Exact two-spinon dynamical correlation function of the one-dimensional Heisenberg model

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We derive the exact contribution of two spinons to the dynamical correlation function of the spin-1/2 Heisenberg model. For this, we use the isotropic limits of the exact form factors that have been recently computed through the quantum affine symmetry of the anisotropic Heisenberg model XXZ. [S0163-1829(96)51142-2]

I. INTRODUCTION

Ever since Niemeijer derived an exact expression for the dynamical correlation function (DCF) at any temperature of the spin-1/2 XY model,[1] there has been a considerable amount of work in trying to extend his result to the more physically interesting case of the isotropic spin-1/2 Heisenberg model (i.e., XXX model). For more details on the definitions of the spin chain models see Refs. 2 and 3. However, so far only approximate, but very accurate results have been computed. For a comprehensive historic review, the importance of the DCF, an account of the existing results, and a list of references on this subject, we recommend Refs. 4–9. In particular, in Ref. 5 an ansatz for the DCF at zero temperature was proposed based on Niemeijer's result, and other approximate numerical and analytical results, but it has never been established whether this ansatz includes contributions from just two spinons or more. The spinon picture has been rigorously studied in the case of the Heisenberg model in Ref. 10, and is of great theoretical and experimental interest. Let us just mention that the main stumbling block in trying to compute the exact DCF for the Heisenberg model is due to the absence of exact results for the form factors. Unfortunately, so far the powerful method of the Bethe ansatz has not been able to provide them. The form factors are well understood now just in two-dimensional quantum field theories with familiar relativistic dispersion relations,[11] but not yet in lattice models. However, an approach based on the concept of exact resolution of dynamics through just infinite-dimensional symmetries, and which is widely used in the context of string theory and conformal field theory, has been recently upgraded to the "massive" XXZ model in Refs. 12 and 13. It provides almost all exact physical quantities (static correlation functions and form factors). Its only shortcomings are that it is based on a relatively complicated symmetry, which is the quantum affine algebra $U_q(\widehat{sl(2)})$, and that the physical quantities it leads to are somewhat complicated to deal with. To be more precise, they typically have a contour multi-integral form. Our main point in this paper is that all these latter problems disappear in the particular case of the two-spinon form factors of the Heisenberg model. Therefore we use them to compute the more interesting quantity of the exact two-spinon DCF at zero temperature.

Our paper is organized as follows: first we briefly review the results related to the diagonalization of the anisotropic XXZ Heisenberg model following Ref. 13. Then we define the two-spinon DCF in the case of XXZ in terms of the form factors of this model. Finally, we take the isotropic limit. As mentioned earlier, the crucial point is that considerable simplifications take place in this case due to the isotropy of the Heisenberg model, and thus allow us to derive a simple exact formula for DCF of a two-spinon. We hope that the exact results contained in this paper will shed some new light on the spinon picture from both theoretical and experimental perspectives.

II. DIAGONALIZATION OF THE ANISOTROPIC HEISENBERG MODEL

The Hamiltonian of the anisotropic (XXZ) Heisenberg model is defined by

$$H_{XXZ}=-\frac{1}{2}\sum_{n=-\infty}^{\infty}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z), \quad (1)$$

where $\Delta=(q+q^{-1})/2$ is the anisotropy parameter. Here $\sigma_n^{x,y,z}$ are the usual Pauli matrices acting at the nth position of the formal infinite tensor product

$$W=\cdots V\otimes V\otimes V\cdots, \quad (2)$$

where V is the two-dimensional representation of $U_q(sl(2))$ quantum group. We consider the model in the antiferromagnetic regime $\Delta<-1$, i.e., $-1<q<0$. Later we take the isotropic limit $q\to-1$ in a special manner. The main point of Refs. 12 and 13 is that the action of H_{XXZ} on W is not well defined due to the appearance of divergences. However, this model is symmetric under the quantum group

$U_q(\widehat{sl(2)})$, and therefore the eigenspace is identified with the following level 0 $U_q(\widehat{sl(2)})$ representation:

$$\mathcal{F}=\sum_{i,j} V(\Lambda_i)\otimes V(\Lambda_j)^*, \tag{3}$$

where Λ_i and $V(\Lambda_i);i=0,1$ are level 1 $U_q(\widehat{sl(2)})$-highest weights and highest weight representations, respectively. Roughly speaking, $V(\Lambda_i)$ is identified with the subspace of the formal semi-infinite space

$$X=\cdots V\otimes V\otimes V, \tag{4}$$

consisting of all linear combinations of spin configurations with fixed boundary conditions such that the eigenvalues of σ_n^z are $(-1)^{i+n}$ in the limit $n\to-\infty$. Similarly, its dual representation $V(\Lambda_i)^*$ is associated with the right semi-infinite tensor product of the V's. The particle picture of this Hamiltonian is given in terms of vertex operators which act as intertwiners of certain $U_q(\widehat{sl(2)})$ representations, and which create the set of eigenstates (spinons) $\{|\xi_1,\ldots,\xi_n\rangle_{\epsilon_1,\ldots,\epsilon_n;i},n\geqslant 0\}$. Here, $i=0,1$ fixes the boundary conditions, ξ_j denotes a spectral parameter living on the unit circle, and $\epsilon_j=\pm 1$ is twice the z component of the spin of a spinon. The actions on $\mathcal{F}$ of H_{XXZ} and the translation operator T, which shifts the spin chain by one site, are given by

$$\begin{gathered}T|\xi_1,\ldots,\xi_n\rangle_i=\prod_{i=1}^{n}\tau(\xi_i)^{-1}|\xi_1,\ldots,\xi_n\rangle_{1-i},\\ T|0\rangle_i=|0\rangle_{1-i}, \\ H_{XXZ}|\xi_1,\ldots,\xi_n\rangle_i=\sum_{i=1}^{n}e(\xi_i)|\xi_1,\ldots,\xi_n\rangle_i,\end{gathered} \tag{5}$$

where

$$\begin{gathered}\tau(\xi)=\xi^{-1}\frac{\theta_{q^4}(q\xi^2)}{\theta_{q^4}(q\xi^{-2})}=e^{-ip(\alpha)},\\ p(\alpha)=\mathrm{am}\left(\frac{2K}{\pi}\alpha\right)+\pi/2,\\ e(\alpha)=\frac{1-q^2}{2q}\xi\frac{d}{d\xi}\ln\tau(\xi)=\frac{2K}{\pi}\sinh\left(\frac{\pi K'}{K}\right)\mathrm{dn}\left(\frac{2K}{\pi}\alpha\right).\end{gathered} \tag{6}$$

Here, $e(\alpha)$ and $p(\alpha)$ are the energy and the momentum of the spinon, respectively, $\mathrm{am}(x)$ and $\mathrm{dn}(x)$ are the usual elliptic amplitude and delta functions, with nome $-q$ and complete elliptic integrals K and K', and

$$\begin{gathered}q=-\exp(-\pi K'/K),\\ \xi=ie^{i\alpha},\\ \theta_x(y)=(x;x)_\infty(y;x)_\infty(xy^{-1};x)_\infty,\\ (y;x)_\infty=\prod_{n=0}^{\infty}(1-yx^n).\end{gathered} \tag{7}$$

This means, $\sigma^{x,y,z}(t,n)$ at time t and position n are related to $\sigma^{x,y,z}(0,0)$ at time 0 and position 0 through

$$\sigma^{x,y,z}(t,n)=\exp(itH_{XXZ})T^{-n}\sigma^{x,y,z}(0,0)T^n\exp(-itH_{XXZ}). \tag{8}$$

The completeness relation reads[13]

$$\mathbb{I}=\sum_{i=0,1}\sum_{n\geqslant 0}\sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1}\frac{1}{n!}\oint\frac{d\xi_1}{2\pi i\xi_1}\cdots\frac{d\xi_n}{2\pi i\xi_n}|\xi_n,\ldots,\xi_1\rangle_{\epsilon_n,\ldots,\epsilon_1;i}\;{}_{i;\epsilon_1,\ldots,\epsilon_n}\langle\xi_1,\ldots,\xi_n|. \tag{9}$$

III. TWO-SPINON DYNAMICAL CORRELATION FUNCTION OF THE HEISENBERG MODEL

First, we will define the dynamical correlation function we are considering in the case of the anisotropic Heisenberg model, where the particle picture is well understood and the form factors are known exactly.[13] However, let us note that the expressions of these form factors are very complicated to lead to a closed formula for the DCF in the anisotropic case. But, in the isotropic limit, i.e., $q\to-1$ one of them simplifies substantially, and using the fact that all the nonvanishing components of the DCF are equal, we find the same closed formula for all of them in this limit.

Let us recall the definition of one of the components of the DCF in the case of the XXZ model. Up to an overall normalization factor it is given by

$$S^{i,+-}(w,k)=\int_{-\infty}^{\infty}dt\sum_{n\in\mathbb{Z}}e^{i(wt+kn)}{}_i\langle 0|\sigma^+(t,n)\sigma^-(0,0)|0\rangle_i, \tag{10}$$

here w and k are the energy and momentum transfer, respectively, and i corresponds to the boundary condition. Later we will find that the DCF is in fact independent of i. Using the completeness relation, the two-spinon contribution is given by

$$\begin{aligned}S_2^{i,+-}(w,k)=\pi\sum_{n\in\mathbb{Z}}\sum_{\epsilon_1,\epsilon_2}&\oint\frac{d\xi_1}{2\pi i\xi_1}\oint\frac{d\xi_2}{2\pi i\xi_2}\exp\{in[k+p(\xi_1)+p(\xi_2)]\}\delta[w-e(\xi_1)-e(\xi_2)]\\ &\times{}_{i+n}\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{\epsilon_2,\epsilon_1;i+n}\;{}_{i;\epsilon_1,\epsilon_2}\langle\xi_1,\xi_2|\sigma^-(0,0)|0\rangle_i.\end{aligned} \tag{11}$$

This can be rewritten as

$$S_2^{i,+-}(w,k)=\pi\sum_{\epsilon_1,\epsilon_2}\oint\frac{d\xi_1}{2\pi i\xi_1}\oint\frac{d\xi_2}{2\pi i\xi_2}\sum_{n\in\mathbb{Z}}\exp\{2in[k+p(\xi_1)+p(\xi_2)]\}\delta[w-e(\xi_1)-e(\xi_2)]\big({}_i\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{\epsilon_2,\epsilon_1;i}\ {}_{i;\epsilon_1,\epsilon_2}\langle\xi_1,\xi_2|\sigma^-(0,0)|0\rangle_i + \exp\{i[k+p(\xi_1)+p(\xi_2)]\}{}_{1-i}\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{\epsilon_2,\epsilon_1;1-i}\ {}_{i;\epsilon_1,\epsilon_2}\langle\xi_1,\xi_2|\sigma^-(0,0)|0\rangle_i\big). \tag{12}$$

The nonvanishing form factors have been computed in Ref. 13, and satisfy the following relations:

$$\begin{aligned}{}_i\langle 0|\sigma^-(0,0)|\xi_n,\dots,\xi_1\rangle_{\epsilon_n,\dots,\epsilon_1;i}&={}_{1-i}\langle 0|\sigma^+(0,0)|\xi_n,\dots,\xi_1\rangle_{-\epsilon_n,\dots,-\epsilon_1;1-i}\\&={}_i\langle 0|\sigma^+(0,0)|-q\xi_1^{-1},\dots,-q\xi_n^{-1}\rangle_{-\epsilon_1,\dots,-\epsilon_n;i},\end{aligned} \tag{13}$$

$${}_{i;\epsilon_1,\dots,\epsilon_n}\langle\xi_1,\dots,\xi_n|\sigma^-(0,0)|0\rangle_i={}_i\langle 0|\sigma^-(0,0)|-q\xi_1,\dots,-q\xi_n\rangle_{-\epsilon_1,\dots,\epsilon_n;i}.$$

Now the isotropic limit $q\to -1$ is performed by first making the following redefinitions:

$$\xi=ie^{\epsilon\beta/i\pi}, \tag{14}$$

$$q=-e^{-\epsilon},\quad \epsilon\to 0^+,$$

with β, the appropriate spectral parameter for the Heisenberg model, being real.

Then, one finds the following exact isotropic limits.[13] (We do not find the same overall coefficient in this limit as that of Ref. 13. We are grateful to Karbach and Müller for their help in simplifying the overall factor in the first relation. See Ref. 14 for further simplifications.)

$$|{}_i\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{--;i}|^2\frac{d\xi_1}{2\pi i\xi_1}\frac{d\xi_2}{2\pi i\xi_2}\to\frac{A(\beta_1-\beta_2)}{8\pi^2\cosh(\beta_1)\cosh(\beta_2)}d\beta_1 d\beta_2,$$

$$\lim[{}_{1-i}\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{--;1-i}]=-\lim[{}_i\langle 0|\sigma^+(0,0)|\xi_2,\xi_1\rangle_{--;i}], \tag{15}$$

$$p(\xi)\to p(\beta)=\cot^{-1}[\sinh(\beta)],\quad -\pi\leqslant p(\beta)\leqslant 0,$$

$$e(\xi)\to e(\beta)=\frac{\pi}{\cosh(\beta)}=-\pi\sin(p(\beta)),$$

where

$$A(\beta)=\exp\left(-\int_0^\infty dt\frac{\{\cosh(2t)\cos(2t\beta/\pi)-1\}\exp(t)}{t\sinh(2t)\cosh(t)}\right). \tag{16}$$

Restricting to the first Brillouin zone (i.e., $0\leqslant k\leqslant 2\pi$ which is assumed in the sequel), integrating the continuous and discrete delta functions, keeping track of the Jacobian factors, the energy-momentum conservation relations, and the isotropic limits of Eq. (13), we find that $S_2^{i,+-}(w,k-\pi)$ is independent of i and simplifies substantially to

$$S_2^{i,+-}(w,k-\pi)=\frac{\Theta(2\pi\sin(k/2)-w)\Theta(w-\pi|\sin(k)|)A(\overline{\beta_1}-\overline{\beta_2})}{2\sqrt{[2\pi\sin(k/2)]^2-w^2}}, \tag{17}$$

where Θ is the Heaviside step function, and for fixed w and k, $(\overline{\beta_1},\overline{\beta_2})$ is a solution to

$$w=e(\overline{\beta_1})+e(\overline{\beta_2}),$$

$$k=-p(\overline{\beta_1})-p(\overline{\beta_2}). \tag{18}$$

Note that $(\overline{\beta_2},\overline{\beta_1})$, $(-\overline{\beta_1},-\overline{\beta_2})$, and $(-\overline{\beta_2},-\overline{\beta_1})$ are all identified with (β_1,β_2).

Let us now make some comments about $S_2^{i,+-}(w,k-\pi)$ as given by Eq. (17). From the isotropy of the Heisenberg model and the inclusion of both sectors $i=0$ and $i=1$, the total two-spinon contribution is obtained through

$$S_2^{+-}(w,k-\pi)=\sum_{i=0}^{1}S_2^{i,+-}(w,k-\pi)=2S_2^{i,+-}(w,k-\pi), \tag{19}$$

from which we derive all the nonvanishing components of the DCF as

$$S_2^{\mu\mu}(w,k-\pi)=2S_2^{+-}(w,k-\pi),\quad \mu=x,y,z. \tag{20}$$

Here we have used

$$\sigma^{\pm} = \frac{\sigma^x \pm i\sigma^y}{2}. \tag{21}$$

Furthermore, from the dispersion relations of two spinons, w, as a function of k, lies between two boundaries: the lower one is given by the famous des Cloizeaux-Pearson dispersion relation, i.e.,

$$w_l = \pi|\sin(k)|, \quad 0 \leqslant k \leqslant 2\pi, \tag{22}$$

whereas the upper one is given by the dispersion relation

$$w_u = 2\pi \sin(k/2), \quad 0 \leqslant k \leqslant 2\pi. \tag{23}$$

Note that despite its square root singularity, $S_2^{+-}(w,k-\pi)$ actually vanishes in the vicinity of the upper boundary. Moreover, it diverges in the vicinity of the lower boundary. It would be interesting to compare our results with presently existing approximate results, and especially the ansatz made for the two-spinon DCF in Ref. 5. In this regard, let us mention that unlike in the latter reference, the upper cutoff at $w = w_u$ appears naturally in our formula. It would also be interesting to investigate the order of contribution of more than two spinons to the DCF, and in particular that of four spinons. Also of interest is to find to what extent the two-spinon DCF satisfy the various sum rules which involve the full DCF.[5] More recently, some of these issues have been treated in Refs. 14–17. In particular, in Ref. 14 the two-spinon DCF of this manuscript is further simplified, in that, it is expressed just in terms of w, w_u, and w_l. Moreover, in Ref. 17 an exact integral formula for the DCF of n spinons is derived. The extension of this work to the Heisenberg model with higher spin is certainly desirable. In this case, the form factors can in principle be computed through the bosonization of the vertex operators which is now available.[18]

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