# Quantum melting of a two-dimensional vortex lattice at zero temperature

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(Received 29 July 1996)

We consider the quantum melting of a two-dimensional flux lattice at temperature $T=0$ in the ''superclean limit.'' In this regime, we find that vortex motion is dominated by the Magnus force. A Lindemann criterion predicts melting when $n_v/n_p \geq \beta$, where $n_v$ and $n_p$ are the areal number densities of vortex pancakes and Cooper pairs, and $\beta \approx 0.1$. A second criterion is derived by using Wigner-crystal and Laughlin wave functions for the solid and liquid phases respectively, and setting the two energies equal. This gives a melting value similar to the Lindemann result. We discuss the numerical value of the $T=0$ melting field for thin layers of a low-$T_c$ superconductor, such as $a$-MoGe, and single layers of high-$T_c$ materials. [S0163-1829(96)50642-9]

## I. INTRODUCTION

Vortices in the layered high-$T_c$ materials have remarkably strong thermal fluctuations, which have been extensively studied.[1] At sufficiently low temperatures, vortex lines are also expected to be subject to *quantum* fluctuations. Quantum effects should manifest themselves in the zero-point motion of vortex lines. If these are large enough, the flux lattice can melt even at temperature $T=0$. Indeed, many experiments suggest that vortex lattice melting, both in high-$T_c$ materials[2–4] and in low-$T_c$ films and multilayers,[5] is strongly influenced by quantum fluctuations.

Several authors have already considered possible quantum melting in high-$T_c$ superconductors. Blatter and Ivlev[6] have examined the influence of quantum fluctuations at finite temperatures. They estimated the shift in the melting curve using a Lindemann criterion, assuming overdamped dynamics. Chudnovsky[7] has studied a hypothetical two-dimensional (2D) quantum vortex liquid state at temperature $T=0$. Onogi and Doniach[8] computed the $T=0$ melting field for a 2D superconductor using quantum Monte Carlo (QMC) techniques without dissipative quantum tunneling. By taking into account a fictitious magnetic field arising from the Magnus force on the vortex pancakes,[9] they also found strong numerical evidence for fractional quantum Hall (FQHE) states in the vortex liquid. Such FQHE states had been predicted by several authors,[10,11] principally in the context of Josephson-junction arrays.

In this paper, we describe two simple models for estimating the conditions for quantum melting of a 2D vortex lattice at $T=0$, explicitly including the fictitious magnetic field. The first estimate is a simple Lindemann criterion. The second involves a simple comparison of internal energies in the crystalline and liquid phases.

## II. FICTITIOUS MAGNETIC FIELD AND LINDEMANN MELTING CRITERION

In our model, the vortex pancakes experience two types of forces: those due to other pancakes and the Magnus force arising from the Cooper pair density. We neglect dissipative forces from the ''viscous'' normal electron background, as may be acceptable in the ''superclean limit.''[12] Of the two remaining forces, the Magnus force usually dominates (see below). The resulting Lindemann melting criterion proves independent of the vortex mass. By contrast, in the opposite limit where the intervortex forces dominate, the melting field depends sensitively on the vortex mass.[8]

The Magnus force[9] is an effective interaction between charges and vortices in relative motion. In a superconducting film of thickness $d$, this force, per unit volume, acting on the vortices, in their rest frame, is $-\mathbf{J}\times B\hat{\mathbf{z}}/c$, where $\mathbf{J}=-2en_p\mathbf{v}/d$ is the pair current density, $n_p$ is the effective areal number density of Cooper pairs, and $\mathbf{v}$ is the vortex velocity relative to the pairs. The force acting on a single two-dimensional (''pancake'') vortex is then

$$\mathbf{F}_p = q_v h \mathbf{v}\times\hat{\mathbf{z}} n_p \equiv \frac{2e}{c}\mathbf{v}\times\hat{\mathbf{z}}B_{\rm eff}. \tag{1}$$

Here $q_v=\pm 1$ is the effective charge of the pancake vortex, $h$ is Planck's constant, $B_{\rm eff}=\Phi_0 n_p$ is the fictitious field, $\Phi_0=hc/2e$ is the flux quantum, and the film is assumed perpendicular to the $z$ axis.

We now wish to show that the intervortex force is typically small compared to the Magnus force. If the London penetration depth is $\lambda$, the direct interaction potential between two pancakes separated by $r$ is

$$\Pi(r) = 2\epsilon_0 d \mathrm{K}_0\left(\frac{r}{\lambda_\perp}\right), \tag{2}$$

where $\lambda_\perp=\lambda^2/d$ is the transverse penetration depth, $\epsilon_0=\Phi_0^2/(16\pi^2\lambda^2)$, and $K_0(x)$ is the modified Bessel function of zeroth order. To estimate the effects of the vortex-vortex interaction, we assume that the vortices are ordered into a triangular lattice, and calculate the change in potential energy per vortex, $\Delta U_{\rm harm}$, due to harmonic vibrations about this lattice. After some algebra, this extra energy is found to be $\Delta U_{\rm harm}=\Sigma_{\mathbf{l}}[\chi(l)/4]\langle|\mathbf{u}_0-\mathbf{u}_{\mathbf{l}}|^2\rangle$. Here $\mathbf{l}$ is a lattice vector of the triangular lattice, $\mathbf{u}_{\mathbf{l}}$ is the displacement of the $\mathbf{l}$th vortex from equilibrium, and $\chi(l)=(\epsilon_0 d/\lambda_\perp^2)[(\lambda_\perp/l)\mathrm{K}_0'(l/\lambda_\perp)+\mathrm{K}_0''(l/\lambda_\perp)]$, where $l=|\mathbf{l}|$, and the primes denote differentiation.

We estimate this energy as follows. First, since the vortex-vortex interaction is assumed small, we neglect

 

$\langle \mathbf{u_0}\cdot\mathbf{u_l}\rangle$. Secondly, in the weak-screening regime where the nearest-neighbor intervortex distance $a_0 \ll \lambda_\perp$, the summation may reasonably be replaced by an integral. With these approximations, and using several identities for derivatives of Bessel functions, we finally obtain

$$\Delta U_{\rm harm} \approx (\epsilon_0 d)\times \pi n_v \langle |\mathbf{u}_0|^2\rangle, \tag{3}$$

where $n_v = 2/(\sqrt{3}a_0^2)$ is the areal vortex density.

Similarly, for a pancake of mass $m_v$ moving in a fictitious field $B_{\rm eff}$, the zero-point energy per pancake $\Delta U_{\rm mag}$ for a pancake in the lowest Landau level is

$$\Delta U_{\rm mag} = \frac{1}{2}\hbar\omega_c^{\rm eff}, \tag{4}$$

where $\omega_c^{\rm eff} = 2eB_{\rm eff}/(m_v c)$.

To show that the zero point motion is usually dominated by $B_{\rm eff}$, we demonstrate that $\hbar\omega_c \ll \hbar\omega_c^{\rm eff}$ where $\omega_c$ is the frequency for zero-point motion of the harmonic lattice in the absence of $B_{\rm eff}$. Now $\omega_c = \sqrt{k/m_v}$, where $k$ is the effective spring constant of the harmonic lattice. It follows from Eq. (3) that $k = 2\epsilon_0 d\pi n_v$.

To compare $\omega_c$ and $\omega_c^{\rm eff}$ we use the London estimate for the penetration depth $\lambda^2(T) = (m_p c^2)/(4\pi q^2 n_p^{\rm 3D}) = (m_p c^2 d)/(4\pi q^2 n_p)$, where $n_p^{\rm 3D}$ is the pair density per unit volume, $m_p$ is the pair mass, and $q$ the pair charge. Then a little algebra reveals that $\omega_c \ll \omega_c^{\rm eff}$ provided that

$$\frac{m_v}{m_p} \ll \frac{2n_p}{n_v}, \tag{5}$$

where $m_p$ is the Cooper pair mass . As will be shown below, $n_v/n_p \approx 0.1$ at the melting point. Then inequality (5) is satisfied so long as $m_v/m_p \ll 20$. Now in $BiSr_2Ca_2Cu_2O_{8+x}$, the mass of a single pancake vortex, assuming a thickness $d \approx 10$ Å (appropriate for a single layer of high-$T_c$ material) has been estimated as one electron mass[8]. Thus, in this regime, the inequality is satisfied and $\Delta U_{\rm harm} \ll \Delta U_{\rm mag}$ as required. Hence, in calculating melting behavior for vortices of this mass, we apparently need consider only $\Delta U_{\rm mag}$. Our results based on including only $\Delta U_{mag}$ do indeed give $n_v/n_p \approx 0.1$, thereby confirming the self-consistency of our approach.

We now obtain a simple Lindemann melting criterion, assuming that the dominant contribution to zero-point vortex motion arises from $B_{\rm eff}$. Although $\omega_c^{\rm eff}$ clearly depends on $m_v$, the zero-point displacement does not. We calculate this displacement assuming the symmetric gauge for the fictitious vector potential, $\mathbf{A}_{\rm eff} = \frac{1}{2}\mathbf{B}_{\rm eff}\times\mathbf{r}$. Then in the lowest Landau level, one finds

$$\langle |\mathbf{u}_0|^2\rangle \equiv \langle (u_x^2 + u_y^2)\rangle = \frac{\Phi_0}{\pi B_{\rm eff}} = \frac{1}{\pi n_p}, \tag{6}$$

*independent* of vortex mass.

According to the Lindemann criterion, melting occurs when the zero-point amplitude is a certain fraction, say $\alpha_L$, of $a_0$. In most conventional materials, $\alpha_L \approx 0.1-0.2$. Since $a_0 = (2\Phi_0/\sqrt{3}B)^{1/2}$, the Lindemann criterion becomes

$$\frac{n_v}{n_p} = \frac{2\pi}{\sqrt{3}}\alpha_L^2 \approx 0.07, \tag{7}$$

using the estimate $\alpha_L^2 \approx 0.02$. Thus, the Lindemann picture predicts quantum melting at $T=0$ at a vortex density of around 7% of the effective areal density of Cooper pairs.

## III. LAUGHLIN LIQUID VERSUS WIGNER CRYSTAL

Next, we describe an alternative way of estimating the melting temperature in a 2D lattice. We treat the pancake vortices as bosons, moving in the effective field $B_{\rm eff}$. To describe the bosons, we use a Wigner crystal (WC) wave function in the solid phase, and a properly symmetrized Laughlin wave function[13] in the liquid. The melting point is determined by requiring the energies $E_{\rm WC}$ and $E_{\rm LL}$ of the solid and liquid states to be equal. A related approach has been used to treat melting of the 2D electron lattice in a magnetic field.[14,15]

The WC wave function is

$$\Psi_{\rm WC} = A\mathcal{S}\left(\prod_{\mathbf{l}} \psi(\mathbf{r_l} - \mathbf{l})\right). \tag{8}$$

Here $\psi(\mathbf{r})$ denotes the zero-momentum single-particle wave function of the lowest Landau level, $\mathcal{S}$ is the symmetrization operator, and $A$ is a normalization constant. We wish to calculate the averaged vortex-vortex interaction energy in this state, i.e. $E_{\rm WC}/(2\epsilon_0 dS) = \langle\Psi|\Sigma_{\mathbf{l_1}}\Sigma_{\mathbf{l_2}\neq\mathbf{l_1}} K_0(|\mathbf{r_{l_1}} - \mathbf{r_{l_2}}|/\lambda_\perp)|\Psi\rangle/(2S)$, where $S$ is the sample surface area.

We simplify the calculation by several approximations. First, since $a_0^2 \gg \langle|\mathbf{u}_0|^2\rangle$, the wave function symmetrization is quantitatively unimportant for calculating $E_{\rm WC}$. Indeed, for large argument, the single-particle wave function $\psi(\mathbf{r})$ decays exponentially, and the overlap integral between $\psi(\mathbf{r}-\mathbf{l}_1)$ and $\psi(\mathbf{r}-\mathbf{l}_2)$ is almost zero, unless $\mathbf{l}_1 = \mathbf{l}_2$. In view of this degree of localization, $E_{\rm WC}$ can be expanded in powers of the small ratio $\langle|\mathbf{u}_0|^2\rangle/\lambda_\perp^2$ keeping only the first two terms. The result is $E_{\rm WC}/(2\epsilon_0 dS) = (n_v/2)\Sigma_{\mathbf{l}\neq\mathbf{0}} K_0(l/\lambda_\perp) + n_v \Delta U_{\rm harm}/2\epsilon_0 d$, where $\Delta U_{\rm harm}$ is given by Eq. (3). The fluctuations $\langle|\mathbf{u}_0|^2\rangle$ appearing in Eq. (3) are the sum of two parts: one due to $B_{\rm eff}$ and the other to the intervortex potential. Of these, the former is typically much larger, as noted above, and has already been evaluated in Eq. (6). We substitute this value into Eq. (3) and hence into the expression for $E_{\rm WC}$. In the limit $a_0 \ll \lambda_\perp$, one can evaluate this sum numerically. The result is very well fitted numerically by the form $\Sigma_{\mathbf{l}\neq\mathbf{0}} K_0(l/\lambda_\perp) \approx n_v \int d^2\mathbf{r} K_0(r/\lambda_\perp) - 0.500\ln(\lambda_\perp^2 n_v) - 1.437$. Collecting all these results, we finally obtain

$$\frac{E_{\rm WC}}{2\epsilon_0 dS} = \frac{n_v^2}{2}\int d^2\mathbf{r} K_0\left(\frac{r}{\lambda_\perp}\right) - \frac{1}{4}n_v \ln(\lambda_\perp^2 n_v) \tag{9}$$

$$-0.719 n_v + \frac{n_v^2}{2n_p}.$$

For the liquid phase, the wave function symmetry matters since the pancakes are delocalized. We use an (unnormalized) trial wave function of the Laughlin form:[13]

$$\Psi_{\mathrm{LL},m}=\prod_{j<k}(z_j-z_k)^m\exp\left(-\frac{1}{4}\sum_l |z_l|^2\right). \tag{10}$$

Here $z_j=x_j+iy_j$ is the position coordinate of the $j$th pancake, and all lengths are expressed in units of the ''magnetic length'' $\ell_0\equiv[\Phi_0/(2\pi B_{\mathrm{eff}})]^{1/2}$. Since the vortex pancakes are bosons, $m$ must be an even integer. In the Laughlin theory of the fractional quantum Hall effect, $1/m$ is the filling fraction of the first Landau level.

Laughlin's prescription for obtaining the minimizing value of $m$ is readily translated to the present problem, in which the role of charges and magnetic field are reversed. The generalized prescription is that the minimizing $m$ occurs when the number density $n_p$ of vortices of the fictitious magnetic field equals $m$ times the number density $n_v$ of fictitious charges, i.e., $m=n_p/n_v$.

We next calculate the internal energy of the Laughlin liquid at various $m$'s. With a change of scale, the vortex-vortex interaction energy of the liquid becomes $E_{\mathrm{LL}}/(2\epsilon_0 dS)=(n_v/2\pi)\int d^2\mathbf{x}\mathrm{K}_0(x/\lambda_\perp\sqrt{\pi n_v})g(x)$, where $g(x)$ is the dimensionless density-density Laughlin-liquid correlation function (normalized to unity at large $x$), and the dimensionless coordinate $\mathbf{x}=\mathbf{r}\sqrt{\pi n_v}$. Since $g(x)$ differs significantly from unity mainly where $x<1$, the interaction energy is conveniently decomposed as

$$\frac{E_{\mathrm{LL}}}{2\epsilon_0 dS}=\frac{n_v^2}{2}\int d^2\mathbf{r}K_0\left(\frac{r}{\lambda_\perp}\right) \tag{11}$$

$$+\frac{n_v}{2\pi}\int d^2\mathbf{x}K_0\left(\frac{x}{\lambda_\perp\sqrt{\pi n_v}}\right)[g(x)-1]$$

$$\approx\frac{n_v^2}{2}\int d^2\mathbf{r}K_0\left(\frac{r}{\lambda_\perp}\right)$$

$$-n_v\int_0^\infty xdx\left[\ln\left(\frac{x}{2\lambda_\perp\sqrt{\pi n_v}}\right)+\gamma\right][g(x)-1],$$

where $\gamma\approx0.577\ldots$ is Euler's constant and we have used the small-$x$ approximation for $K_0(\mathrm{x})$.

As noted by Laughlin, the correlation function $g(\mathrm{r})$ for the Laughlin-liquid state is just that of the 2D one-component classical plasma (OCP), in which the particles interact logarithmically. The last term on the right is, to within a factor, just the internal energy of the OCP. We can therefore use standard numerical results for the OCP, as obtained by Monte Carlo methods by Caillol *et al.*[16] Using the analytical fit of these authors to their own numerical results for the integral $\int_0^\infty xdx\ln x[g(x)-1]$, we find

$$-\int_0^\infty xdx\left[\ln\left(\frac{x}{2\lambda_\perp\sqrt{\pi n_v}}\right)+\gamma\right][g(x)-1] \tag{12}$$

$$=-\int_0^\infty xdx\ln x[g(x)-1]-\frac{1}{4}\ln(4\pi\lambda_\perp^2 n_v)+\frac{\gamma}{2}$$

$$\approx-0.3755+0.440\left(\frac{n_v}{2n_p}\right)^{0.74}-\frac{1}{4}\ln(\lambda_\perp^2 n_v)$$

$$-\frac{1}{4}\ln(4\pi)+\frac{\gamma}{2}.$$

Hence, the energy of the Laughlin liquid can be written as

$$\frac{E_{\mathrm{LL}}}{2\epsilon_0 dS}=\frac{n_v^2}{2}\int d^2\mathbf{r}K_0\left(\frac{r}{\lambda_\perp}\right)-\frac{1}{4}n_v\ln(\lambda_\perp^2 n_v) \tag{13}$$

$$-0.720n_v+0.440n_v\left(\frac{n_v}{2n_p}\right)^{0.74}.$$

This differs from the Wigner crystal energy basically only in the last term on the right-hand side. The first three terms on the right-hand side of Eqs. (9) and (13) represent the energy of the static Wigner crystal, while the last term in each equation represents the deviation of the internal energy from those values in the solid and liquid states, which have different structure factors.

Finally, the zero-temperature melting transition is defined by the equation $E_{\mathrm{WC}}=E_{\mathrm{LL}}$, or $n_v/2n_p\approx0.440(n_v/2n_p)^{0.74}$, or equivalently

$$\frac{n_v}{n_p}\approx0.09. \tag{14}$$

This result agrees remarkably well with the Lindemann criterion.

## IV. DISCUSSION

We now evaluate these predictions for two materials, using a simplified approximation for $n_p$. As noted by Ao and Thouless,[9] $n_p$ is not simply the areal density of Cooper pairs, but that of *superconducting* Cooper pairs—that is, those not pinned by lattice disorder. Since it is unclear how to evaluate this quantity, we simply use the London equation to estimate $n_p$ at zero field. To get $n_p(B)$, we use the Ginzburg-Landau approximation $\lambda(B,0)=\lambda(0,0)/[1-B/B_{c2}]^{1/2}$, where $B_{c2}$ is the $T=0$ upper critical field, and $\lambda(B,T)$ is the penetration depth. The melting condition, from either the Lindemann criterion or equating solid and liquid energies, is $n_v/n_p=\beta$, where $\beta\approx0.1$. Substituting the above expressions into this melting condition, we obtain for the melting field $B_m$

$$\frac{B_m}{B_{c2}}=\frac{B_0}{B_0+B_{c2}}, \tag{15}$$

where $B_0=\beta m_p c^2 d\Phi_0/[4\pi\lambda^2(0,0)q^2]$.

First, we apply this result to an amorphous MoGe film, an extensively studied 2D extreme type-II superconductor. An amorphous $Mo_{0.43}Ge_{0.57}$ film of thickness 30 Å has $\lambda(0,0) \approx 8000$ Å and $B_{c2} \approx 10^4$ G.[17] Taking $B \approx H$ (a good approximation in the extreme type-II limit), and using $\beta = 0.1$, we find $B_0 \approx 7 \times 10^4$ G, and therefore, $B_m / B_{c2} \approx 0.8 - 0.9$. This is consistent with the observations of Ephron *et al.*,[17] who find a superconducting-insulating transition at around 10 kG, quite close to the estimated $B_{c2}$. The transition in Ref. 17 is undoubtedly not uncomplicated quantum melting, since it occurs in highly disordered samples. Indeed, it is undoubtedly better described as a continuous phase transition from a vortex glass to a Cooper pair glass.[18] Nonetheless, it is gratifying that our predicted field, estimated for a clean sample, falls rather close to the observed transition.

Of equal interest is possible quantum melting in high-$T_c$ superconductors. Since our model is strictly 2D, we consider only a single layer of a high-$T_c$ material. The result may conceivably be extrapolated to the most anisotropic $CuO_2$-based materials, such as $BiSr_2Ca_2Cu_2O_{8+x}$. Assuming $d = 10$ Å and $\lambda(0,0) = 1400$ Å, we obtain $B_0 \approx 1.5 \times 10^6$ G. Estimating $B_{c2} = 3 \times 10^6$ G, we find $B_m \approx 10^6$ G. Since $T_c$ is smaller and $\lambda(0,0)$ is larger in an underdoped sample, we may expect $B_m$ also to decrease in such materials.

Finally, we comment on the connection between our results and the calculations of Ref. 8. While these authors find FQHE-like commensuration effects in the flux liquid state, their observed melting scales with $m_v$ as if there were no influence of $B_{\rm eff}$ on $B_m$. (They consider only mass ratios $m_v/m_p \geq 10$.) Our simplified analytical calculations suggest that $B_{\rm eff}$ may dominate the melting behavior for sufficiently light pancake masses ($m_v \ll 40 m_e$). Presumably, this influence of $B_{\rm eff}$ would also show up in QMC studies at sufficiently small $m_v$.

To conclude, we have calculated the quantum melting criterion for a 2D vortex lattice at $T = 0$, by comparing the internal energies of the vortex solid and vortex fluid in a hypothetical superclean limit. We find that, at sufficiently low vortex masses, melting behavior seems to be dominated by a fictitious magnetic field acting on the vortices and produced by the Cooper pair density. The calculated melting field is close to the superconducting-insulating transition observed in certain thin films of amorphous MoGe, and may be within reach of pulsed magnetic fields in some underdoped $CuO_2$-based high-$T_c$ materials.

## ACKNOWLEDGMENTS

One of us (D.S.) gratefully acknowledges conversations with Professor S. Doniach, as well as the hospitality of the Stanford Department of Applied Physics, where this calculation was initiated. Work supported by NSF Grant No. DMR94-02131 and by the Department of Energy through the Midwest Superconductivity Consortium at Purdue University, Grant No. DE-FG90-02ER-45427.

[1] G. Blatter *et al.*, Rev. Mod. Phys. **66**, 1125 (1994), and references therein.
[2] R. de Andrade and O. F. de Lima, Phys. Rev. B **51**, 9383 (1995).
[3] A. Schilling *et al.*, Physica B **194/196**, 1555 (1994).
[4] P. Y. Fukuzumi *et al.*, Phys. Rev. Lett. **76**, 684 (1996).
[5] C. Attanasio *et al.*, Phys. Rev. B **53**, 1087 (1996).
[6] G. Blatter and B. Ivlev, Phys. Rev. Lett. **70**, 2621 (1993).
[7] E. M. Chudnovsky, Phys. Rev. B **51**, 15 351 (1995).
[8] H. Onogi and S. Doniach, Solid State Commun. **98**, 1 (1996).
[9] P. Ao and D. J. Thouless, Phys. Rev. Lett. **70**, 2158 (1993).
[10] M. Y. Choi, Phys. Rev. B **50**, 10 088 (1994).
[11] A. Stern, Phys. Rev. B **50**, 10 092 (1994)
[12] G. Blatter *et al.*, Rev. Mod. Phys. **66**, 1153 (1994).
[13] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[14] K. Maki and X. Zotos, Phys. Rev. B **28**, 4349 (1983).
[15] P. K. Lam and S. M. Girvin, Phys. Rev. B **30**, 473 (1984).
[16] J. M. Caillol *et al.*, J. Stat. Phys. **28**, 325 (1982).
[17] D. Ephron *et al.*, Phys. Rev. Lett. **76**, 1529 (1996).
[18] M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).