij} - rJ_0)), \quad 0 \leq r \leq 1, \quad (1)$$

which ensures^{20,21} that the critical temperature $\beta_c = 1/k_B T_c$ of the corresponding two-dimensional system is given by

$$\sinh(2\beta_c J_0) \sinh(2\beta_c r J_0) = 1. \quad (2)$$

We have used three values of r in calculations: $r=0.5$, 0.25, and 0.1; the two smallest values have been chosen for the purpose of comparing with recent Monte-Carlo simulations where ν and γ are evaluated.²⁵ A wide range of disorder is thus covered.

The procedure for evaluation of the largest Lyapunov exponent Λ_L^0 for a strip of width L and length $N \gg 1$ is well known.^{16,26} The average free energy per site is then $f_L^{\text{av}}(T) = -(1/L) \Lambda_L^0$ in units of $k_B T$.

From finite-size scaling, the initial susceptibility of a strip at the critical temperature of the corresponding infinite system, $\chi_L(T_c)$ must vary as¹³

$$\chi_L(T_c) = \left. \frac{\partial^2 f_L^{\text{av}}(T_c)}{\partial h^2} \right|_{h=0} = L^{\gamma/\nu} Q(0), \quad (3)$$

where h is a uniform external field and $Q(0)$ is a constant.¹² As $f_L^{\text{av}}(T)$ is expected to have a normal distribution,^{15,26} so will χ_L . Thus the fluctuations are Gaussian, and relative errors must die down with sample size (strip length) N as $1/\sqrt{N}$. The intervals (of external field values, in this case) used in obtaining finite differences for the calculation of numerical derivatives must be strictly controlled, so as not to be an important additional source of errors. We have managed to minimize these latter effects by using δh typically of order 10^{-4} in units of J when estimating $f_L^{\text{av}}(T_c; h=0, \pm \delta h)$ for the derivative in Eq. (3).

A succession of estimates, $(\gamma/\nu)_L$, for the ratio γ/ν is then obtained from Eq. (3) as follows:

$$\left(\frac{\gamma}{\nu} \right)_L = \frac{\ln[\chi_L(T_c)/\chi_{L-1}(T_c)]}{\ln[L/(L-1)]}. \quad (4)$$

Least-squares fits for plots of $(\gamma/\nu)_L$ against $1/L^2$ (see, e.g., Ref. 19 for a discussion of suitable powers of $1/L$ for extrapolation) provide the following results: $\gamma/\nu = 1.748 \pm 0.012$, 1.749 ± 0.008 , and 1.746 ± 0.013 , respectively for $r=0.50$, 0.25 , and 0.10 ; the latter two estimates agree with 1.74 ± 0.03 , 1.73 ± 0.05 , obtained in Ref. 25.

The overall picture is thus consistent with $\gamma/\nu = \frac{7}{4}$ for all degrees of disorder. Taken together with the results of Ref. 19, and using the scaling relation $\gamma/\nu = 2 - \eta$, this confirms the view that (1) the conformal invariance relation²² $\eta = L/\pi \xi_L(T_c)$ still holds for disordered systems, provided that an averaged correlation length is used; and that (2) the appropriate correlation length to be used is that coming from the slope of semilog plots of correlation functions against distance.¹⁹

We now present results for the exponent ν . The first difference to the free energy calculation described above is that the correlation functions are expected to have a *log-normal* distribution^{14,15} rather than a normal one. Thus self-averaging is not present, and fluctuations for a given sample do not die down with increasing sample size. However, we have seen that overall averages (i.e., central estimates) from different samples do get closer to each other as the various samples' sizes increase. Accordingly, in what follows the error bars quoted arise from fluctuations among four central estimates, each obtained from a different impurity distribution. Similar procedures seem to have been followed in Monte-Carlo calculations of correlation functions in finite $(L \times L)$ systems.⁵

The direct calculation of correlation functions, $\langle \sigma_0 \sigma_R \rangle$, follows the lines of Section 1.4 of Ref. 11, with standard adaptations for an inhomogeneous system.¹⁹ For fixed distances up to $R=100$, and for strips with the same length as those used for averaging the free energy, the correlation functions are averaged over an ensemble of 10^4 – 10^5 different estimates to yield $\langle \sigma_0 \sigma_R \rangle$.

The average correlation length, ξ^{av} , is in turn defined by

TABLE I. Critical exponent ν from Eq. (8).

L	$r=0.50$	0.25	0.10
5	0.928 ± 0.004	0.993 ± 0.015	1.13 ± 0.11
6	0.962 ± 0.010	1.029 ± 0.024	1.15 ± 0.06
7	0.981 ± 0.020	1.040 ± 0.036	1.14 ± 0.06
8	0.997 ± 0.025	1.053 ± 0.040	1.15 ± 0.06
9	1.000 ± 0.030	1.052 ± 0.026	1.13 ± 0.06
10	1.009 ± 0.033	1.063 ± 0.016	1.15 ± 0.07
11	1.012 ± 0.032	1.062 ± 0.017	1.15 ± 0.08
12	1.016 ± 0.039	1.064 ± 0.032	1.14 ± 0.11
Extrap.	1.037 ± 0.016	1.083 ± 0.014	1.14 ± 0.06
Ref. 25		1.09 ± 0.01	1.23 ± 0.02

$$\overline{\langle \sigma_0 \sigma_R \rangle} \sim \exp(-R/\xi^{\text{av}}), \quad (5)$$

and is calculated from least-squares fits of straight lines to semilog plots of the average correlation function as a function of distance, in the range $10 \leq R \leq 100$.

We can then apply the usual finite-size scaling (FSS) arguments^{12,13} to obtain estimates ν_L of the exponent ν . Assuming a simple power-law divergence—i.e., ignoring, for the time being, less-divergent terms such as power-law or logarithmic corrections—of the correlation length in the form $\xi \sim t^{-\nu}$, with t being some reduced distance to the critical point, its FSS ansatz becomes

$$\xi_L^{\text{av}} = L \mathcal{F}(z), \quad (6)$$

where $z = tL^{1/\nu}$ and \mathcal{F} is a scaling function. Since ν does not appear explicitly in the expression for $\xi_L^{\text{av}}(T_c)/L$, one resorts to the temperature derivative of the correlation length, which can also be cast in a similar scaling form,

$$\mu_L \equiv \frac{d\xi_L^{\text{av}}}{dt} = L^{1+(1/\nu)} \mathcal{G}(z), \quad (7)$$

with $\mathcal{G} \equiv d\mathcal{F}/dz$. μ_L at T_c [see Eq. (2)] is calculated numerically from values of ξ_L^{av} evaluated at $T_c \pm \delta T$, with $\delta T/T_c = 10^{-3}$. For systems of sizes L and $L-1$, one obtains the estimates

$$\frac{1}{\nu_L} = \frac{\ln(\mu_L/\mu_{L-1})_{T=T_c}}{\ln(L/L-1)} - 1. \quad (8)$$

Note that this is slightly different from the usual fixed-point calculation,^{10,11} and is more convenient in the present case where the exact critical temperature is known. Our data for each pair of $(L, L-1)$ strips are shown in Table I, together with results of extrapolations against $1/L^2$ for each separate sequence corresponding to different values of r . Taken at face value, the data show a systematic trend towards values of ν slightly larger than the pure-system value of 1, though the variation is smaller than that shown in Ref. 25.

Before accepting this trend as an indication of the weak-universality scenario, we must test for corrections caused by less-divergent terms as being responsible for the observed

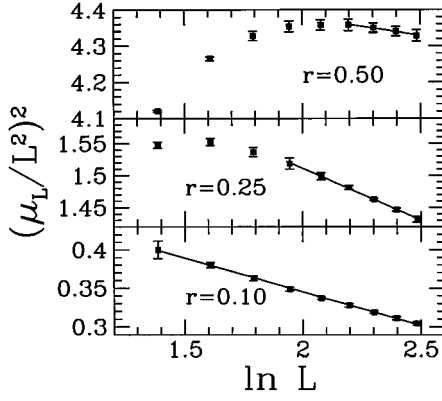


FIG. 1. Finite-size scaling plots of logarithmic corrections [Eq. (11)]. Straight lines are least-squares fits of data, respectively, for $L=9-12$ ($r=0.5$); $7-12$ ($r=0.25$); and $4-12$ ($r=0.1$). For data in this figure, $\mu_L = d\xi_L/dK$, with $K \equiv J/T$.

disorder dependence of ν . We first recall that logarithmic corrections have already been proposed for the bulk correlation length in the form³

$$\xi \sim t^{-\nu} [1 + C \ln(1/t)]^{\tilde{\nu}}, \quad (9)$$

with $\nu=1$ and $\tilde{\nu}=\frac{1}{2}$, and C is a (disorder-dependent) constant. For the same reasons as above, estimates for ν can only be consistently tested through the temperature derivative of ξ :

$$\mu \equiv \frac{d\xi}{dt} \sim t^{-(1+\nu)} [1 + C \ln(1/t)]^{\tilde{\nu}}, \quad (10)$$

plus less-divergent terms.

For finite systems, logarithmic corrections are expected to show on scales larger than a disorder-dependent characteristic length $L_C \sim \exp(1/C)$.³ A finite-size scaling ansatz for the behavior at T_c can be obtained by a suitable generalization of the standard procedures for pure power-law singularities (see, e.g., Ref.13). Assuming $\nu=1$ and $\tilde{\nu}=\frac{1}{2}$ one then has, to dominant order

$$\frac{\mu_L}{L^2} \sim (1 - b \ln L)^{1/2}, \quad (11)$$

where $b \sim 1/\ln L_C$. Figure 1 shows the results for $(\mu_L/L^2)^2$ as a function of $\ln L$, for different values of r . In each case, log-corrected behavior sets in for suitably large L , exactly in the manner predicted by theory: the data stabilize onto a straight line only for $L \geq L_C$, which decreases with increasing disorder.³

This crossover effect is similar to that found by Wang *et al.*²⁷ in their fitting of specific heat data to the double logarithmic divergence predicted by theory.³ However, the increasing broadening of the specific heat maximum with disorder in finite-sized systems has been interpreted as evidence against double-logarithmic corrections.^{7,28} Here, instead, we deal with a case of single-log corrections to a divergence much stronger than that of the specific heat. It is thus easier to separate between corrections and the dominant power-law behavior, as made evident by the consistent fits displayed in Fig. 1. Though $\xi_L(T)$ and the susceptibility

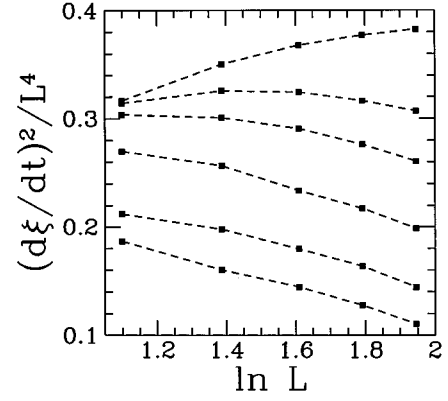


FIG. 2. Finite-size scaling plots of logarithmic corrections [Eq. (11)] for site-diluted Ising model. Top to bottom: concentration of magnetic sites = 1.0, 0.95, 0.90, 0.80, 0.70, 0.65. See Ref. 18 for details on the calculation of ξ_L .

$\chi_L(T)$ were calculated through Monte Carlo simulations in Refs. 7 and 25, no attempt seems to have been made to fit the corresponding data to a form similar to Eq. (11). It must be recalled that by examining the behavior at the exact T_c , only finite-size effects play any role in tuning the crossover; by contrast, when T_c is not known exactly, one cannot be sure whether the (thermal) crossover towards critical behavior has already occurred. As a final check of our data, we have also tried less-divergent power-law corrections, but the fittings were always much poorer than those assuming logarithmic corrections. In view of these facts, the estimates provided by Eq. (8) should then be regarded as *effective* exponents, since strong universality still holds.

We have also applied the ideas behind Eq. (11) to the two-dimensional site-diluted Ising model, using the calculational scheme proposed in Ref. 18. Results for the pure and diluted cases (for concentrations of magnetic sites in the range $p=0.65-0.95$) and $L=3-7$ are depicted in Fig. 2. It can be seen that the qualitative trend clearly changes towards a $\ln L$ dependence similar to that found in the random-bond case as soon as dilution is introduced. The small curvature in the plots must be at least partly attributed to imprecise knowledge of the exact critical line [and to the approximate nature of that calculational scheme itself, which is asymptotically exact only as $T \rightarrow 0$ (Ref. 18)]. Again, pure-system exponents with logarithmic corrections (as opposed to dilution-dependent ones^{7,8}) seem to describe the behavior of site-diluted Ising magnets in two dimensions.

In conclusion, our data independently confirm that the conformal invariance result $\xi^{av} = L/\pi\eta$ is still valid for the two-dimensional random-bond Ising model, with $\eta = \frac{1}{4}$ as in the pure case. The apparent dependence of ν with disorder was found to be due to logarithmic corrections, which become more important the farther one moves away from the pure (i.e., $r=1$) system. The weak-universality scenario, though quite appealing for the possibility of demanding new underlying concepts to be explained, does not seem to hold in the two-dimensional random-bond Ising model. A similar picture most likely holds for the site-diluted model as well. The results presented here do not necessarily imply, however, that conformal invariance or strong universality should be valid for *any* type of disorder. In the problems treated here

correlations can still freely propagate, unlike cases where, e.g., frustration is allowed; we are currently investigating these issues for spin-glass-like systems.

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